Oyama, T.
Osaka J. Math.
10 (1973), 379-439

ON MULTIPLY TRANSITIVE GROUPS XI

TUYOSI OYAMA

(Received December 3, 1971)
(Revised February 10, 1973)

1. Introduction

In this paper we shall prove the following

Theorem. *Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$. If a Sylow 2-subgroup of a stabilizer of four points in G fixes exactly seven points, then G must be A_7 or M_{23}.*

To prove the theorem we need the following Lemma 1, which will be proved in the section 3.

Lemma 1. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, 23\}$ and assume the following:*

(1) *To any two distinct points i, j, of Ω there corresponds a subset $\Delta(i, j)$ of Ω.*
(2) $|\Delta(i, j)|\geq 1$, $\Delta(i, j)=\Delta(j, i)$ *and* $\Delta(i, j)\not\ni i, j$.
(3) *If* $\Delta(i, j)\ni k$, *then* $\Delta(i, k)\ni j$.
(4) *G is a subgroup of M_{23}.*
(5) *Let $a\in G$ and $\Delta(i, j)=\{k_1, k_2, \cdots, k_t\}$. Then $\Delta(i, j)^a=\Delta(i^a, j^a)=\{k_1^a, k_2^a, \cdots, k_t^a\}$.*
(6) *If $\Delta(i, j)\ni k$, then there is an involution a of G such that $I(a)\supset\{i, j, k\}$.*
(7) *Let a be an involution of G. If $I(a)\supset\{i, j\}$, then $|\Delta(i, j)\cap I(a)|=1$.*

Then G is isomorphic to a subgroup of M_{11} which is one of the following groups:

(i) $G\simeq M_{11}$ *and the lengths of the G-orbits are* 11, 12.
(ii) $G\simeq M_{10}$ *and the lenghts of the G-orbits are* 1, 10, 12.
(iii) $G\simeq M_{10}^*$, *where M_{10}^* is a subgroup of M_{10} of index 2, and the lengths of the G-orbits are* 1, 10, 6, 6.

This work was done at Osaka University and supported by Japan Society for the Promotion on Science. The author wishes to thank Professor Hirosi Nagao for his useful suggestions.

(iv) *$G\cong N_{M_{11}}(M_9)$ and the lengths of the G-orbits are* 2, 9, 12.
(v) *$G\cong N_{M_{11}}(M_9)^*$, where $N_{M_{11}}(M_9)^*$ is a subgoup of $N_{M_{11}}(M_9)$ of index* 2, *and the lengths of the G-orbits are* 2, 9, 6, 6.
(vi) *$G\cong PSL(2, 11)$ and the lengths of the G-orbits are* 11, 1, 11.
(vii) *$G\cong S_5$ and the lengths of the G-orbits are* 5, 6, 2, 10.

We shall use the same notations as in [3].

2. Proof of the theorem

The proof of the theorem is obtained in the following way: If a Sylow 2-subgroup of the stabilizer of four points in G is semiregular, then G is A_7 or M_{23} by [3]. Here we say that P is semiregular when P is "semiregular" on $\Omega-I(P)$ in the usual sense. Hence, by way of contradiction, we assume that P is not semiregular. Let R be a subgroup of P such that its order is maximal among all subgroups of P which fix more than twenty-three points and set $N=N_G(R)^{I(R)}$. Then N has an involution a which fixes exactly twenty-three points and then $C_N(a)^{I(a)}$ is a subgroup of M_{23}. Let $(i_1 i_2)$ be a 2-cycle of a, and for $\{i, j\}\subset I(a)$, let $\Delta(i, j)$ be the intersection of $I(a)$ and the $G_{i_1 i_2 i j}$-orbit of odd length (>1) which is uniquely determined by $\{i_1, i_2, i, j\}$. Then $(C_N(a)_{i_1 i_2})^{I(a)}$ and $\{\Delta(i, j)\}$ satisfy the assumption of Lemma 1, and hence $(C_N(a)_{i_1 i_2})^{I(a)}$ is one of the permutation groups listed in Lemma 1. Thus we see that N satifies the conditions of Lemma 3 and we have a contradiction. The above result on $(C_N(a)_{i_1 i_2})^{I(a)}$ is useful in determining $C_N(a)^{I(a)}$.

Let P be a Sylow 2-subgroup of the stabilizer G_{1234}. If P is semiregular, then G is A_7 or M_{23} by [3]. Hence from now on we assume that P is not semiregular and proceed by way of contradiction.

2.1. *For any point t of $\Omega-I(P)$ $|I(P_t)|\geq 23$. In particular if $|I(P_t)|=23$, then $N_G(G_{I(P_t)})^{I(P_t)}=M_{23}$.*

Proof. Let $|I(P_t)|$ be the smallest number such that $t\in\Omega-I(P)$ and Q a Sylow 2-subgroup of $G_{I(P_t)}$. Then $I(Q)=I(P_t)$. For any four points i, j, k, l of $I(P_t)$ let P' be a Sylow 2-subgroup of G_{ijkl} containing Q. Then P' is conjugate to P. Hence by minimality of $|I(P_t)|$, $N_{P'}(Q)^{I(Q)}$ is a semiregular 2-group fixing $I(P')$ pointwise. Thus $N_G(Q)^{I(Q)}=M_{23}$ and $|I(Q)|=23$ by Theorem 1 in [4].

In particular if $|I(P_t)|=23$, then by the same argument as above $N_G(Q)^{I(Q)}=M_{23}$, where Q is a Sylow 2-subgroup of $G_{I(P_t)}$. Hence by the Frattini argument, $N_G(G_{I(P_t)})=G_{I(P_t)}\cdot N_G(Q)$ and so $N_G(G_{I(P_t)})^{I(P_t)}=N_G(Q)^{I(P_t)}=M_{23}$.

2.2. *Let $t\in\Omega-I(P)$. If $|I(P_t)|=23$, then t is a point of a G_{1234}-orbit of even length.*

Proof. By (2.1), $N_G(G_{I(P_t)})^{I(P_t)}=M_{23}$. Hence a Sylow 2-subgroup of $(N_G(G_{I(P_t)})^{I(P_t)})_{1234}$ is a regular normal subgroup of $(N_G(G_{I(P_t)})^{I(P_t)})_{1234}$. On the other hand since $t\in\Omega-I(P)$, $N_P(P_t)^{I(P_t)}$ is a nonidentity 2-subgroup of $(N_G(G_{I(P_t)})^{I(P_t)})_{1234}$. Hence any nonidentity 2-subgroup of $(N_G(G_{I(P_t)})^{I(P_t)})_{1234}$ fixes exactly $I(P)$ pointwise.

Suppose by way of contradiction that t is a point of a G_{1234}-orbit of odd length. Let Q be a Sylow 2-subgroup of $G_{I(P_t)}$ containing P_t and P' be a Sylow 2-subgroup of G_{1234t} containing Q. Since the index of G_{1234t} in G_{1234} is odd, P' is a Sylow 2-subgroup of G_{1234}. Hence $N_{P'}(Q)^{I(Q)}$ is a nonidentity 2-subgroup of $(N_G(G_{I(P_t)})^{I(P_t)})_{1234}$. Thus $I(P')=I(P)$. This shows that $t\in I(P)$, contrary to the assumption.

2.3. *Since P is not semiregular, $|\Omega|>23$. Hence there is a subgroup of P which has more than twenty-three fixed points. Let R be a subgroup of P such that its order is maximal among all subgroups of P which have more than twenty-three fixed points. Here R may be the identity. Then R satisfies the following conditions*:

(1) $|I(R)|>22$.

(2) *Any imvolution of $N_G(R)^{I(R)}$ fixes exactly seven or twenty-three points. In particular any central involution of a Sylow 2-subgroup in $N_G(R)^{I(R)}$ fixes exactly twenty-three points.*

(3) *The stabilizer of any four points of $I(R)$ in $N_G(R)^{I(R)}$ has an involution fixing exactly twenty-three points.*

Proof. Since Sylow 2-subgroups of the stabilizer of four points in G are conjugate in G, R is a Sylow 2-subgroup of $G_{I(R)}$. Clearly $|I(R)|>23$.

For any four points i, j, k and l of $I(R)$ let P' be a Sylow 2-subgroup of G_{ijkl} containing R. Since R is a Sylow 2-subgroup of $G_{I(R)}$, $N_{P'}(R)^{I(R)}\neq 1$. Thus $(N_G(R)^{I(R)})_{ijkl}$ has a nonidentity 2-subgroup. Let Q be a Sylow 2-subgroup of $(N_G(R)^{I(R)})_{ijkl}$ and x an involution of Q. Then by assumption $|I(Q)|\leq 23$ and $|I(x)|\leq 23$. Hence by assumption and (2.1), $|I(Q)|=7$ or 23 and $N_{N_G(R)^{I(R)}}(Q)^{I(Q)}\leq A_7$ or M_{23} respectively and $|I(x)|=7$ or 23.

Now if any involution of $N_G(R)^{I(R)}$ fixing at least four points fixes exactly seven points, then $|I(R)|=23$ by Theorem 1 in [4], which is a contradiction. Thus there is an involution fixing exactly twenty-three points in $N_G(R)^{I(R)}$. To complete the proof of (2.3) it is sufficient to prove the following lemma.

Lemma 2. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$, $n>23$. Let P be a Sylow 2-subgroup of the stabilizer of any four points in G. Assume that P satisfies the following conditions*:

(i) *P fixes exactly seven or twenty-three points.*

(ii) *If $|I(P)|=7$, then $N_G(P)^{I(P)}\leq A_7$ and if $|I(P)|=23$, then $N_G(P)^{I(P)}\leq M_{23}$.*

(iii) *Let x be any involution of P. Then* $|I(x)|=7$ *or* 23. *Moreover if* $|I(x)|=23$, *then* $N_G(G_{I(x)})^{I(x)} \leq M_{23}$.

Then

(1) *any involution of G fixes exactly seven or twenty-three points and any central involution of a Sylow 2-subgroup of G fixes exactly twenty-three points,*

(2) *the stabilizer of four points in G has an involution fixing exactly twenty-three points.*

Proof. If any involution of G fixing at least four points fixes exactly seven points, then $|\Omega|=23$ by Theorem 1 in [4], contrary to the assumption. Thus there is an involution fixing twenty-three points.

Let y be an involution. Since $|I(P)|=7$ or 23, $|\Omega|$ is odd and so $|I(y)|$ is odd. Suppose that $|I(y)|=1$ and $y=(i_1 i_2)(i_3 i_4)\cdots$. Since $y \in N_G(G_{i_1 i_2 i_3 i_4})$, there is a Sylow 2-subgroup P' of $G_{i_1 i_2 i_3 i_4}$ such that $y \in N_G(P')$. By assumption, $N_G(P')^{I(P')} \leq A_7$ or M_{23}. Hence $|I(y^{I(P')})|=3$ or 7, which is a contradiction. Thus $|I(y)| \neq 1$. If $|I(y)| \geq 4$, then by assumption, $|I(y)|=7$ or 23. Thus $|I(y)|=3$, 7 or 23.

Now we remark that if y and y' are involutions such that $|I(y)|=23$ and $yy'=y'y$, then $|I(y'^{I(y)})|=7$ or 23 since $y'^{I(y)} \in N_G(G_{I(y)})^{I(y)} \leq M_{23}$.

Since there is an involution fixing exactly twenty-three points, let a be an involution such that $|I(a)|=23$. We may assume that

$$a = (1)\,(2)\cdots(23)\,(24\ 25)\cdots.$$

Since $a \in N_G(G_{1\,2\,24\,25})$, there is an involution b of $G_{1\,2\,24\,25}$ commuting with a. Then by the remark above $|I(b^{I(a)})|=7$ or 23. Furthermore b fixes 24, 25 which are not contained in $I(a)$. Thus by assumption, $|I(b)|=23$. Hence we may assume that

$$b = (1)\,(2)\cdots(7)\,(8\ 9)\,(10\ 11)\cdots(22\ 23)\,(24)\,(25)\cdots(39)\cdots.$$

Since $ab=ba$, we may assume that

$$a = (1)\,(2)\cdots(23)\,(24\ 25)\,(26\ 27)\cdots(38\ 39)\cdots.$$

Since $\langle a, b\rangle < N_G(G_{8\,9\,24\,25})$, there is an involution c of $G_{8\,9\,24\,25}$ commuting with a and b. Similarly $|I(c)|=23$. Since $N_G(G_{I(a)})^{I(a)}$, $N_G(G_{I(b)})^{I(b)}$ and $N_G(G_{I(c)})^{I(c)}$ are subgroups of M_{23}, we may assume that

$$\begin{aligned} c = {} & (1)\,(2\ 3)\,(4\ 5)\,(6)\,(7)\,(8)\,(9)\,(10\ 11)\,(12\ 14)\,(13\ 15)\,(16)\,(17)\,(18\ 19) \\ & (20\ 22)\,(21\ 23)\,(24)\,(25)\,(26)\,(27)\,(28\ 29)\,(30\ 31)\,(32\ 34)\,(33\ 35) \\ & (36\ 38)\,(37\ 39)\,(40)\,(41)\cdots(51)\cdots. \end{aligned}$$

Consequently we may assume that

$$a = (1)\,(2)\cdots(23)\,(24\ 25)\,(26\ 27)\cdots(50\ 51)\cdots,$$

$b=(1)(2)\cdots(7)(8\ 9)(10\ 11)\cdots(22\ 23)(24)(25)\cdots(39)(40\ 41)(42\ 43)$
$(44\ 46)(45\ 47)(48\ 50)(49\ 51)\cdots$.

Now let u be a central involution of a Sylow 2-subgroup of G containing $\langle a, b, c\rangle$. Then by the remark above $|I(u)|=7$ or 23. Suppose by way of contradiction that $|I(u)|=7$. Since u commutes with a, b and c, $I(u)$ is contained in $I(a)$, $I(b)$ and $I(c)$. This is a contradiction since $|I(a)\cap I(b)\cap I(c)|=3$ and $|I(u)|=7$. Thus $|I(u)|=23$. By the conjugacy of Sylow 2-subgroups of G, any central involution of a Sylow 2-subgroup of G fixes exactly twenty-three points. Furthermore this shows that there is no involution fixing three points by the remark above. Thus (1) holds.

Next by assumption, for any four points there is an involution x fixing these four points. Then $|I(x)|=7$ or 23. If $|I(x)|=7$, then by what we have proved above, there is an involution x' fixing exactly twenty-three points and commuting with x. Since $|I(x)|=7$, $I(x')\supset I(x)$. Thus (2) holds.

From now on R denotes the 2-group as in (2.3).

2.4. *Let i_1, i_2, i_3, i_4 be any four points of Ω. Then $G_{i_1 i_2 i_3 i_4}$ has exactly one orbit of odd length* $(\neq 1)$.

Proof. Let P' be a Sylow 2-subgroup of $G_{i_1 i_2 i_3 i_4}$. Since $|I(P')|=7$, we may assume that $I(P')=\{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$. Then from $N_G(P')^{I(P')}=A_7$, there is an element

$$x=(i_1)(i_2)(i_3)(i_4)(i_5\ i_6\ i_7)\cdots.$$

Since $x\in G_{i_1 i_2 i_3 i_4}$, three points i_5, i_6, i_7 belong to the same $G_{i_1 i_2 i_3 i_4}$-orbit. Since P' is a Sylow 2-subgroup of $G_{i_1 i_2 i_3 i_4}$, this orbit is of odd length. Furthermore by the conjugacy of Sylow 2-subgroups of $G_{i_1 i_2 i_3 i_4}$, this orbit is only one $G_{i_1 i_2 i_3 i_4}$-orbit of odd length $(\neq 1)$.

2.5. *Let a be a 2-element of $N_G(R)$ such that $a^{I(R)}$ is an involution fixing twenty-three points and let $(i_1\, i_2)$ be a 2-cycle of $a^{I(R)}$. Set $\Gamma=I(a^{I(R)})$. For any two points i, j of Γ $G_{i\, j\, i_1 i_2}$ has exactly one orbit of odd length $(\neq 1)$ by (2.5). Hence we denote the intersection of the $G_{i\, j\, i_1 i_2}$-orbit of odd length $(\neq 1)$ and Γ by $\Delta_{i_1 i_2}(i, j)$ or merely $\Delta(i, j)$. Then we have*

(1) *$|\Delta(i, j)|\geq 1$ and $\Delta(i, j)=\Delta(j, i)$.*
(2) *If $\Delta(i, j)\ni k$, then $\Delta(i, k)\ni j$.*
(3) *If $\Delta(i, j)\ni k$, then there is a 2-element x of $N_G(R)$ such that $x^{I(R)}$ is an involution commuting with $a^{I(R)}$, $|I(x^{I(R)})|=23$ and $I(x^{I(R)})\supset \{i, j, k, i_1, i_2\}$.*
(4) *Let x be a 2-element of $N_G(R)$ such that $x^{I(R)}$ is an involution fixing*

twenty-three points and commuting with $a^{I(R)}$. *If* x *fixes* i, j, i_1, i_2, *then* $|\Delta(i, j)\cap I(x^{\Gamma})|=1$.

Proof. Since $\langle a, R\rangle < N_G(G_{i\,j\,i_1\,i_2})$ and $G_{i\,j\,i_1\,i_2}$ has exactly one orbit of odd length ($\neq 1$), $\langle a, R\rangle$ fixes this $G_{i\,j\,i_1\,i_2}$-orbit as a set. Furthermore since $\langle a, R\rangle$ is a 2-group, $\langle a, R\rangle$ has fixed points in this $G_{i\,j\,i_1\,i_2}$-orbit. Thus $|\Delta(i, j)|\geq 1$. Clearly $\Delta(i, j)=\Delta(j, i)$. Thus (1) holds.

By the 4-fold transitivety of G, $|G_{i_1 i_2 i\,j}: G_{i_1 i_2 i\,j\,k}|=|G_{i_1 i_2 i\,k}: G_{i_1 i_2 i\,j\,k}|$, where $k\notin\{i_1, i_2, i, i, j\}$. Thus (2) holds.

Let k be a point of $\Delta(i, j)$. Since $a\in N_G(G_{i_1\,i_2\,i\,j\,k})$, a normalizes some Sylow 2-subggroup P' of $G_{i_1\,i_2\,i\,j\,k}$ containing R. Since k is a point of the $G_{i_1\,i_2\,i\,j}$-orbit of odd length, P' is also a Sylow 2-subgroup of $G_{i_1\,i_2\,i\,j}$. Set $Q=N_{P'}(R)$. Since R is a Sylow 2-subgroup of $G_{I(R)}$, $Q^{I(R)}\neq 1$. Then because $a\in N_G(Q)$ and $I(R)^a=I(R)$, there is an element x in Q such that $x^{I(R)}$ is an involution and commutes with $a^{I(R)}$. Since $I(Q)\supseteq I(P')\supseteq\{i, j, k, i_1, i_2\}$, $I(x)\supseteq\{i, j, k, i_1, i_2\}$. Furthermore since $N_G(a)^{\Gamma}\leq M_{23}$, $|I(x^{\Gamma})|=7$ or 23. On the other hand $x^{I(R)}$ fixes two points k, l which are not in Γ. Hence by (2.3), $|I(x^{I(R)})|=23$. Thus (3) holds.

Let x be a 2-element of $N_G(R)$ satisfying the assumption of (4). Since $a\in N_G(\langle x, x^a, R\rangle)$, there is a Sylow 2-subgroup $\bar{R}$ of $\langle x, x^a, R\rangle$ such that $a\in N_G(\bar{R})$. Then since $\bar{R}^{I(R)}=\langle x\rangle^{I(R)}$, there is a 2-element y of $N_G(R)$ such that $y^{I(R)}=x^{I(R)}$ and $\bar{R}=\langle y, R\rangle$. Since $a\in N_G(G_{i\,j\,i_1\,i_2})$, a normalizes a Sylow 2-subgroup P' of $G_{i\,j\,i_1\,i_2}$ containing $\bar{R}$. Set $I(P')=\{i, j, i_1, i_2, k_1, k_2, k_3\}$. Since $N_G(P')^{I(P')}=A_7$, we may assume that $a^{I(P')}=(i)(j)(i_1\,i_2)(k_1\,k_2)(k_3)$. Since k_3 is fixed by the Sylow 2-subgroup P' of $G_{i\,j\,i_1\,i_2}$, k_3 belongs to the $G_{i\,j\,i_1\,i_2}$-orbit of odd length ($\neq 1$). Since $I(x^{I(R)})=I(y^{I(R)})$ and any point of $I(y^{I(R)})-I(P')$ belongs to a $G_{i\,j\,i_1\,i_2}$-orbit of even length by (2.2) $\Delta(i, j)\cap I(x^{\Gamma})=\{k_3\}$. Thus (4) holds.

2.6. *Let* a *be 2-element of* $N_G(R)$ *such that* $a^{I(R)}$ *is an involution fixing twenty-three points and let* $(i_1\,i_2)$ *be any 2-cycle of* $a^{I(R)}$. *Set* $\Gamma=I(a^{I(R)})$ *and* $N=N_G(R)^{I(R)}$. *Then* $(C_N(a^{I(R)})_{i_1\,i_2})^{\Gamma}$ *is one of the groups as in Lemma* 1.

Proof. By (2.5), $\Delta_{i_1\,i_2}(i, j)$ in (2.5) and $(C_N(a^{I(R)})_{i_1\,i_2})^{\Gamma}$ satisfy the assumptions of Lemma 1. Hence $(C_N(a^{I(R)})_{i_1\,i_2})^{\Gamma}$ is one of the goups as in Lemma 1.

By the assumption of the theorem, (2.3) and (2.6), $N=N_G(R)^{I(R)}$ satisfies the following conditions:

(1) Let P be a Sylow 2-subgroup of the stabilizer of any four points in N. Then $|I(P)|=7$ and $N_N(P)^{I(P)}\leq A_7$ or $|I(P)|=23$ and $N_N(P)^{I(P)}\leq M_{23}$.

(2) Any involution u of P fixes exactly seven or twenty-three points. If $|I(u)|=23$, then $N_N(u)^{I(u)}\leq M_{23}$ and for any 2-cycle $(i\,j)$ of u $(C_N(u)_{i\,j})^{I(u)}$ is one of the groups as in Lamma 1,

In Lemma 3 we shall show that there is no group satisfying the above conditions (1) and (2). Then we shall complete the proof of the theorem.

Lemma 3. *Let G be a permutation group on $\Omega=\{1, 2, \cdots, n\}$, $n>23$. Then there is no group which satisfies the following conditions:*

(1) *Let P be a Sylow 2-subgroup of the stabilizer of any four points in G. Then $|I(P)|=7$ and $N_G(P)^{I(P)}\leq A_7$ or $|I(P)|=23$ and $N_G(P)^{I(P)}\leq M_{23}$.*

(2) *Any involution u of P fixes exactly seven or twenty-three points. If $|I(u)|=23$, then $C_G(u)^{I(u)}\leq M_{23}$ and for any 2-cycle $(i\,j)$ of u $(C_G(u)_{i\,j})^{I(u)}$ is one of the groups as in Lemma 1.*

Proof. The proof will be given in various steps. Suppose by way of contradiction that G is a counter-example to Lemma 3. The following (i) and (ii) follow immediately from Lemma 2.

(i) *Any involution of G fixes exactly seven or twenty-three points and any central involution of a Sylow 2-subgroup of G fixes exactly twenty-three points.*

(ii) *The stabilizer of any four points in G has an involution fixing exactly twenty-three points.*

(iii) *Let Q be a 2-group fixing exactly twenty-three points. Then Q is semiregular and elementary abelian.*

Proof. By assumption, Q is semiregular. Suppose that Q has an element $x=(i\,j\,k\,l)\cdots$. Then $x\in N_G(G_{i\,j\,k\,l})$. Hence x normalizes a Sylow 2-subgroup P' of $G_{i\,j\,k\,l}$. By assumption, $N_G(P')^{I(P')}\leq A_7$ or M_{23}. Hence x has a 2-cycle, which is a contradiction. Thus Q is elementary abelian.

(iv) *Let a be an involution fixing twety-three points and let $(i\,j)$ be a 2-cycle of a. Then any involution of $C_G(a)_{i\,j}$ fixes exactly twenty-three points. If x and y are two distinct involutions of $C_G(a)_{i\,j}$ such that $xy=yx$, then $|I(x)\cap I(y)\cap I(a)|=3$.*

Proof. Since $C_G(a)^{I(a)}$ is a subgroup of M_{23}, two commuting involutions with distinct fixed points have three fixed points in common. On the other hand $(C_G(a)_{i\,j})^{I(a)}$ is one of the groups as in Lemma 1 which is isomorphic to a subgroup of M_{11}, where M_{11} is embedded in M_{23} and has orbits of lengths 11 and 12, and two commuting invoutions in M_{11} of degree 11 have distinct fixed points. Hence two commuting distinct involutions of $(C_G(a)_{i\,j})^{I(a)}$ have exactly three fixed points in common. Thus $x^{I(a)}=y^{I(a)}$ or $|I(x)\cap I(y)\cap I(a)|=3$. If $x^{I(a)}=y^{I(a)}$, then xy is an involution fixing $I(a)\cap\{i, j\}$ pointwise. Thus $|I(xy)|>23$, contrary to the assumption (2). Hence $|I(x)\cap I(y)\cap I(a)|=3$.

Since any involution of $C_G(a)_{i\,j}$ fixes seven or twenty-three points of $I(a)$ and at least two points i, j which are not in $I(a)$, by the assumption (2), any involution of $C_G(a)_{i\,j}$ fixes exactly twenty-three points.

(v) *We may assume that there are three involutions a, b and c of the following forms such that $|I(a)|=|I(b)|=|I(c)|=23$ and $\langle a, b, c\rangle$ is elementary abelian.*

$$a=(1)\,(2)\cdots(23)\,(24\ 25)\,(26\ 27)\cdots(50\ 51)\cdots,$$
$$b=(1)\,(2)\cdots(7)\,(8\ 9)\,(10\ 11)\cdots(22\ 23)\,(24)\,(25)\cdots(39)\,(40\ 41)\,(42\ 43)\,(44\ 46)\,(45\ 47)\,(48\ 50)\,(49\ 51)\cdots,$$
$$c=(1)\,(2\ 3)\,(4\ 5)\,(6)\,(7)\,(8)\,(9)\,(10\ 11)\,(12\ 14)\,(13\ 15)\,(16)\,(17)\,(18\ 19)\,(20\ 22)\,(21\ 23)\,(24)\,(25)\,(26)\,(27)\,(28\ 29)\,(30\ 31)\,(32\ 34)\,(33\ 35)\,(36\ 38)\,(37\ 39)\,(40)\,(41)\cdots(51)\cdots.$$

Proof. By (ii), there is an involution a fixing twenty-three points. We may assume that

$$a=(1)\,(2)\cdots(23\ )(24\ 25)\cdots.$$

By (iv), there is an involution b of $C_G(a)_{24\,25}$. We may assume that

$$b=(1)\,(2)\cdots(7)\,(8\ 9)\,(10\ 11)\cdots(22\ 23)\,(24)\,(25)\cdots(39)\cdots.$$

Since $\langle a, b\rangle \leq N_G(G_{8\,9\,24\,25})$, there is an involution c commuting with a and b. By (iv), c fixes exactly twenty-three points. Hence we may assume that

$$c=(1)\,(2\ 3)\,(4\ 5)\,(6)\,(7)\,(8)\,(9)\,(10\ 11)\,(12\ 14)\,(13\ 15)\,(16)\,(17)\,(18\ 19)\,(20\ 22)\,(21\ 23)\,(24)\,(25)\,(26)\,(27)\,(28\ 29)\,(30\ 31)\,(32\ 34)\,(33\ 35)\,(36\ 38)\,(37\ 39)\,(40)\,(41)\cdots(51)\cdots.$$

Hence we my assume that

$$a=(1)\,(2)\cdots(23)\,(24\ 25)\,(26\ 27)\cdots(50\ 51)\cdots,$$
$$b=(1)\,(2)\cdots(7)\,(8\ 9)\,(10\ 11)\cdots(22\ 23)\,(24)\,(25)\cdots(39)\,(40\ 41)\,(42\ 43)\,(44\ 46)\,(45\ 47)\,(48\ 50)\,(49\ 51)\cdots.$$

From now on a, b and c denote the elements as in (v).

(vi) *Set $T=(C_G(a)_{24\,25})^{I(a)}$. Let*

$$\bar{b}=b^{I(a)},$$
$$\bar{c}=c^{I(a)},$$
$$x_1=(1)\,(2)\,(8)\,(3\ 9)\,(10\ 12)\,(11\ 15)\,(13\ 14)\,(4)\,(18\ 19)\,(16)\,(5\ 17)\,(20)\,(6\ 21)\,(23)\,(7\ 22),$$
$$x_2=(1)\,(2)\,(10)\,(3\ 11)\,(8\ 12)\,(9\ 14)\,(13\ 15)\,(18)\,(4\ 19)\,(5)\,(16\ 17)\,(20)\,(6\ 21)\,(22)\,(7\ 23),$$

$x_3 = (1\ 2)(3)(8)(9)(12\ 13)(10\ 14)(11\ 15)(4)(5\ 6)(7)(16\ 20)(17\ 21)$
$(18\ 19)(22)(23)$,
$x_4 = (1\ 2)(3)(8\ 13)(9\ 12)(10\ 11)(14)(15)(5)(4\ 7)(6)(16)(17)(20\ 21)$
$(18\ 23)(19\ 22)$,
$x_5 = (2)(8)(10)(1\ 12)(9\ 13)(3\ 15)(14\ 11)(4)(7)(17)(18)(5\ 23)(6\ 21)$
$(16\ 22)(19\ 20)$,
$x_6 = (3)(13)(10)(2\ 14)(1\ 8)(9\ 11)(12\ 15)(4)(5)(17)(22)(6\ 18)(7\ 20)$
$(16\ 21)(19\ 23)$,
$y_1 = x_1^{\bar{c}x_3} = (2)(3)(8)(1\ 9)(10\ 15)(11\ 13)(12\ 14)(20)(22)(4\ 21)(5\ 23)$
$(6)(17)(18\ 19)(7\ 16)$,
$y_2 = x_2^{\bar{c}x_3} = (2)(3)(15)(1\ 14)(8\ 10)(9\ 13)(11\ 12)(4)(22)(20\ 21)(5\ 23)$
$(18)(16)(6\ 19)(7\ 17)$,
$y_3 = x_4^{\bar{c}x_3} = (2\ 3)(1)(12)(13)(8\ 11)(9\ 10)(14\ 15)(4)(5)(20)(21)$
$(22\ 23)(6\ 7)(16\ 19)(17\ 18)$.

Then T is isomorphic to one of the following groups, and hence we may identify T with one of these groups:

(1) $\langle \bar{b}, \bar{c}, x_1, x_2, x_3 \rangle$, *which is isomorphic to* M_{11} *and has orbits* $\{1, 2, 3, 8, 9, \cdots, 15\}$ *and* $\{4, 5, 6, 7, 16, 17, \cdots, 23\}$.
(2) $\langle \bar{b}, \bar{c}, x_1, x_2, x_3x_4 \rangle$, *which is isomrphic to* M_{10} *and has orbits* $\{1\}$, $\{2, 3, 8, 9, \cdots, 15\}$ *and* $\{4, 5, 6, 7, 16, 17, \cdots, 23\}$.
(3) $\langle \bar{b}, \bar{c}, x_1, x_2 \rangle$, *which is isomorphic to* M_{10}^* *and has orbits* $\{1\}$, $\{2, 3, 8, 9, \cdots, 15\}$, $\{4, 5, 16, 17, 18, 19\}$ *and* $\{6, 7, 20, 21, 22, 23\}$.
(4) $\langle \bar{b}, \bar{c}, y_1, y_2, x_3x_4 \rangle$, *which is isomorphic to* $N(M_9)$ *and has orbits* $\{2, 3\}$, $\{1, 8, 9, \cdots, 15\}$ *and* $\{4, 5, 6, 7, 16, 17, \cdots, 23\}$.
(5) $\langle \bar{b}, \bar{c}, y_1, y_2, y_3 \rangle$,*which is isomrphic to* $N(M_9)^*$ *and has orbits* $\{2, 3\}$, $\{1, 8, 9, \cdots, 15\}$, $\{4, 5, 20, 21, 22, 23\}$ *and* $\{6, 7, 16, 17, 18, 19\}$.
(6) $\langle \bar{b}, \bar{c}, x_5 \rangle$, *which is isomorphic to* $PSL(2, 11)$ *and has orbits* $\{1, 2, 3, 8, 9, \cdots, 15\}$, $\{7\}$ *and* $\{4, 5, 6, 16, 17, \cdots, 23\}$.
(7) $\langle \bar{b}, \bar{c}, x_6 \rangle$, *which is isomorphic to* S_5 *and has orbits* $\{1, 8, 9, 10, 11\}$, $\{2, 3, 12, 13, 14, 15\}$, $\{4, 5\}$ *and* $\{6, 7, 16, 17, \cdots, 23\}$.

Proof. By the assumption (2), T is isomorphic to one of the following groups: M_{11}, M_{10}, M_{10}^*, $N(M_9)$, $N(M_9)^*$, $PSL(2, 11)$ or S_5. In (3.16) and (3.17) of the section 3 we shall show that these groups have the generators and the orbits as in (vi).

(vii) *Let Q be a 2-subgroup fixing exactly twenty-three points. If Q is a Sylow 2-subgroup of $G_{I(Q)}$, then Q is of order two and $N_G(Q)^{I(Q)} \leq M_{23}$.*

Proof. Let $I(Q) = \{1, 2, \cdots, 23\}$. Suppose that the order of Q is at least four. Then by (iii), Q has two involutions x and y. We may assume that

$$x = (1)\,(2)\cdots(23)\,(24\ 25)\,(26\ 27)\cdots,$$
$$y = (1)\,(2)\cdots(23)\,(24\ 26)\,(25\ 27)\cdots.$$

Since $\langle x, y\rangle < N_G(G_{24\ 25\ 26\ 27})$, there is an involution z of $G_{24\ 25\ 26\ 27}$ fixing twenty-three points and commuting with x and y. By the assumption (2), we may assume that

$$z = (1)\,(2)\cdots(7)\,(8\ 9)\,(10\ 11)\cdots(22\ 23)\,(24)\,(25)\,(26)\,(27)\cdots.$$

Then $\langle x, y\rangle < C_G(z)_{8\ 9}$ and $|I(x)\cap I(y)\cap I(z)|=7$, contrary to (iv). Thus the order of Q is two.

By the assumption (2), $N_G(Q)^{I(Q)}\leq M_{23}$.

From now on set $T=(C_G(a)_{24\ 25})^{I(a)}$ and $C=C_G(a)$. Then by (vii), $\langle a\rangle$ is a Sylow 2-subgroup of $G_{I(a)}$.

(viii) *Let x and y be involutions such that both x and y belong to $C_G(a)\cap C_G(b)$ and $x^{I(a)}=y^{I(a)}$. Then $x^{I(b)}=y^{I(b)}$ or $(ya)^{I(b)}$.*

Proof. Since $(xy)^{I(a)}=1$ and $xy\in C_G(b)$, $|I(xy)\cap I(b)|=7$ or 23. Hence $(xy)^{I(b)}$ is an involution or the identity. If $(xy)^{I(b)}$ is the identity, then $x^{I(b)}=y^{I(b)}$. Next suppose that $(xy)^{I(b)}$ is an involution. Assume that $a^{I(b)}\neq(xy)^{I(b)}$. Since $xy\in C$, $\langle a, xy\rangle^{I(b)}$ is of order four. Hence $G_{I(a)}$ has a subgroup $\langle a, xy\rangle$ of order divisible by four, contrary to (vii). Thus $a^{I(b)}=(xy)^{I(b)}$ and so $x^{I(b)}=(ya)^{I(b)}$

(ix) *Let $(i\ j)$ be any 2-cycle of a. If $(C_{i\ j})^{I(a)}$ is not isomrophic to M_{11}, then $C^{I(a)}$ contains $(C_{i\ j})^{I(a)}$ properly.*

Proof. We may assume that $(i\ j)=(24\ 25)$. Then $C_{24\ 25}>\langle b, c\rangle$. By assumption, $T=(C_{24\ 25})^{I(a)}$ is one of the groups other than (1) in (vi). Suppose by way of contradiction that $T=C^{I(a)}$. Then $(C_{32\ 33})^{I(a)}$ is a subgroup of T and isomorphic to one of the groups other than (i) in Lemma 1, where 32 and 33 are fixed by b and a has a 2-cycle (32 33).

Now we show that $C_{32\ 33}$ has an element x such that $x^{I(a)}=c^{I(a)}$. Since none of groups M_{10}*, $N(M_9)$*, $PSL(2, 11)$ or S_5 as Lemma 1 contains any group as Lemma 1 properly, if T is isomorphic to M_{10}*, $N(M_9)$*, $PSL(2, 11)$ or S_5, then $(C_{32\ 33})^{I(a)}=T$. Hence $C_{32\ 33}$ has an element x such that $x^{I(a)}=c^{I(a)}$. Next suppose that T is isomorphic to M_{10}. Then $(C_{32\ 33})^{I(a)}$ is isomorphic to M_{10} or M_{10}*. On the other hand if a subgroup of T is isomorphic to M_{10}*, then this subgroup is the group (3) in (vi). Thus $(C_{32\ 33})^{I(a)}=T$ or $(C_{32\ 33})^{I(a)}$ is the group (3) in (vi), and so $C_{32\ 33}$ has an element x such that $x^{I(a)}=c^{I(a)}$. Finally suppose that T is isomorphic to $N(M_9)$. Then $(C_{32\ 33})^{I(a)}$ is isomorphic to $N(M_9)$ or $N(M_9)$*. Similarly if a subgroup of T is isomorphic to $N(M_9)$*, then this subgroup is the group (5) in (vi). Thus $(C_{32\ 33})^{I(a)}=T$ or $(C_{32\ 33})^{I(a)}$ is the group (5) in (vi), and so $C_{32\ 33}$

has an element x such that $x^{I(a)}=c^{I(a)}$. Thus in any case $C_{32\ 33}$ has an element x such that $x^{I(a)}=c^{I(a)}$.

Then x is an involution on $I(a)\cup\{32, 33\}$. Since x^2 fixes $I(a)\cup\{32, 33\}$ pointwise and $|I(a)\cup\{32, 33\}|=25$, x^2 is odd order. Hence x^s is an involution, where s is a suitable odd integer. Set $c'=x^s$. Then $C_{32\ 33}$ has an involution c' such that $c'^{I(a)}=c^{I(a)}$. Then bc' is an involution on $I(a)\cup\{32, 33\}$. Since $(bc')^2$ fixes $I(a)\cup\{32, 33\}$ pointwise and $|I(a)\cup\{32, 33\}|=25$, $(bc')^2$ is of odd oder. Hence bc' is of order $2r$, where r is an odd integer. Set $c''=b(bc')^r$. Then c'' is an involution commuting with b and $c''^{I(a)}=c^{I(a)}$. Since $c^{I(b)}=(32\ 34)\ (33\ 35)\cdots$, $c''^{I(b)}=(32)\ (33)\cdots$ and $a^{I(b)}=(32\ 33)\cdots$, $(cc'')^{I(b)}\neq 1$, $a^{I(b)}$, contrary to (viii).

(x) *Let x be a 2-element of C such that $x^{I(a)}$ is an involution. If there is an involution y in C such that $|I(y)|=23$ and $x^{I(a)}$ and $y^{I(a)}$ are conjugate in $C^{I(a)}$, then x is an involution fixing twenty-three points.*

Proof. By assumption, there is an element z of C such that $(x^z)^{I(a)}=y^{I(a)}$. Let R be a Sylow 2-subgroup of $\langle x^z, y, a\rangle$. Since $R^{I(a)}=\langle y\rangle^{I(a)}$, the order of $R^{I(a)}$ is two. By (vii), $R_{I(a)}=\langle a\rangle$. Thus the order of R is four. This implies that both $\langle x^z, a\rangle$ and $\langle y, a\rangle$ are Sylow 2-subgroups of $\langle x^z, y, a\rangle$. Hence x^z is conjugate to y or ya. Thus x is conjugae to the involution y or ya which fixes exactly twenty-three points by (iv). Hence x is an involution fixing twenty-three points.

(xi) *Let x and y be distinct involutions of $C-\langle a\rangle$ such that $xy=yx$, $I(x^{I(a)})=I(y^{I(a)})$ and $x\neq ya$. If z is an involution of $C-\langle a\rangle$ fixing twenty-three points and commuting with x and y, then $I(x^{I(a)})=I(z^{I(a)})$.*

Proof. Suppose by way of contradiction that $I(x^{I(a)})\neq I(z^{I(a)})$. Set $H=\langle x, y, a\rangle$. Then H is an elementary abelian group of order eight. Since $I(H^{I(a)})=I(x^{I(a)})\neq I(z^{I(a)})$, any element of $H-\langle a\rangle$ fixes the same three points of $I(a)\cap I(z)$ and four points of $I(z)-(I(z)\cap I(a))$. Since $|H-\langle a\rangle|=6$ and $|I(z)-(I(z)\cap I(a))|=16$, there are two involutions u and v of $H-\langle a\rangle$ such that u and v fix the same points i, j of $I(z)-(I(z)\cap I(a))$, where $(i\ j)$ is a 2-cycle of a. Then u and v are involutions of $C_{i\ j}$ and $I(u^{I(a)})=I(v^{I(a)})$, contrary to (iv).

(xii) *$C^{I(a)}$ is intransitive and has no orbit of length twenty-two. Furthermore it is impossible that $C^{I(a)}$ has two orbits Γ and Δ such that $|\Gamma|=2$, $|\Delta|=21$ and $(C^{I(a)})_\Gamma$ is doubly tarnsitive on Δ.*

Proof. Assume by way of contradiction that (xii) is false. Then we show first that there is a 2-element

$$x=(1)\ (2)\ \cdots\ (7)\ (8\ 16)\ (9\ 17)\ \cdots$$

in C such that $x^{I(a)}$ commutes with $\bar{b}$ and $\bar{c}$. First suppose that $C^{I(a)}$ is transitive. Then $C^{I(a)}$ is M_{23} or contained in a group of order 23.11 (see [1], p. 235). The latter case does not occur since $C^{I(a)}$ has an ivnolution. Thus $C^{I(a)}$ is M_{23}. Since

$$\bar{b}=(1)\,(2)\cdots(7)\,(8\ 9)\,(16\ 17)\cdots$$

and

$$\bar{c}=(1)\,(2\ 3)\,(4\ 5)\,(6)\,(7)\,(8)\,(9)\,(16)\,(17)\cdots,$$

C has a 2-element x as above. Next suppose that $C^{I(a)}$ has an orbit of length twenty-two. Then $C^{I(a)}$ is a subgroup of M_{22}. Since M_{22} has no proper transitive subgroup on twenty-two points (see [1], p. 235), $C^{I(a)}$ is M_{22}. Since $T=(C_{24\,25})^{I(a)}$ is contained in $C^{I(a)}$ and $C^{I(a)}$ has an orbit of length one, T is isomrphic to M_{10}, $M_{10}{}^*$ or $PSL(2, 11)$ by (vi). Hence by (vi), the $C^{I(a)}$-orbit of length one is $\{1\}$ or $\{7\}$. Hence similarly to the case above $C^{I(a)}$ has a 2-element x as above. Finally suppose that $C^{I(a)}$ has two orbits Γ and Δ such that $|\Gamma|=2$, $|\Delta|=21$ and $(C^{I(a)})_\Gamma$ is doubly transitive on Δ. Since T is contained in $C^{I(a)}$ and $C^{I(a)}$ has an orbit of length two, T is isomorphic to $N(M_9)$, $N(M_9)^*$ or S_5 by (vi). Hence by (vi), Γ is $\{2, 3\}$ or $\{4, 5\}$ and contained in $I(\bar{b})$. Let $\bar{M}$ be a normalizer of a Sylow 2-subgroup of $(C^{I(a)})_{I(\bar{b})}$ in $(C^{I(a)})_\Gamma$. Since a Sylow 2-subgroup of $(C^{I(a)})_{I(\bar{b})}$ is a Sylow 2-subgroup of a stabilizer of two points of Δ in $(C^{I(a)})_\Gamma$ by a lemma of *E.* Witt ([6], Theorem 9.4), $\bar{M}$ is doubly transitive on $I(\bar{b})-\Gamma$. Hence $\bar{M}$ has an element $\bar{u}$ of order five and fixing exactly one point of $I(a)-I(\bar{b})$. Since a Sylow 2-subgroup of $(C^{I(a)})_{I(\bar{b})}$ is $(C^{I(a)})_{I(\bar{b})}$ and a semiregular 2-group on $I(a)-I(\bar{b})$, $\bar{u}$ is a nonidentity automorphism of $(C^{I(a)})_{I(\bar{b})}$ induced by conjugation by $\bar{u}$. Since $(C^{I(a)})_{I(\bar{b})}$ is of order at most sixteen and $\bar{u}$ is of order five, $(C^{I(a)})_{I(\bar{b})}$ is of order sixteen. On the other hand since $\bar{c}$ is a transposition on $I(\bar{b})-\Gamma$, the normalizer of $(C^{I(a)})_{I(\bar{b})}$ in $C^{I(a)}$ is S_5 on $I(\bar{b})-\Gamma$. Thus $(C^{I(a)})_\Gamma$ is doubly transitive on Δ and a stabilizer of two points of Δ in $(C^{I(a)})_\Gamma$ is of order divisible by $3\cdot 16$. Hence $C^{I(a)}=N_{M_{23}}(M_{21})$, and so similarly to the case above C has a 2-element x as above.

Let R be a Sylow 2-subgroup of $\langle a, b, c, x\rangle$ containing $\langle a, b, c\rangle$. Then $R^{I(a)}=\langle a, b, c, x\rangle^{I(a)}$. Hence there is an element x' of R such that $x'^{I(a)}=x^{I(a)}$. Since $C^{I(a)}=M_{23}$, M_{22} or $N_{M_{23}}(M_{21})$, any involution of $C^{I(a)}$ is conjugate to $\bar{b}$ or $\bar{c}$ in $C^{I(a)}$. Hence by (x), R is elementary abelian. Since $I(\bar{b})=I(x'^{I(a)})$, from (xi) we have that $I(\bar{b})=I(\bar{c})$, which is a contradiction.

(viii) *It is impossible that $C^{I(a)}$ has an orbit of length fourteen.*

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length fourteen. Then by (vi), T is isomorphic to $N(M_9)$ or $N(M_9)^*$, which has orbits of lengths 2, 9, 12 or 2, 9, 6, 6 respectively. Hence $C^{I(a)}$ has orbits of lengths 9, 14. On the other hand M_{23} has not a maximal subgroup having orbits of lengths 9, 14 (see [1], p. 235). Hence if $C^{I(a)}$ has orbits of lengths 9, 14, then the maximal subgroup of M_{23} containing $C^{I(a)}$ is transitive and so of order 23. 11 (see [1], p. 235). This is a contradiction since the order of $C^{I(a)}$ is even, Thus (xiii) holds.

(xiv) *It is impossible that $C^{I(a)}$ has an orbit of length fifteen.*

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length fifteen. Then by (vi), T is isomorphic to $N(M_9)^*$ or S_5, which has orbits of lengths 2, 9, 6, 6 or 5, 6, 2, 10 respectively. Hence $C^{I(a)}$ has orbits of lengths 2, 6, 15 or 8, 15.

First we show that if $C^{I(a)}$ has orbits of lengths 2, 6, 15 then C^{Δ} is S_6, where Δ is the $C^{I(a)}$-orbit of length six. In the case $T \simeq N(M_9)^*$, we may assume that $\Delta = \{4, 5, 20, 21, 22, 23\}$. In the acse $T \simeq S_5$, $\Delta = \{2, 3, 12, 13, 14, 15\}$. Suppose that $(C^{I(a)})_{\Delta} \neq 1$. Since $|\Delta| = 6$ and $C^{I(a)} \leq M_{23}$, $(C^{I(a)})_{\Delta}$ is an elementary abelian 2-group and so $(C^{I(a)})_{\Delta}$ has exactly one orbit of length one in $I(a) - \Delta$. On the other hand $(C^{I(a)})_{\Delta}$ is a normal subgroup of $C^{I(a)}$. Hence $C^{I(a)}$ has an orbit of length one in $I(a) - \Delta$, which is a contradiction. Thus $(C^{I(a)})_{\Delta} = 1$. Hence C^{Δ} is isomorphic to $C^{I(a)}$ and so contains T^{Δ} properly. First assume that T is isomorphic to $N(M_9)^*$. Since $C^{I(a)}$ has an orbit of length fifteen, C^{Δ} has an element of order five. Since $|\Delta| = 6$, C^{Δ} is doubly transitive. Furthermore C^{Δ} is of order divisible by $5 \cdot |T^{\Delta}|$ and has an odd permutation c^{Δ}. Hence by [5], C^{Δ} is isomorphic to S_6. Next assume that T is isomorphic to S_5. Since C^{Δ} contains T^{Δ} properly, C^{Δ} is S_6 by [5].

Next we show that if $C^{I(a)}$ has orbits of lengths 8, 15, then $C^{\Delta'}$ contains A_8, where Δ' is the $C^{(Ia)}$-orbit of length eight. In the case $T \simeq N(M_9)^*$ we may assume that $\Delta' = \{2, 3, 4, 5, 20, 21, 22, 23\}$. In the case $T \simeq S_5$ $\Delta' = \{4, 5, 2, 3, 12, 13, 14, 15\}$. Take two points 2, 4 of Δ'. In the case $T \simeq N(M_9)^*$, $T_2 = \langle y_1, \bar{c} y_3 \rangle$ has orbits of lengths 1, 6 on $\Delta' - \{2\}$ and $T_4 = \langle b, y_2, y_3 \rangle$ has no orbit of length one in $\Delta' - \{4\}$. In the case $T \simeq S_5$, $T_4 = \langle \bar{b}, x_6, x_6^{\bar{c}} \rangle$ has orbits of lengths 1, 6 on $\Delta' - \{4\}$ and $T_2 = \langle \bar{b}, (\bar{b} x_6 \bar{c})^{x_6} \rangle$ has no orbit of length one in $\Delta' - \{2\}$. Since Δ' is an orbit of C, C_2 is conjugate to C_4. Hence C_2 is transitive on $\Delta' - \{2\}$ and so $C^{\Delta'}$ is doubly transitive on Δ'. Furthermore since $C^{I(a)}$ has an orbit of length fifteen, $C^{I(a)}$ has an element of order five. Since an element of order five fixes exactly three points and $|\Delta'| = 8$, $C^{\Delta'}$ has an element of order five. Thus by [5], $C^{\Delta'}$ contains A_8.

Hence there is an involution d in C such that d is conjugate to b and has the following form: If T is isomorphic to $N(M_9)^*$, then

$$d = (2)(3)(4)(5)(20\ 23)(21\ 22)\cdots,$$

and if T is isomorphic to S_5, then

$$d = (2)(3)(4)(5)(12\ 15)(13\ 14)\cdots.$$

Since $\langle \bar{b}, d^{I(a)} \rangle < (C^{I(a)})_{2345}$, $I(\bar{b}) = I(d^{I(a)})$ and $(bd)^{I(a)}$ is an involution. Hence $(bd)^s$ is a 2-element $(\neq 1)$, where s is a suitable odd integer. Since $((bd)^s)^{I(a)}$ is conjugate to $\bar{b}$ and of order two, $(bd)^s$ is an involution by (x). Thus $(bd)^s$ is a central involution of a dihedral group $\langle b, d\rangle$. Set $d'=b(bd)^s$. Then d' is an involution commuting with b and has the same form as d on $I(a)$. Furthermore $(cd')^{I(a)}=(1)\,(6)\,(7)\,(2\ 3)\,(4\ 5)\,(20\ 21)\,(22\ 23)\cdots$ or $(1)\,(6)\,(7)\,(2\ 3)\,(4\ 5)\,(12\ 13)\,(14\ 15)\cdots$. Thus $(cd')^{I(a)}$ is an involution. Hence by the same argument as above, $d''=c(cd')^r$ is an involution commuting with c and $d''^{I(a)}=d'^{I(a)}=d^{I(a)}$, where r is a suitable odd integer. Since b commutes with c and d', b commutes with d''. Furthermore $I(\bar{b})=I(d''^{I(a)})$ and $d''\neq ba$. On the other hand c is an involution commuting with b and d'' and fixing twenty-three points. Hence by (xi) $I(\bar{b})=I(\bar{c})$, which is a contradiction.

(xv) *It is impossible that $C^{I(a)}$ has an orbit of length eight.*

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length eight. Then by (vi), T is isomorphic to $N(M_9)^*$ or S_5, which has orbits of lengths 2, 9, 6, 6 or 5, 6, 2, 10 respectively. Since $C^{I(a)}$ has no orbit of length fiftten by (xiv), $C^{I(a)}$ has orbits of lengths 8, 9, 6 or 5, 8, 10. Since the $C^{I(a)}$-orbit of length eight is Δ' as in the proof of (xiv), by the same argument as is used in (xiv) $C^{\Delta'}$ is doubly transitive on Δ'. Hence $C^{I(a)}$ has an element of order seven which fixes exactly two points. On the other hand $C^{I(a)}$ has an orbit of length six or five, which must be fixed by an element of order seven pointwise. Thus we have a contradiction.

(xvi) *It is impossible that $C^{I(a)}$ has an orbit of length sixteen.*

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length sixteen. Then by (vi), T is isomorphic to M_{10}^* or S_5, which has orbits of lengths 1, 10, 6, 6 or 5, 6, 2, 10 respectively. Hence $C^{I(a)}$ has orbits of lengths 1, 6, 16 or 2, 5, 16 or 7, 16. Let Δ be the $C^{I(a)}$-orbit of length sixteen. Set $\Gamma=I(a)-\Delta$ and $E=(C^{I(a)})_\Gamma$. Since $|\Gamma|=7$, C^Γ is a subgroup of S_7 and so a Sylow 2-subgroup of C^Γ is of order at most sixteen. On the other hand since $(C^{I(a)})_2$ has an involution $\bar{b}$, where $2\in\Delta$, and $|\Delta|=16$, the order of $C^{I(a)}$ is divisible by $16\cdot 2$. Hence C^Γ is not isomorphic to $C^{I(a)}$. Since C^Γ is isomrphic to $C^{I(a)}/E$, $E\neq 1$.

Now we show that the order of E is sixteen and any two involutions of E are conjugate. Since T is isomorphic to M_{10}^* or S_5, Γ contains a T-orbit of length six or five on which T is A_6 or S_5 respectively. Hence $C^{I(a)}$ has an element $\bar{u}$ of order three and fixing four points of Γ and an element $\bar{v}$ of order five and fixing two points of Γ. Since $|I(\bar{u})|=5$ and $|I(\bar{v})|=3$, $\bar{u}$ and $\bar{v}$ fix exactly one point of Δ. Since Δ is a $C^{I(a)}$-orbit, we may assume that $\bar{u}$ and $\bar{v}$ fix the same point i of Δ. Then $\langle\bar{u}, \bar{v}\rangle$ is transitive on $\Delta-\{i\}$ and so $C^{I(a)}$ is doubly transitive on Δ. On the other hand E is a normal subgroup of $C^{I(a)}$ and semiregular on Δ. Hence E is regular on Δ. Thus the order of E is sixteen. Furthermore since $\langle\bar{u}, \bar{v}\rangle$ acts transitively on $E-\{1\}$ by conjugation, any two involutions of E are conjugate.

Let x and x' be 2-elements of C such that $x^{I(a)}=x'^{I(a)}\in E-\{1\}$. Then x^2 and x'^2 are contained in $\langle a\rangle$ which is of order two. Hence the orders of x and x' are two or four. Assume that the order of x is four and the order of x' is two. Then $\langle x, x'\rangle^{I(a)}=\langle x\rangle^{I(a)}$ and $\langle x, x'\rangle_{I(a)}\geq\langle x^2\rangle=\langle a\rangle$ by (vii). Hence a Sylow 2-subgroup of $\langle x, x'\rangle$ is of order four. This implies that both $\langle x\rangle$ and $\langle x', a\rangle$ are Sylow 2-subgroups of $\langle x, x'\rangle$ and not conjugate, which is a contradiction. Thus x and x' are of the same order. Then since any two involutions of E are conjugate, every 2-elements of $C_\Gamma-\langle a\rangle$ have the same order. Let Q be a Sylow 2-subgroup of C_Γ. Then $Q^{I(a)}=E$ and $Q_{I(a)}=\langle a\rangle$. Hence Q is of order $16\cdot 2$. If every elements of $Q-\langle a\rangle$ are of order four, then a is the only one involution of Q. Hence Q is a cyclic or generalized quaternion group. This is impossible since $Q^{I(a)}$ is an elementary abelian goup of order sixteen and isomorphic to $Q/\langle a\rangle$. Hence Q is elementary abelian. Since b normalizes C_Γ, we may assume that b normalizes Q. Since b fixes exactly four points of Δ and $Q^{I(a)}$ is regular on Δ, $C_{Q^{I(a)}}(\bar{b})$ is of order four. Hence there are two involutions y and z in $Q-\langle a\rangle$ such that both $y^{I(a)}$ and $z^{I(a)}$ commute with $\bar{b}$ and $y^{I(a)}\neq z^{I(a)}$. Suppose that b commutes with neither y nor z. Then since $(y\cdot y^b)^{I(a)}=(z\cdot z^b)^{I(a)}=1$, $y^b=ya$ and $z^b=za$. Hence $(yz)^b=yaza=yz$ and $yz\neq a$. Hence we may assume that b commutes with y. Then y fixes exactly three points of $I(b)\cap I(a)$ and four points of $I(b)-(I(b)\cap I(a))$. Hence there is a 2-cycle $(i\,j)$ of a such that $C_{i\,j}>\langle b, y\rangle$. Since $(C_{i\,j})^{I(a)}$ is a subgroup of $C^{I(a)}$, Γ is unions of $C_{i\,j}$-orbits. Since $|\Gamma|=7$, by the assumption (2) $(C_{i\,j})^{I(a)}$ is isomorphic to $M_{10}{}^*$ or S_5 and has orbits of lengths 1, 6 or 2, 5 on Γ respectively. Therefore $((C_{i\,j})^{I(a)})_\Gamma=1$. However $y^{I(a)}$ fixes Γ pointwise. Thus we have a contradiction.

(xvii) *It is impossible that $C^{I(a)}$ has an orbit of length seven.*

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length seven. Then by (vi) T is isomorphic to $M_{10}{}^*$ or S_5, which has orbits of lengths 1, 10, 6, 6 or 5, 6, 2, 10 respectively. Hence $C^{I(a)}$ has orbits of lengths 7, 10, 6 or 7, 16. Then $C^{I(a)}$ has an element of order seven. Since an element of order seven fixes exactly two points, it is impossible that $C^{I(a)}$ has orbits of lengths 7, 10, 6. By (xvi), the second case does not occur. Thus (xvii) holds.

(xviii) *It is impossible that $C^{I(a)}$ has an orbit of length nine.*

Proof. Assume by way of contradiction that $C^{I(a)}$ has an orbit of length nine. Then by (vi), T is isomorphic to $N(M_9)$ or $N(M_9)^*$, which has orbits of lengths 2, 9, 12 or 2, 9, 6, 6 respectively. By (xiii) and (xv), $C^{I(a)}$ has no orbit of length fourteen or eight. Hence $C^{I(a)}$ has orbits of lengths 2, 9, 12 or 2, 9, 6, 6, where the orbit of length two is $\{2, 3\}$ and the orbit of length nine is $\{1, 8, 9, \cdots, 15\}$. Since $C^{(Ia)}$ is a subgroup of M_{23}, the order of $(C^{I(a)})_{2\,3\,1}$ is a divisor of $20\cdot 48$. Since an element of order five fixes exactly three points and there are $C^{I(a)}$-orbits of lengths 2, 12 or 2, 6, 6, $C^{I(a)}$ has no element of order five. Suppose that $(C^{I(a)})_{231}$ has an element of order three. Since $(C^{I(a)})_1 \geq (C^{I(a)})_{231}$ and $\{1, 8, 9, \cdots, 15\}$ is a $C^{I(a)}$-orbit of length nine, $C^{I(a)}$ is of order divisible by 3^3. This is a contradiction because M_{23} has no subgroup of order 3^3. Thus $(C^{I(a)})_{231}$ is a 2-group. Furthermore T is primitive on $\{1, 8, 9, \cdots, 15\}$. Hence by [5], the order of $C^{I(a)}$ is 144 and so $C^{I(a)}$ is isomorphic to $N(M_9)$. Hence by (ix), T is not isomorphic to $N(M_9)$. Hence T is isomorphic to $N(M_9)^*$. Take $(C_{32\,33})^{I(a)}$, where 32, 33 are fixed by b and a has a 2-cycle (32 33). Then by (ix), $(C_{32\,33})^{I(a)}$ is isomorphic to $N(M_9)^*$ and so $(C_{32\,33})^{I(a)}=T$. Thus $C_{32\,33}$ has an element x such that $\bar{c}=x^{I(a)}$. Then by the same argument as is used in (ix), we have a contradiction. Thus (xviii) holds.

(xix) *It is impossible that $C^{I(a)}$ has an orbit of length thirteen or seventeen.*

Proof. M_{23} has no element of order thirteen or seventeen. Hence (xix) holds.

(xx) *If $C^{I(a)}$ has an orbit of length eleven, then $C^{I(a)}$ is isomorphic to M_{11}.*

Proof. An element of order eleven has two 11-cycles and fixes one point. Hence by (xii), $C^{I(a)}$ has orbits of lengths 11, 1, 11 or 11, 12. First assume that $C^{I(a)}$ has orbits of lengths 11, 1, 11. Then by (vi), T is isomorphic to $PSL(2, 11)$. Hence by (ix), $C^{I(a)}$ contains T properly. This is a contradiction since $PSL(2, 11)$ is a maximal subgroup of M_{22} (see [1], p. 235). Next assume that $C^{I(a)}$ has orbits of lengths 11, 12. Then the maximal subgroup of M_{23} containing $C^{I(a)}$ is isomorphic to M_{11} or of order $23 \cdot 11$ (see [1], p. 235). However the latter case dose not occur since $C^{I(a)}$ is of even order. Suppose that $C^{I(a)}$ is isomorphic to a proper subgroup of M_{11}. Since the maximal subgroup of M_{11} whose order is divisible by 11 is $PSL(2, 11)$, $C^{I(a)}$ is isomorphic to a subgroup of $PSL(2, 11)$. Hence T is isomorphic to $PSL(2, 11)$ and so $C^{I(a)}=T$, contrary to (ix). Hence $C^{I(a)}$ is isomorphic to M_{11}.

(xxi) $C^{I(a)}$ is isomorphic to M_{11}.

Proof. T is one of the seven groups of (vi). In the following we treat seven cases separately.

(1) Let T be isomorphic to M_{11}. Then T is a maximal subgroup of M_{23} (see [1] p. 235). Hence if $C^{I(a)}$ contains T properly, then $C^{I(a)}$ is M_{23}, contrary to (xii). Thus $C^{I(a)}=T$ and so $C^{I(a)}$ is isomorphic to M_{11}.

(2) Let T be isomorphic to M_{10}. Then the lengths of the T-orbits are 1, 10, 12. By (xii) and (xix), $C^{I(a)}$ has no orbit of length 22, 23 or 13. Hence $C^{I(a)}$ has orbits of lengths 1, 10, 12 or 11, 12. By (ix), $C^{I(a)}$ contains T properly. Hence the first case does not occur because M_{10} is a maximal subgroup of M_{22} (see [1], p. 235). In the second case $C^{I(a)}$ is isomorphic to M_{11} by (xix).

(3) Let T be isomorphic to M_{10}^*. Then the lengths of the T-orbits are 1, 10, 6, 6. By (xii), (xvi), (xvii) and (xix), $C^{I(a)}$ has no orbit of length 23, 22, 16, 7, 13 or 17. Hence the lentghs of the $C^{I(a)}$-orbits are one of the following:

(a) 1, 10, 6, 6, (b) 1, 10, 12, (c) 11, 6, 6, (d) 11, 12.

First consider the case (a). Since $C^{I(a)}$ contains T properly, $C^{I(a)}$ is contained in $N_{M_{22}}(E)$ (see [1], p. 235), where E is an elementary abelian 2-group of order sixteen and fixing seven points of $I(a)$. Set $\Delta=I(E)$. Since $C^{I(a)}$ contains T properly and $|\Delta|=7$, $(C^{I(a)})_\Delta$ is a nonidentity elementary abelian 2-group. Since $C^{I(a)}/(C^{I(a)})_\Delta=A_6$, $(C^{I(a)})_\Delta$ has an automorphisum group which is isomorphic to A_6. Hence $(C^{Ia)})_\Delta$ is of order sixteen and has an orbit of length sixteen. This is a contradiction. Thus the case (a) does not occur.

Next consider the case (b). Then $C^{I(a)}$ is isomorphic to M_{10}. Take $(C_{32\,33})^{I(a)}$, where 32, 33 are fixed by b and a has a 2-cycle (32 33). Then by (ix), $(C_{32\,33})^{I(a)}$ is isomorphic to M_{10}^* and so $(C_{32\,33})^{I(a)}=T$. Thus $C_{32\,33}$ has an element x such that $\bar{c}=x^{I(a)}$. Then by the same argument as in (ix), we have a contradiction. Thus the case (b) does not occur.

Finally consider the cases (c) and (d). Since $C^{I(a)}$ has an orbit of length eleven, $C^{I(a)}$ is isomorphic to M_{11} by (xx). Thus the lengths of the $C^{I(a)}$-orbits are 11, 12 and the case (c) does not occur.

(4) Let T be isomorphic to $N(M)_9$. Then the lengths of the T-orbits are 2, 9, 12. By (xii), (xiii) and (xviii), $C^{I(a)}$ has no orbit of length 23, 14 or 9. Hence the lengths of the $C^{I(a)}$-orbits are one of the following:

(a) 2, 21, (b) 11, 12.

First consider the case (a). Since $T_1=\langle\bar{b}, \bar{c}, x_3x_4\rangle$, the T_1-orbits on $I(a)-\{1, 2, 3\}$ are $\{8, 9, \cdots, 15\}$, $\{4, 5, 6, 7\}$ and $\{16, 17, \cdots, 23\}$, whoes lengths are 8, 4 and 8 respectively. Since $T_6=\langle\bar{b}, \bar{c}, y_1\rangle$, the T_6-orbits on $I(a)-\{6, 2, 3\}$ are $\{1, 8, 9\}$, $\{10, 11, \cdots, 15\}$, $\{4, 5, 20, 21, 22, 23\}$, $\{7, 16, 17\}$ and $\{18, 19\}$, whose lengths are 3, 6, 6, 3, and 2 respectively. Since 1 and 6 are contained in the same $C^{I(a)}$-orbit, $(C^{I(a)})_1$ and $(C^{I(a)})_6$ are conjugate in $C^{I(a)}$ and so the $(C^{I(a)})_1$-orbits and $(C^{I(a)})_6$-orbits have the same lengths. Thus $(C^{I(a)})_1$ is transitive or has orbits of lengths eigtht and twelve on $I(a)-\{1, 2, 3\}$.

Suppose that $(C^{(Ia)})_1$ has two orbits on $I(a)-\{1, 2, 3\}$. Then $\{8, 9, \cdots, 15\}$ and $\{4, 5, 6, 7, 16, 17, \cdots, 23\}$ or $\{4, 5, \cdots, 15\}$ and $\{16, 17, \cdots, 23\}$ are the $(C^{I(a)})_1$-orbits. In the first case $\langle(C^{I(a)})_1, T\rangle$ has an orbit $\{1, 8, 9, \cdots, 15\}$ of length nine and $\langle(C^{I(a)})_1, T\rangle_1=(C^{I(a)})_1$ has an orbit of length twelve. Hence $\langle(C^{I(a)})_1, T\rangle$ has a subgroup of order 3^3. This is a contradiction since M_{23} has no subgroup of order 3^3. In the second case set $\Delta=\{4, 5, \cdots, 15\}$. Let $\bar{x}$ be an involution of $C^{I(a)}$ and conjugate to $\bar{b}$ in $(C^{I(a)})_1$. Since $\bar{b}$ fixes three points 2, 3, 1 and exactly four points of the $(C^{I(a)})$-orbit Δ, $\bar{x}$, fixes three points 2, 3, 1 and exactly four points of Δ. We may assume that $\bar{x}$ fixes a point 9 of Δ which is not fixed by $\bar{b}$. Then $I(\bar{b})\neq I(\bar{x})$ and so $|I(\bar{b})\cap I(\bar{x})|\leq 3$. Hence $I(\bar{b})\cap I(\bar{x})=\{1, 2, 3\}$. Since $\bar{b}$ fixes four points 4, 5, 6, 7 of Δ and $\bar{x}$ fixes a point 9 of Δ, $\bar{x}$ fixes exactly three points of Δ-$\{4, 5, 6, 7, 9\}=\{8, 10, 11, \cdots, 15\}$. On the other hand y_1 is an element of T and fixes $\{1, 2, 3, 9\}$ as a set which is contained in $I(\bar{x})$. Hence y_1 fixes $I(\bar{x})$ as a set. Then since $y_1=(8)(10\ 15)(11\ 13)(12\ 14)$ on $\{8, 10, 11, \cdots, 15\}$ and $|I(\bar{x})\cap\{8, 10, 11, \cdots, 15\}|=3$, $I(\bar{x})\ni 8$ and so $I(\bar{x})\supset\{1, 2, 3, 8, 9\}$. Thus $\bar{b}$ fixes $I(\bar{x})$ as a set because $\{1, 2, 3, 8, 9\}^{\bar{b}}=\{1, 2, 3, 8, 9\}$. Hence $\langle\bar{b}, y_1\rangle$ fixes $I(\bar{x})$ as a set. However since $\bar{x}$ has fixed points in the $\langle\bar{b}, y_1\rangle$-orbits $\{10, 11, \cdots, 15\}$, $I(\bar{x})\supset\{10, 11, \cdots, 15\}$, which is a cntradiction.

Thus $(C^{I(a)})_1$ is transitive on $I(a)-\{1, 2, 3\}$ and so $C^{I(a)}$ is doubly transitive on $I(a)-\{2, 3\}$. Since $(C^{I(a)})_{2\,3}$ is a normal subgroup of $C^{I(a)}$, $(C^{I(a)})_{2\,3}$ is transitive on $I(a)-\{2, 3\}$. Furthermore the order of $C^{I(a)}$ is divisible by $21\cdot 20$. Hence $C^{I(a)}$ has an element of five fixing exactly three points. Hence $(C^{I(a)})_{1\,2\,3}$ has an element of order five which has no fixed points in $I(a)-\{1, 2, 3\}$. On the other hand $T_{1\,2\,3}=\langle\bar{b}, x_3x_4, (x_3x_4)^{\bar{c}}\rangle$ has three orbits $\{8, 9, \cdots, 15\}$, $\{4, 5, 6, 7\}$ and $\{16, 17, \cdots, 23\}$, whose lengths are 8, 4 and 8 respectively. Hence $(C^{Ia})_{1\,2\,3}$ is transitive on $I(a)-\{1, 2, 3\}$. Thus $(C^{I(a)})_{2\,3}$ is doubly transitive on $I(a)-\{2, 3\}$, contrary to (xii).

In the case (b) by (xix), $C^{I(a)}$ is isomrphic to M_{11}.

(5) Let T be isomorphic to $N(M_9)^*$. Then the lengths of the T-orbits are 2, 9, 6, 6. By (xii), (xiii), (xiv), (xv), (xviii) and (xix), $C^{I(a)}$ has no orbit of length 23, 14, 15, 8, 9 or 17. Hence the lengths of the $C^{I(a)}$-orbits are one of the following:

(a) 2, 21, (b) 11, 6, 6, (c) 11, 12.

First consider the case (a). Set $\Delta=\{1, 4, 5, \cdots, 23\}$ which is the $C^{I(a)}$-orbit of length twenty-one. Since $T_1=\langle\bar{b}, \bar{c}, y_3\rangle$, the T_1-orbits on $I(a)-\{2, 3, 1\}$ are $\{8, 9, 10, 11\}$, $\{12, 13, 14, 15\}$, $\{4, 5\}$, $\{6, 7\}$, $\{16, 17, 18, 19\}$ and $\{20, 21, 22, 23\}$, whose lengths are 4, 4, 2, 2, 4 and 4 respectively. Since $T_4=\langle\bar{b}, y_2, y_3\rangle$, the T_4-orbits on $I(a)-\{2, 3, 4\}$ are $\{1, 14, 15\}$, $\{8, 9, \cdots, 13\}$, $\{5, 22, 23\}$, $\{6, 7, 16, 17, 18, 19\}$ and $\{20, 21\}$, whose lengths are 3, 6, 3, 6 and 2 respectively. Since 1 and 4 belong to the $C^{I(a)}$-orbit Δ, $(C^{I(a)})_1$ and $(C^{I(a)})_4$ are conjugate in $C^{I(a)}$ and so the $(C^{I(a)})_1$-orbits and the $(C^{I(a)})_4$-orbits have the same lengths. Thus the lengths of the $(C^{I(a)})_4$-orbits on $\Delta-\{4\}$ are one of the following:

(a.1) 2, 6, 12, (a.2) 2, 18, (a.3) 8, 6, 6, (a.4) 8, 12, (a.5) 14, 6, (a.6) 20.

First consider the cases (a.1) and (a.2). Since a point 4 belongs to the $C^{I(a)}$-orbit of length twenty-one and $\{2, 3\}$ is a $C^{I(a)}$-orbit of length two, $|C^{I(a)}:(C^{I(a)})_{2\,3\,4}|=2\cdot 21$. Hence $|C^{Ia)}:(C^{I(a)})_{2\,3\,4\,i\,j}|\leq 4\cdot 21$, where $\{i, j\}$ is the $(C^{I(a)})_4$-orbit of length two in Δ. On the other hand T is of order $9\cdot 8$. Hence

$(C^{I(a)})_{234ij}$ is of order divisible by two and so has an involution $\bar{x}$. Then since $(C^{I(a)})_4$ fixes a subset $\{2, 3, 4, i, j\}$ of $I(\bar{x})$ with cardinality five, $(C^{I(a)})_4$ fixes $I(\bar{x})$. Hence $(C^{I(a)})_4$ fixes $I(\bar{x})-\{2, 3, 4, i, j\}$. Thus $I(\bar{x})-\{2, 3, 4, i, j\}$ is a union of two $(C^{I(a)})_4$-orbits of length one or a $(C^{I(a)})_4$-orbit of length two in $\Delta-\{4, i, j\}$, which is a contradiction.

Next consider the cases (a.3) and (a.4). Let i be a point of $\Delta\cap I(\bar{b})$ and Γ be the $(C^{I(a)})_i$-orbit of length eight. Then we show that $|\Gamma\cap I(\bar{b})|\neq 2$. Suppose by way of contradiction that $|\Gamma\cap I(\bar{b})|=2$. Set $\{j, k\}=\Gamma\cap I(\bar{b})$. Since $|C^{I(a)}: (C^{I(a)})_{23i}|=2\cdot 21$ and T is of order $9\cdot 8$, $(C^{I(a)})_{23i}$ is of order divisible by 3 and so has an element $\bar{u}$ of order three. Since Γ is a $(C^{I(a)})_i$-orbit of length eight and $\bar{u}$ fixes exactly five points, $\bar{u}$ fixes exactly two points of Γ. Hence we may assume that $\bar{u}$ fixes the point j of $\Gamma\cap I(\bar{b})$. Then $\bar{u}$ fixes a subset $\{2, 3, i, j\}$ of $I(\bar{b})$ with cardinality four. Hence $\bar{u}$ fixes $I(\bar{b})$. Since $I(\bar{b})$ contains $\{2, 3, 4, i, j, k\}$, $\bar{u}$ has a 3-cycle $(k\ k'\ k'')$ in $I(\bar{b})$. On the other hand $\bar{u}$ is an element of $(C^{I(a)})_i$ and k is a point of the $(C^{I(a)})_i$-orbit Γ. Hence $\{k, k', k''\}$ is contained in Γ and so contained in $\Gamma\cap I(\bar{b})$. This is a contradiction since $\Gamma\cap I(\bar{b})=\{j, k\}$. Thus $|\Gamma\cap I(\bar{b})|\neq 2$.

Since T_4-orbits on $\Delta-\{4\}$ are $\{1, 14, 15\}$, $\{8, 9, \cdots, 13\}$, $\{5, 22, 23\}$, $\{6, 7, 16, 17, 18, 19\}$ and $\{20, 21\}$, the $(C^{I(a)})_4$-orbit of length eigth is $\{20, 21, 1, 14, 15, 5, 22, 23\}$, $\{20, 21, 6, 7, 16, 17, 18, 19\}$ or $\{20, 21, 8, 9, 10, 11, 12, 13\}$. Hence by what we have proved above, the $(C^{I(a)})_4$-orbit of length eight is $\{20, 21, 8, 9, 10, 11, 12, 13\}$. Similarly since $T_6=\langle\bar{b}, \bar{c}, y_1\rangle$ has orbits $\{1, 8, 9\}$, $\{10, 11, 12, 13, 14, 15\}$, $\{4, 5, 20, 21, 22, 23\}$, $\{18, 19\}$, and $\{7, 16, 17\}$ on $\Delta-\{6\}$, by what we have proved above the $(C^{I(a)})_6$-orbit of length eight is $\{18, 19, 10, 11, 12, 13, 14, 15\}$. Therefore $|(C^{I(a)})_4: (C^{I(a)})_{4\,12}|=|(C^{I(a)})_6: (C^{I(a)})_{6\,12}|=8$. Since $|(C^{I(a)})_4|=|(C^{I(a)})_6|=|(C^{I(a)})_{12}|$, 4 and 6 belong to the $(C^{I(a)})_{12}$-orbit of length eight. On the other hand $T_{12}=T_1^{y_2\bar{c}}$ has orbits $\{4, 20, 22, 23\}$ and $\{6, 7, 16, 19\}$. Hence the $(C^{I(a)})_{12}$-orbit of length eight is $\{4, 20, 22, 23, 6, 7, 16, 19\}$. Since $\Gamma=\{20, 21, 8, 9, 10, 11, 12, 13\}$ is a $(C^{I(a)})_4$-orbit of length eight, by the same argument as is used above for $\bar{u}$, $(C^{I(a)})_{234}$ has an element $\bar{v}$ which is of order three and fixes 12 and exactly one more point of Γ. Since $\bar{v}$ is also an element of $(C^{I(a)})_{12}$, $\bar{v}$ fixes exactly two points of the $(C^{I(a)})_{12}$-orbit of length eight. Hence $\bar{v}$ fixes exactly five points 2, 3, 4, 12, 20 because the $(C^{I(a)})_4$-orbit of length eight and $(C^{I(a)})_{12}$-orbit of length eight have exactly one point 20 in common. On the other hand T has an element $\bar{b}y_2=(2)(3)(4)(20)(21)(9\ 10\ 12)(8\ 13\ 11)\cdots$. Thus $\langle\bar{v}, \bar{b}y_2\rangle\leq(C^{I(a)})_{2\,3\,4\,20}$. Since $\bar{v}$ fixes one point 12 and have two 3-cycles and $\bar{b}y_2=(21)(9\ 10\ 12)(18\ 13\ 11)$ on $\Gamma-\{20\}$, $(C^{I(a)})_{2\,3\,4\,20}$ is transitive or has tow orbits $\{21, 9, 10, 12\}$ and $\{8, 11, 13\}$ on $\Gamma-\{20\}$. In the first case sine $|\Gamma-\{20\}|=7$, $(C^{I(a)})_{2\,3\,4\,20}$ has an element of order seven. This is a contradiction since an element of order seven fixes exactly two points. In the second case since $|\{21, 9, 10, 12\}|=4$ and $|\{8, 11, 13\}|=3$, there is an involution $\bar{x}$ fixing {2, 3, 4, 20, 8, 11, 13} pointwise. This is a contradiction sicne y_1 fixes four points 2, 3, 4, 8 of $I(\bar{x})$ but does not fix $I(\bar{x})$.

In the case (a.5), there is an element of order seven which fixes a $(C^{I(a)})_4$-orbit of length six pointwise. This is a contradiction.

Finally consider the case (a.6). Then $C^{I(a)}$ is doubly transitive on Δ. Since $(C^{I(a)})_{23}$ is normal in $C^{I(a)}$, $(C^{I(a)})_{23}$ is transitive on Δ. Since $(C^{I(a)})_1$ has an orbit of length twenty, $(C^{I(a)})_1$ has an element of order five, which fixes the $(C^{I(a)})_1$-orbit {2,3} pointwise. Thus $(C^{I(a)})_{123}$ has an element of order five. On the other hand since $T_{123}=\langle\bar{b}, \bar{c}y_3\rangle$, the lengths of the T_{123}-orbits on Δ-{1} are 4, 4, 2, 2, 4, 4. Hence $(C^{I(a)})_{123}$ is transitive or has two orbits of length 10 on Δ-{1}. Furthermore since $T_{234}=\langle\bar{b}, (y_2y_3)^2\rangle$, the lengths of the T_{234}-orbits are 3, 6, 2, 3, 3 on Δ-{4}. Since $(C^{I(a)})_{231}$ is conjugate to $(C^{I(a)})_{234}$, $(C^{I(a)})_{234}$ has no orbit of length ten in Δ-{4}. Thus $(C^{I(a)})_{23}$ is doubly transitve on Δ. contrary to (xii).

(6) Let T be isomorphic to $PSL(2, 11)$. Then the lengths of the T-orbits are 11, 1, 11. By (xii), $C^{I(a)}$ has no orbit of length twenty-three or twenty-two. Hence $C^{I(a)}$ has an orbit of length eleven. Thus by (xx), $C^{I(a)}$ is isomorphic to M_{11}.

(7) Let T be isomorphic to S_5. Then the lengths of the T-orbits are 5, 6, 2, 10. By (xii), (xiv), (xv), (xvi), (xvii), and (xix), $C^{I(a)}$ has no orbit of length 23, 15, 8, 16, 7, 13 or 17. Hence the lengths of the $C^{I(a)}$-orbits are one of the following:

(a) 5, 6, 2, 10, (b) 5, 6, 12, (c) 5, 18, (d) 11, 2, 10, (e) 11, 12,
(f) 2, 21.

First consider the cases (a) and (b). Let Δ be the $C^{I(a)}$-orbit of length six. Suppose that $(C^{I(a)})_\Delta\neq 1$. Since $|\Delta|=6$, $(C^{I(a)})_\Delta$ is a 2-group and fixes exactly one more point i which is not contained in Δ. Since $(C^{I(a)})_\Delta$ is normal in $C^{I(a)}$, $C^{I(a)}$ fixes the point i, which is a contradiction. Hence $(C^{I(a)})_\Delta=1$ and so $C^{I(a)}\simeq C^\Delta$. By (iv), $C^{I(a)}$ contains T properly. Hence C^Δ contains T^Δ properly. Since T^Δ is isomorphic to S_5, C^Δ is S_6. Let j be a point of the $C^{I(a)}$-orbit of length five. Then $|C^{I(a)}:(C^{I(a)})_j|=5$. This is a contradiction since S_6 has no subgroup of index five.

Next consider the case (c). Since the $C^{I(a)}$-orbit of length eighteen contains 4 and the order of T_4 is $|S_5|/2$, the order of $C^{I(a)}$ is divisible by 3^3. This is a contradiction since the order of M_{23} is not divisible by 3^3.

In the cases (d) and (e) by (xx), $C^{I(a)}$ is isomorphic to M_{11}. Thus the lengths of the $C^{I(a)}$-orbits are 11, 12 and the case (d) does not occur.

Finally consider the case (f). Then the $C^{I(a)}$-orbit of length two is {4,5}. Set $\Delta=I(a)-\{4, 5\}$. Since $T_1=\langle\bar{b}, \bar{c}, \bar{c}^{x_6}\rangle$, the T_1-orbits on $\Delta-\{1\}$ are {8, 9, 10, 11}, {2, 3, 12, 13, 14, 15}, {6, 7, 20, 21, 22, 23} and {16, 17, 18, 19}, whose lengths are 4, 6, 6 and 4 respectively. Since $T_3=\langle\bar{b}, (\bar{b}\bar{c}^{x_6})^{\bar{c}x_6\bar{c}}\rangle$, the T_3-orbits on

$\Delta-\{3\}$ are $\{1, 8, 9, 10, 11\}$, $\{2, 12, 13, 14, 15\}$ and $\{6, 7, 16, 17, 18, 19, 20, 21, 22, 23\}$, whose lengths are 5, 5 and 10 respectively. Since $(C^{(I(a)})_3$ is conjugate to $(C^{I(a)})_1$, $(C^{I(a)})_3$ is transitive or has two orbits of length ten on $\Delta-\{3\}$. Suppose that $(C^{I(a)})_3$ has two orbits of length ten. Then $\{1, 2, 8, 9, \cdots, 15\}$ and $\{6, 7, 16, 17, \cdots, 23\}$ are $(C^{I(a)})_3$-orbits of length ten. Hence the $\langle (C^{I(a)})_3, T\rangle$-orbits are $\{1, 2, 3, 8, 9, \cdots, 15\}$, $\{4, 5\}$ and $\{6, 7, 16, 17 \cdots, 23\}$, whose lengths are 11, 2 and 10 respectively. This is a contradiction since an element of order eleven has two 11-cycles. Hence $(C^{I(a)})_3$ is transitive on $\Delta-\{3\}$ and so $C^{I(a)}$ is doubly transitive on Δ. Since $(C^{I(a)})_{45}$ is normal in $C^{I(a)}$, $(C^{I(a)})_{45}$ is transitive on Δ. Since $T_{451}=\langle \bar{b}, \bar{c}, \bar{c}^{x_6}\rangle$ and $T_{453}=\langle \bar{b}, ((\bar{b}\bar{c}^{x_6})^{\bar{c}x_6\bar{c}})^2\rangle$, T_{451} and T_{453} has the same orbits as T_1 and T_3 on Δ respectively. Hence in the same way above $(C^{I(a)})_{45}$ is doubly transitvie on Δ, contrary to (xii).

(xxii) *If $C^{I(a)}$ is isomorphic to M_{11}, then we have a contradiction.*

Proof. Suppose by way of contradiction that $C^{I(a)}$ is isomorphic to M_{11}. If T is one of the groups of (vi) other than $PSL\ (2, 11)$, then T has an element of order four. Hence first we assume that T is not isomorphic to $PSL(2, 11)$. Then $C_{24\,25}$ has a 2-element

$$u=(1)\,(2)\,(3)\,(8\ 10\ 9\ 11)\,(12\ 14\ 13\ 15)\,(4\ 5)\,(6\ 7)\,(16\ 18\ 17\ 19)\,(20\ 22\ 21\ 23)\,(24)\,(25)\cdots.$$

Then $\langle u, b\rangle^{I(a)\cup\{24,25\}}$ is a cyclic group of order four and $\langle u, b\rangle_{I(a)\cup\{24,25\}}$ is of odd order because $|I(a)\cup\{24, 25\}|=25$. Hence a Sylow 2-subgroup of $\langle u, b\rangle$ containing b is a cyclic group of order four and has a generator which has the same form as u on $I(a)\cup\{24, 25\}$. Hence we may assume that $u^2=b$. Then $u^{I(b)}=(1)\,(2)\,(3)\,(4\ 5)\,(6\ 7)\,(24)\,(25)\cdots$ and so $u^{I(b)}$ is of order two. On the other hand $c^{I(b)}$ is an involution of $C_G(b)^{I(b)}$ and $(cu)^{I(b)}$ is also an involution of $(C_G(b)^{I(b)})$ because $(cu)^{I(b)}=(1)\,(2\ 3)\,(4)\,(5)\,(6\ 7)\,(24)\,(25)\cdots$. Thus $C_G(b)^{I(b)}$ has an elementary abelian group $\langle a, u, c\rangle^{I(b)}$ of order eight. On the other hand by the same argument as is used for a $C_G(b)^{I(b)}$ is isomorphic to M_{11}. This is a contradiction since M_{11} has no elementary abelian group of order eight. Thus T must be isomorphic to $PSL(2, 11)$. In the same way we have that for any 2-cycle $(i\,j)$ of a $(C_{i\,j})^{I(a)}$ is isomorphic to $PSL(2, 11)$. Since $C^{I(a)}$ is isomorphic to M_{11}, C has a 2-element

$$v=(1)\,(2)\,(3)\,(8\ 10\ 9\ 11)\,(12\ 14\ 13\ 15)\,(4\ 5)\,(6\ 7)\,(16\ 18\ 17\ 19)\,(20\ 22\ 21\ 23)\cdots.$$

Then $(v^2)^{I(a)}=\bar{b}$. Hence by (x), v^2 is an involution fixing twenty-three points. Let Q be a Sylow 2-subgroup of $\langle b, v\rangle$ containing $\langle b\rangle$. Since $Q^{I(a)}=\langle v\rangle^{I(a)}$, there is an element w in Q such that $w^{I(a)}=v^{I(a)}$. Then w^2b is a 2-element of C and fixes at least twenty-three points. Thus $w^2b=1$ or a and so $w^2=b$ or ab, Thus $w^{I(b)}$ or $w^{I(ab)}$ is an involution fixing seven points. Since $I(a)\cap I(b)\cap I(w)=I(a)\cap I(ab)\cap I(w)=\{1, 2, 3\}$, there are two points i, j in $I(b)$ or $I(ab)$ such that a has a 2-cycle $(i\,j)$ and w fixes i, j. Thus $w\in C_{i\,j}$. Hence $w^{I(a)}$is of order four and contained in $(C_{i\,j})^{I(a)}$. However since $(C_{i\,j})^{I(a)}$ is isomorphic to $PSL(2, 11)$, $(C_{i\,j})^{I(a)}$ has no element of order four. Thus we have a contradiction.

Thus we complete the proof of Lemma 3. Hence this completes the proof of the theorem.

3. Proof of Lemma 1

In this section we assume that G is a permutation group as in Lemma 1.

3.1. *Let x be an element of G.*

(i) *If x is an involution, then $|I(x)|=7$.*
(ii) *If x is of order three, then $|I(x)|=5$.*
(iii) *If x is of order four, then $|I(x)|=3$ and x has two 2-cycles.*
(iv) *If x is of order five, then $|I(x)|=3$.*
(v) *If x is of order six, then $|I(x)|=1$ and x has two 2-cycles and two 3-cycles.*
(vi) *Let y be an element of G. If x and y are of order two and $|I(x)\cap I(y)|\geq 4$, then $I(x)=I(y)$.*

Proof. This follows immediately from the assumption (4).

3.2. *Let a be an involution of G. Then we have a projective plane $P(a)$ of order two with the following incidence structure*:

(i) *The set of points of $P(a)$ is $I(a)$.*
(ii) *For two distinct points i, j of $I(a)$ the line containing i, j is $\{\Delta(i, j)\cup\{i, j\}\}\cap I(a)$.*

Proof. By (i) of (3.1), $|I(a)|=7$. By the assumption (7), each line contains three points. Hence if $\Delta(i, j)\cap I(a)=\{k\}$, then a line containing any two points of $\{i, j, k\}$ contains the remaining one point of $\{i, j, k\}$ by the assumption (3). Thus any two distinct points are contained in one and only one line.

Next let L_1 and L_2 be distinct lines. Suppose that L_1 has no point in common with L_2. Then there is exactly one point i neither on L_1 nor on L_2. Let j be a point on L_1 and let k_1, k_2 be two distinct points on L_2. Then the line L containing j and k_1 is distinct from L_1 and L_2. Hence L contains i. In the same way the line L' containing j and k_2 contains i. Thus two distinct lines L and L' have two points i, j in common, which is a contradiction. Hence any two distinct lines contain one and only one point in common. Thus (3.2) is proved, From now on for an involution a a projective plane defined in (3.2) is called a plane $P(a)$ or merely a plane.

3.3. *If $\Delta(i, j) \ni k$, then i, j, k are collinear in every plane containing i, j, k.*

Proof. Let a be any involution such that $I(a) \supset \{i, j, k\}$. By the assumption (7), $I(a) \cap \Delta(i, j) = \{k\}$. Hence i, j, k are collinear in the plane $P(a)$.

3.4. *For an involution a $N(G_{I(a)})^{I(a)} \leq PSL(3, 2)$.*

Proof. Let x be any element of $N(G_{I(a)})$ and let i, j be any two distinct points of $I(a)$. Then by the assumption (5), $\{\{\Delta(i, j) \cup \{i, j\}\} \cap I(a)\}^x = \{\Delta(i^x, j^x) \cup \{i,^x j^x\}\} \cap I(a)$. Thus $x^{I(a)}$ is a collineation of the plane $P(a)$. Hence $N(G_{I(a)})^{I(a)} < PSL(3, 2)$.

3.5. *If an involution a has a 2-cycle (i, j), then there is an involution of $G_{i\,j}$ commuting with a.*

Proof. By the assumptions (1) and (6), $G_{i\,j}$ has an involution. Since $a \in N_G(G_{i\,j})$, there is an involution of $G_{i\,j}$ commuting with a.

3.6. *Let a and b be involutions such that $I(a) \cap I(b) = \{i, j, k\}$. If $\Delta(i, j) \ni k$, then ab is of odd order.*

Proof. Suppose by way of contradiction that ab is of even order. Then there is an involution c of $G_{i\,j\,k}$ commuting with a and b. Clearly c fixes $I(a)$. Since i, j, k are noncollinear in $P(a)$ and fixed by c, c fixes $I(a)$ pointwise by (3.4). Thus $I(a) = I(c)$. In the same way $I(b) = I(c)$. Hence $I(a) = I(b)$, which is a contradiction.

3.7. *Let a and b be involutions such that $I(a) \supset \{i, j, k\}$ and $I(b) \supset \{i, j, l\}$. If $\Delta(i, j) \ni k, l$ and $k \neq l$, then ab is of odd order.*

Proof. Suppose by way of contradiction that ab is of even order. Then there is an involution c of $G_{i\,j}$ commuting with a and b. Clearly c fixes $I(a)$. Since i, j, k are collinear in $P(a)$ and c fixes i, j, c fixes k. Similarly c fixes l. Thus $\Delta(i, j) \cap I(c) \supseteq \{k, l\}$, contrary to the sssumption (7).

3.8. *$\Delta(i, j)$ is an orbit of $G_{i\,j}$.*

Proof. Let k be a point of $\Delta(i, j)$ and l a point of a $G_{i\,j}$-orbit containing k. Then there is an element x of $G_{i\,j}$ such that $l = k^x$. Since $\Delta(i, j)^x = \Delta(i, j)$, $\Delta(i, j) \ni k^x = l$. Conversely let $\Delta(i, j) \ni k, l$ and $k \neq l$. By the assumption (6), there are involutions a and b such that $I(a) \supset \{i, j, k\}$ and $I(b) \supset \{i, j, l\}$. By (3.7), ab is of odd order. Hence $a^x = b$ for some element x of $G_{i\,j}$. On the other hand $\Delta(i, j) \cap I(a) = \{k\}$ and $\Delta(i, j) \cap I(b) = \{l\}$. Hence $k^x = l$. Thus $\Delta(i, j)$ is an orbit of $G_{i\,j}$.

3.9. $|\Delta(i, j)| > 1$ *for some* i, j *of* Ω.

Proof. Suppose by way of contradiction that $|\Delta(i, j)| = 1$ for any two points i, j. Let $\Delta(i, j) = \{k\}$ and set $\Delta'(i, j) = \Delta(i, j) \cup \{i, j\}$. Then $\Delta'(i, j) = \Delta'(j, k) = \Delta'(k, i) = \{i, j, k\}$. Conversely if $\Delta'(i', j') = \Delta'(i, j)$, then $\{i', j'\} \subset \{i, j, k\}$. Hence $\Delta(i', j')$ is one of the sets $\Delta(i, j)$, $\Delta(j, k)$ or $\Delta(k, i)$. Thus the number of combinations of twenty-three points taken two at a time is divisible by three. which is a contradiction. Thus $|\Delta(i, j)| > 1$ for some i, j of Ω.

3.10. $|\Delta(i, j)|$ *is odd and at most nine.*

Proof. By the assumption (6), there is an involution x fixing i and j. Then x fixes $\Delta(i, j)$. Since $|\Delta(i, j) \cap I(x)| = 1$, $|\Delta(i, j)|$ is odd.

Now let $|\Delta(i, j)| > 1$. Then for distinct points k_1, l_1 of $\Delta(i, j)$ there are involutions a and b such that $I(a) \supset \{i, j, k_1\}$ and $I(b) \supset \{i, j, l_1\}$. By (i) of (3.1) $I(a) = \{i, j, k_1, k_2, \cdots, k_5\}$ and $I(b) = \{i, j, l_1\, l_2, \cdots, l_5\}$. Then by the assumption (7) $k_t, l_t \notin \Delta(i, j)$, $2 \leq t \leq 5$. Since $I(a) \neq I(b)$, $|I(a) \cap I(b)| \leq 3$ by (vi) of (3.1). Hence $|\{k_2, k_3, k_4, k_5\} \cap \{l_2, l_3, l_4, l_5\}| \leq 1$. Furthermore $I(b^a) \supset \{i, j\}$. If $I(b^a) = I(a)$, then b^a commutes with a. Consequently $aba \cdot a = a \cdot aba$, and so $(ab)^2 = 1$, contrary to (3.7). Thus $I(b^a) \neq I(a)$. Moreover from $k_1^a = k_1 \neq l_1$, $l_1^a \neq k_1$. Hence in the same way $|\{k_2, k_3, k_4, k_5\} \cap \{l_2, l_3, l_4, l_5\}^a| \leq 1$. Next if $I(b^a) = I(b)$, then similarly $aba \cdot a = b \cdot aba$, and so $(ab)^4 = 1$, contrary to (3.7). Thus $I(b^a) \neq I(b)$. Since $l_1^a \neq l_1$, in the same way $|\{l_2, l_3, l_4, l_5\} \cap \{l_2, l_3, l_4, l_5\}^a| \leq 1$. Thus $|\{k_2, k_3, k_4, k_5\} \cap \{l_2, l_3, l_4, l_5\} \cap \{l_2, l_3, l_4, l_5\}^a| \geq 12 - 3 = 9$. Since i, j, k_t, l_t and l_t^a, where $2 \leq t \leq 5$, do not belong to $\Delta(i, j)$, $|\Delta(i, j)| \leq 23 - (2 + 9) = 12$. Since $|\Delta(i, j)|$ is odd, $|\Delta(i, j)| \leq 11$. If $|\Delta(i, j)| = 11$, then $G_{i\,j}$ has an element of order eleven by (3.8). This is a contradiction since an element of order eleven in M_{23} fixes exactly one point. Thus (3.10) is proved.

3.11. *G has no element of order seven.*

Proof. Suppose by way of contradiction that G has an element x of order seven. Then we may assume that

$$x = (1)\,(2)\,(3\ 4 \cdots 9)\,(10\ 11 \cdots 16)\,(17\ 18 \cdots 23).$$

Set $\Gamma_1 = \{3, 4, \cdots, 9\}$, $\Gamma_2 = \{10, 11, \cdots, 16\}$ and $\Gamma_3 = \{17, 18, \cdots, 23\}$. Since $1 \leq |\Delta(1, 2)| \leq 9$, we may assume that $\Delta(1, 2) = \Gamma_1$ by (3.8). By the assumption (6), there is an involution a fixing 1, 2, 3. Since $|I(a)| = 7$ and $|I(a) \cap \Gamma_1| = 1$, a fixes exactly four points of $\Gamma_2 \cup \Gamma_3$, say i_1, i_2, i_3, i_4. Set $H = \langle a, x \rangle$. Then $\{1\}$, $\{2\}$ and Γ_1 are H-orbits.

Suppose that $\Gamma_2 \cap \Gamma_3$ is an H-orbit. Since $|\Gamma_2 \cup \Gamma_3| = 14$ and $a \in H_{i_1}$, $|H| = |i_1^H| \cdot |H_{i_1}| = 14 \cdot 2r$, where r is some integer. On the other hand $|H| = |3^H| \cdot |H_3| = |\Gamma_1| \cdot |H_3|$. Hence $|H_3| = 4r$. Thus there is an element b different from a and 1 in a Sylow 2-subgroup of H_3 containing a. From $I(b)\supseteq\{1,2,3\}$, we get that the order of b is four or two. First assume that b is of order four. Then by (iii) of (3.1), b fixes exactly three points and has two 2-cycles. Since b fixes $\Gamma_1-\{3\}$ and $|\Gamma_1-\{3\}|=6$, b has a 2-cycle on Γ_1. Thus b^2 is an involution such that $I(b^2)\supset\{1, 2\}$ and $|\Delta(1, 2)\cap I(b^2)|>1$, contrary to the assumption (7). Next assume that b is of order two. By the same reason, ab is also of order two. Thus a and b have the same 2-cycle on Γ_1. Hence $|\Delta(1, 2)\cap I(ab)|>1$, contrary to the assumption (7).

Therefore Γ_2 and Γ_3 are two distinct H-orbits. Since $I(a)\cap\{\Gamma_2\cup\Gamma_3\}=\{i_1, i_2, i_3, i_4\}$ and $|\Gamma_2|=|\Gamma_3|=7$, we may assume that $I(a)\cap\Gamma_3=\{i_1, i_2, i_3\}$. Then for some integer s $c=a^{x^s}$ fixes i_1 and does not fix 3. By (3.7), ac is of odd order. Since $I(ac)\supset\{1, 2, i_1\}$, the order of ac is three or five. First assume that ac is of order three. Then by (ii) of (3.1), $|I(ac)|=5$. Since $|\Gamma_i|=7$, $|I(ac)\cap\Gamma_i|=1$, $1\leq i\leq 3$. However $|I(a)\cap I(c)\cap\Gamma_1|=0$. Hence $|I(ac)\cap\Gamma_1|\neq 1$. Thus we have a contradiction. Next assume that ac is of order five. Since $|\Gamma_i|=7$, $|I(ac)\cap\Gamma_i|\geq 2$, $1\leq i\leq 3$. This is a contradiction since $|I(ac)|=3$. Thus G has no element of order seven.

3.12. $|\Delta(i, j)|=1, 3, 5$ *or* 9.

Proof. This follows immediately from (3.10) and (3,11).

3.13. *If there is a plane containing three distinct points i, j, k, then the number of planes containing i, j, k is one, three or five.*

Proof. We may assume that $\{i, j, k\}=\{1, 2, 3\}$. For a point t of $\Omega-\{1, 2, 3\}$ the number of planes containing $\{1, 2, 3, t\}$ is at most one by (vi) of (3.1). Since $|\Omega-\{1, 2, 3\}|=20$ and there are seven points in a plane, the number of planes containing 1, 2, 3 is at most $20/4=5$.

Suppose by way of contradiction that the number of planes containing 1, 2, 3 is even. Then we may assume that there is an involution x of the following form

$$x=(1)\,(2)\cdots(7)\,(8\;9)\,(10\;11)\cdots.$$

We denote a set of planes containing 1, 2, 3 by S. Then x is regarded as a permutation on S. Since $|S|$ is even and x fixes $P(x)$, x fixes at least one more plane of S different from $P(x)$. Let $P(y)\neq P(x)$ be a plane of S fixed by x. We may assume that $I(y)=\{1, 2, 3, 8, 9, 10, 11\}$. Then

$$xy=(1)\,(2)\,(3)\,(8\;9)\,(10\;11)\cdots.$$

Hence the order of xy is two or four.

First assume that the order of xy is two. Then $P(xy)$ is a plane of S different from both $P(x)$ and $P(y)$. Since $|S|$ is even and at most five, $|S|=4$, namely

$S=\{P(x), P(y), P(xy), P(z)\}$, where z is an involution. We may set $I(xy)=\{1, 2, 3, 12, 13, 14, 15\}$ and $I(z)=\{1, 2, 3, 16, 17, 18, 19\}$. Then the group $\langle x, y, z\rangle$ fixes $\{I(x), I(y), I(xy), I(z)\}$ as a set. Hence $\langle x, y, z\rangle$ fixes $\{20, 21, 22, 23\}$ as a set. Then x, y, xy and z have two 2-cycles on $\{21, 21, 22, 23\}$ respectivly. Hence there is an involution u of $\langle x, y\rangle$ such that u and z are the same form on $\{20, 21, 22, 23\}$. This implies that uz is an involution fixing $\{1, 2, 3, 20, 21, 22, 23\}$ pointwise. Thus $P(uz)\in S$, which is a contradiction.

Next assume that xy is of order four. If xy fixes some plane of S different from $P(y)$, then xy fixes three points 1, 2, 3 and has one 4-cycle on the set of points of this plane. This contradicts (3.4). Thus xy fixes only $P(y)$ of S. Hence $|S|$ is odd, which is a contradiction. Thus $|S|=1$, 3 or 5.

3.14. *G is intransitive and has no orbit of length twenty-two.*

Proof. First suppose by way of contradiction that G is transitive. Then G is M_{23} or contained in a group of order $23\cdot 11$ (see [1], p. 235). Since G has an involution, G must be M_{23}. Then by (3.8) for any two points i, j of Ω $|\Delta(i, j)|=21$, contrary to (3.12).

Next suppose by way of contradiction that G has orbits of lengths one and twenty-two. Then G is a subgroup of M_{22}. Since M_{22} has no proper subgroup which is transitive on twenty-two points (see [1], p. 235), G is M_{22}. Let $\{i\}$ be the G-orbit of length one. Then by (3.8) for anty point j of $\Omega-\{i\}$ $|\Delta(i. j)|=21$, which is also a contradiction.

3.15. *If G has an orbit of length eleven, then G is* (*i*) *or* (*vi*) *of Lemma* 1.

Proof. By assumption G has an element of order eleven consisting of one fixed point and two 11-cycles. Hence the lengths of G-orbits are 11, 1, 11 or 11, 12.

First assume that G has three orbits Γ_1, Γ_2 and Γ_3, where $|\Gamma_1|=1$ and $|\Gamma_2|=|\Gamma_3|=11$. By the assumption (4), G^{Γ_2} is isomorphic to G. Since for any two points i, j of Γ_2 there is an involution fixing i, j, $(G^{\Gamma_2})_{i\,j}\neq 1$. Hence G^{Γ_2} is doubly transitive on Γ_2 by Theorem 11.6 and Theorem 11.7 in [6]. By [5], G^{Γ_2} is isomorphic to $PSL(2, 11)$ or contains M_{11}. Since the lengths of G-orbits are 11, 1, 11, G is isomorphic to $PSL(2, 11)$ (see [1], p. 235).

Next assume that G has two orbits Γ_1 and Γ_2, where $|\Gamma_1|=11$ and $|\Gamma_2|=12$. By the similar reason above, G^{Γ_1} is isomorphic to G and G is isomorphic to $PSL(2, 11)$ or contains M_{11}. Since the lengths of G-orbits are 11, 12, G is isomorphic to M_{11} (see [1], p. 235).

3.16. *If there are two points i, j such that* $|\Delta(i, j)|=9$, *then G is one of the groups* (*i*), (*ii*), (*iii*), (*iv*) *or* (*v*) *of Lemma* 1, *and we have the following*:

(i) *If G is isomorphic to* M_{11}, *then G is isomorphic to* $\langle a, b, c, y_1, x_1\rangle$ *as a permutation group.*

(ii) *If G is isomorphic to* M_{10}, *then G is isomorphic to* $\langle a, b, c, y_1, x_1 y_2\rangle$ *as a permutation group.*

(iii) *If G is isomorphic to* M_{10}^*, *then G is isomorphic to* $\langle a, b, c, y_1\rangle$ *as a permutation group.*

(iv) *If G is isomorphic to* $N(M_9)$, *then G is isomorphic to* $\langle a, b^{y_1 x_1}, c^{y_1 x_1}, y_1, (x_1 y_2)^{x_1}\rangle$ *as a permutation group.*

(v) *If G is isomorphic ot* $N(M_9)^*$, *then G is isomorphic to* $\langle a, b^{y_1 x_1}, c^{y_1 x_1}, y_1, y_2^{y_1 x_1}\rangle$ *as a permutation group.*

Here

$$a=(1)\,(2)\,(3)\,(8\;9)\,(10\;11)\,(12\;13)\,(14\;15)\,(4)\,(18\;19)\,(5)\,(16\;17)\,(6)\,(20\;21)\,(7)\,(22\;23),$$
$$b=(1)\,(2)\,(8)\,(3\;9)\,(10\;12)\,(11\;15)\,(13\;14)\,(4)\,(18\;19)\,(16)\,(5\;17)\,(20)\,(6\;21)\,(23)\,(7\;22),$$
$$c=(1)\,(2)\,(10)\,(3\;11)\,(8\;12)\,(9\;14)\,(13\;15)\,(18)\,(4\;19)\,(5)\,(16\;17)\,(20)\,(6\;21)\,(22)\,(7\;23),$$
$$x_1=(1\;2)\,(3)\,(8)\,(9)\,(12\;13)\,(10\;14)\,(11\;15)\,(4)\,(5\;6)\,(7)\,(16\;20)\,(17\;21)\,(18\;19)\,(22)\,(23),$$
$$y_1=(1)\,(2\;3)\,(8)\,(9)\,(10\;11)\,(12\;14)\,(13\;15)\,(4\;5)\,(6)\,(7)\,(16)\,(17)\,(18\;19)\,(20\;22)\,(21\;23),$$
$$y_2=(1\;2)\,(3)\,(8\;13)\,(9\;12)\,(10\;11)\,(14)\,(15)\,(5)\,(4\;7)\,(6)\,(19\;22)\,(18\;23)\,(16)\,(17)\,(20\;21).$$

Proof. Let $|\Delta(1,2)|=9$ and $\Delta(1,2)=\{3, 8, 9, \cdots, 15\}$. Then we may assume that there is an involution a of the form

$$a=(1)\,(2)\cdots(7)\,(8\;9)\,(10\;11)\,(12\;13)\,(14\;15)\,(16\;17)\,(18\;19)\,(20\;21)\,(22\;23).$$

For any point i of $\Delta(1,2)$ there is a plane containing points 1, 2, i, in which there is no point of $\Delta(1,2)-\{i\}$ by the assumptions (6) and (7). Hence this plane has four points in $\Omega-\{\Delta(1,2)\cup\{1,2\}\}$. Since $|\Omega-\{\Delta(1,2)\cup\{1,2\}\}|=12$ the number of planes containing 1, 2, i is at most three. Suppose that there are exactly three planes containing 1, 2, i, say P_1, P_2 and P_3. Let P' be a plane containing 1, 2, j, where $j\in\Delta(1,2)$ and $j\neq i$. Then P' also has four points in $\Omega-\{\Delta(1,2)\cup\{1,2\}\}$. Hence P' and P, where P is one of P_1, P_2, P_3, have at least two points of $\Omega-\{\Delta(1,2)\cup\{1,2\}\}$ in common. Since 1, 2 are the points of both P' and P, P' and P have at least four points in common. Thus $P'=P$, which is a contradiction. Therefore for any point i of $\Delta(1,2)$ there is exactly one plane containing 1, 2, i by (3.13).

Let b be an involution fixing 1, 2, 8. Then ab is of odd order by (3.7). Since ab fixes $\Delta(1,2)$ with length nine, the order of ab is not five by (vi) of (3.1).

Hence the order of ab is three. Since b fixes $\Delta(1, 2)$ as a set and ab fixes exactly five points by (ii) of (3.1), we may assume that b is of the form

$$b = (1)\ (2)\ (8)\ (3\ 9)\ (10\ 12)\ (11\ 15)\ (13\ 14)\ (4)\ (18\ 19)\ (16)\ (5\ 17)\ (20)\ (6\ 21)\ (23)\ (7\ 22).$$

Then

$$ab = (1)\ (2)\ (3\ 9\ 8)\ (10\ 15\ 13)\ (11\ 12\ 14)\ (4)\ (18)\ (19)\ (5\ 17\ 16)\ (6\ 21\ 20)\ (7\ 22\ 23).$$

Since an element of order three of $PSL\ (3, 2)$ fixes exactly one point, ab does not fix a plane containing 1, 2, 4. Thus the lengths of $\langle ab\rangle$-orbits on the set of planes containing 1, 2, 4 are three. Hence by (3.13) the planes containing 1, 2, 4 are $P(a)$, $P(a^{ab})=P(bab)$ and $P(a^{(ab)^2})=P(b)$. Let c be an involution fixing 1, 2, 10. Then in the same way ac and bc are of order three. Hence

$$c = (1)\ (2)\ (10\ )(3\ 11)\ (8\ 12)\ (9\ 14)\ (13\ 15)\cdots.$$

Then

$$abc = (1)\ (2)\ (3\ 14)\ (8\ 11)\ (9\ 12)\ (10\ 13)\ (15)\cdots.$$

Thus $(abc)\ (abc) = 1$ and so $c(ab)c=ba=(ab)^{-1}$. This implies that c fixes {4, 18, 19}. Hence $c=(4)\ (18\ 19)\cdots$, $(18)\ (4\ 19)\cdots$ or $(19)\ (4\ 18)\cdots$. If $c=(4)\ (18\ 19)\cdots$, then $P(c)$ is a plane containing 1, 2, 4 and different from $P(a)$, $P(bab)$ and $P(b)$, which is a contradiction. Then we may assume that $c=(18)\ (4\ 19)\cdots$ since in the case $c=(19)\ (4\ 18)\cdots$ the proof is similar. Thus we may assume that c is of the form

$$c = (1)\ (2)\ (10)\ (3\ 11)\ (8\ 12)\ (9\ 14)\ (13\ 15)\ (18)\ (4\ 19)\ (5)\ (16\ 17)\ (20)\ (6\ 21)\ (22)\ (7\ 23).$$

Then

$$\begin{aligned}
ac &= (1)\ (2)\ (3\ 11\ 10)\ (8\ 14\ 13)\ (9\ 12\ 15)\ (4\ 19\ 18)\ (5)\ (16\ )(17)\ (6\ 21\ 20)\ (7\ 23\ 22),\\
b^c &= (1)\ (2)\ (12)\ (3\ 13)\ (8\ 10)\ (9\ 15)\ (11\ 14)\ (19)\ (4\ 18)\ (17)\ (5\ 16)\ (20)\ (6\ 21)\ (7)\ (22\ 23),\\
ab^c &= (1)\ (2)\ (3\ 13\ 12)\ (8\ 15\ 11)\ (9\ 10\ 14)\ (4\ 18\ 19)\ (5\ 16\ 17)\ (6\ 21\ 20)\ (7)\ (22)\ (23),\\
a^{bc} &= (1)\ (2)\ (14)\ (3\ 15)\ (8\ 13)\ (9\ 10)\ (11\ 12)\ (19)\ (4\ 18)\ (16)\ (5\ 17)\ (6)\ (20\ 21)\ (22)\ (7\ 23),\\
aa^{bc} &= cb = (1)\ (2)\ (3\ 15\ 14)\ (8\ 10\ 12)\ (9\ 13\ 11)\ (4\ 18\ 19)\ (5\ 17\ 16)\ (6)\ (20)\ (21)\ (7\ 23\ 22).
\end{aligned}$$

We use frequently these elements in the following proofs.

Since $\{1, 2, 4\}^c = \{1, 2, 19\}$ and $\{1, 2, 4\}^{ca} = \{1, 2, 18\}$, the number of planes containing 1, 2, 19 or 1, 2, 18 is also three. Thus for any $i \in I(ab) - \{1, 2\}$ the number of planes containing 1, 2, i is three. Since ac, aa^{bc} and ab^c are of order three and ac fixes 1, 2, 5, 16, 17, aa^{bc} fixes 1, 2, 6, 20, 21 and ab^c fixes 1, 2, 7, 22, 23, by the same argument as is used for the planes containing 1, 2, 4, we have that for any point i of $\{4, 5, 6, 7, 16, 17, \cdots, 23\}$ the number of planes containing 1, 2, i is three.

Set $\Delta=\Delta(1, 2)$, $\Gamma_1=\{4, 18, 19\}$, $\Gamma_2=\{5, 16, 17\}$, $\Gamma_3=\{6, 20, 21\}$ and $\Gamma_4=\{7, 22, 23\}$. In the plane $P(a)$ 1, 2, 3 are collinear, Hence from now on we may assume that $\Delta(1, 4)\ni 5$, $\Delta(1, 6)\ni 7$ and $(2, 4)\ni 6$ in $I(a)$.

Now we show that for any point i of $\{1, 2\}$ and j of $\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ $|\Delta(i, j)|=3$ or 5. If we prove that $|\Delta(1, 4)|=3$ or 5, then since $\Delta(1, 18)=\Delta(1, 4)^{ca}$ and $\Delta(1, 19)=(\Delta 1, 4)^c$, $|\Delta(1, 18)|=|\Delta(1, 19)|=|\Delta(1, 4)|=3$ or 5, and in the remaining cases the proofs are similar. By (3.12), $|\Delta(1, 4)|=1$, 3, 5 or 9. Since $\Delta(1, 4)\ni 5$, $\Delta(1, 4)\supseteq\{5\}^{\langle ab\rangle}=\Gamma_2$. On the other hand $\Delta(1, 4)\not\ni 2, 3, 6, 7$ in $I(a)$. Hence any point of $\{2, 3, 6, 7\}^{\langle ab\rangle}$ does not belong to $\Delta(1, 4)$. Thus $\Delta(1, 4)=\Gamma_2, \Gamma_2\cup\{18, 19\}$ or $\Gamma_2\cup\{10, 11, \cdots, 15\}$. Suppose that $\Delta(1, 4)=\Gamma_2\cup\{10, 11, \cdots, 15\}$. Then there is an involution x fixing 1, 4, 10 and the form of x is determined by the same argument as is used for c. Hence the $\langle a, b, x\rangle$-orbits have the same lengths as the lengths of the $\langle a, b, c\rangle$-orbits. Furthermore $\Delta(1, 4)$ is a $G_{1\,4}$-orbit and $\langle a, b\rangle$ has orbits $\{1\}$, $\{2\}$, $\{3, 9, 8\}$, $\{10, 11, \cdots, 15\}$, $\{4\}$, $\{18, 19\}$, Γ_2, Γ_3 and Γ_4. Hence the $\langle a, b, x\rangle$-orbits are $\{1\}$, $\{4\}$, $\Gamma_2\cup\{10, 11, \cdots, 15\}$, $\{2, 18, 19\}$, $\{3, 8, 9\}$, Γ_3 and Γ_4. Hence the $\langle a, b, c, x\rangle$-orbits are $\{1\}$, $\Delta\cup\Gamma_2$, $\{2\}\cup\Gamma_1$, Γ_3 and Γ_4. Since $|\{2\}\cup\Gamma_1|=4$, the order of $\langle a, b, c, x\rangle$ is divisible by four. On the other hand $|\Gamma_3|=3$. Hence $\langle a, b, c, x\rangle_{\{1\}\cup\Gamma_3}$ has an involution y. Since bc fixes four points 1, 6, 20, 21 of $I(y)$, bc fixes $I(y)$ pointwise by (3.4). Thus $|I(bc)|>5$, which is a contradiction. Hence $\Delta(1, 4)=\Gamma_2$ or $\Gamma_2\cup\{18, 19\}$. Thus for any point i of $\{1, 2\}$ and j of $\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ $|\Delta(i, j)|=3$ or 5. Since $\Delta(1, 5)^{ac}=\Delta(1, 5)$ and $\Delta(1, 5)\ni 4$, $\Delta(1, 5)=\Gamma_1$ or $\Gamma_1\cup\{16, 17\}$. Since $\Delta(1, 6)^{aa^{bc}}=\Delta(1, 6)$ and $\Delta(1, 6)\ni 7$, $\Delta(1, 6)=\Gamma_4$ or $\Gamma_4\cup\{20, 21\}$. Since $\Delta(1, 7)^{ab^c}=\Delta(1, 7)$ and $\Delta(1, 7)\ni 6$, $\Delta(1, 7)=\Gamma_3$ or $\Gamma_3\cup\{22, 23\}$. In the same way $\Delta(2, 4)=\Gamma_3$ or $\Gamma_3\cup\{18, 19\}$, $\Delta(2, 5)=\Gamma_4$ or $\Gamma_4\cup\{16, 17\}$, $\Delta(2, 6)=\Gamma_1$ or $\Gamma_1\cup\{20, 21\}$ and $\Delta(2, 7)=\Gamma_2$ or $\Gamma_2\cup\{22, 23\}$.

Furthermore since for $j\in\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ $|\Delta(1, j)|=3$ or 5 and $|\Delta(1, 2)|=9$, the points 2 and j of $\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ do not belong to the same G_1-orbit. Hence G_1 has an orbit $\{2\}$ or $\{2\}\cup\Delta$. Similarly G_2 has an orbit $\{1\}$ or $\{1\}\cup\Delta$. Using this result we prove that any involution fixing $I(a)$ fixes $\{1, 2, 3\}$. Let x be an involution fixing $I(a)$. Then x is one of the following forms: $x=(1)(2)\cdots$, $(1\ 2)\cdots$, $(1)(2\ i)\cdots$, $(1\ i)(2)\cdots$ or $(1\ i)(2\ j)\cdots$, where $i, j\in\{3, 4, \cdots, 7\}$. By the incidence structure of $P(a)$, if $x=(1)(2)\cdots$ or $(1\ 2)\cdots$, then x fixes $\{1, 2, 3\}$. Assume that $x=(1)(2\ i)\cdots$, where $i\in\{3, 4, \cdots, 7\}$. Then $x\in G_1$. Since the G_1-orbit containing 2 is $\{2\}$ or $\{2\}\cup\Delta(1, 2)$, $x=(1)(2\ 3)\cdots$. In the same way if $x=(2)(1\ i)\cdots$, where $i\in\{3, 4, \cdots, 7\}$, then $i=3$. Next assume that $x=(1\ i)(2\ j)\cdots$, where $i, j\in\{3, 4, \cdots, 7\}$. Since 1, 2, 3 are collinear in $P(a)$, $i, j\neq 3$.

Let $i=4$. Then by the incidence structure of $P(a)$, $x=(1\ 4)\ (2\ 7)\ (3)\ (5)\ (6)\cdots$. Since G_1 has an orbit $\{2\}$ or $\{2\}\cup\Delta(1, 2)$, $G_4=G_1^x$ has an orbit $\{7\}$ or $\{7\}\cup\Delta(4, 7)$. Since $G_4>\langle a, b\rangle$, G_4-orbits consist of unions of some $\langle a, b\rangle$-orbits. Thus $\{7\}$ is not a G_4-orbit. Since $|\{7\}\cup\Delta(4, 7)|=10$, if $\{7\}\cup\Delta(4, 7)$ is a G_4-orbit, then $\{7\}\cup\Delta(4, 7)$ contains at least one $\langle a, b\rangle$-orbit of length one. However the $\langle a, b\rangle$-orbits of length one are $\{1\}$, $\{2\}$ and $\{4\}$. Since $1, 4, 7\in\{2\}\cup\Delta(1, 2)$, $\{1, 4, 7\}^x=\{1, 4, 2\}$ and $\{\{2\}\cup\Delta(1, 2)\}^x=\{7\}\cup\Delta(4, 7)$, any point of $\{1, 2, 4\}$ does not belong to $\{7\}\cup\Delta(4, 7)$. This contradiction shows that $x\neq(1\ 4)\ (2\ 7)\cdots$. Finally by the incidence structure of $P(a)$, for $i=5$, 6 or 7 $x=(1\ 5)\ (2\ 6)\ (3)\ (4)\ (7)\cdots$, $(1\ 6)\ (2\ 5)\ (2\ )\ (4\ )\ (7)\cdots$or $(1\ 7)\ (2\ 4)\ (3)\ (5)\ (6)\cdots$ respectively. Then similarly to the case $i=4$, we have a contradiction and so $i\neq5, 6, 7$. Thus any involution fixing $I(a)$ fixes $\{1, 2, 3\}$.

By (3.5), there is an involution fixing 8, 9 and commuting with a. Hence from now on let x be an involution fixing 8, 9 and commuting with a. Then by the assertion above, x fixes $\{1, 2, 3\}$. Hence x is one of the following forms:

(i) $x=(1)\ (2)\ (3)\ (8)\ (9)\cdots$,
(ii) $x=(1\ 2)\ (3)\ (8)\ (9)\cdots$,
(iii) $x=(1)\ (2\ 3)\ (8)\ (9)\cdots$,
(iv) $x=(2)\ (1\ 3)\ (8)\ (9)\cdots$,

(i) Assume that $x=(1)\ (2)\ (3)\ (8)\ (9)\cdots$. Then $P(x)$ and $P(a)$ are distinct planes containing $\{1, 2, 3\}$. This is a contradiction since there is no plane containing 1, 2, 3 except $P(a)$.

(ii) Assume that $x=(1\ 2)\ (3)\ (8)\ (9)\cdots$. Then by the incidence structure of $P(a)$, $x=(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\cdots$, or $(1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8)\ (9)\cdots$.

(ii.i) Let $x=(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\cdots$. Then $\Delta(1, 4)^x=\Delta(2, 7)$. Since $\Delta(1, 4)=\Gamma_2$ or $\Gamma_2\cup\{18, 19\}$ and $\Delta(2, 7)=\Gamma_2$ or $\Gamma_2\cup\{22, 23\}$, $\Gamma_2^x=\Gamma_2$. Thus $x=(16)\ (17)\cdots$ or $(16\ 17)\cdots$. First assume that

$$x=(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\ (16)\ (17)\cdots.$$

Then

$$bx=(1\ 2)\ (3\ 9)\ (8)\ (16)\ (5\ 17)\ (22\ 4\ 7\cdots)\cdots.$$

Thus $(bx)^2$ is not the identity element fixing eight points, which is a contradiction. Next assume that

$$x=(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\ (16\ 17)\cdots.$$

Then $\Delta(1, 5)^x=\Delta(2, 5)$. Since $\Delta(1, 5)=\Gamma_1$ or $\Gamma_1\cup\{16, 17\}$ and $\Delta(2, 5)=\Gamma_4$ or $\Gamma_4\cup\{16, 17\}$, $\Gamma_1^x=\Gamma_4$. Thus $x=(18\ 22)\ (19\ 23)\cdots$ or $(18\ 23)\ (19\ 22)\cdots$. If

$$x=(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\ (18\ 22)\ (19\ 23)\ (16\ 17)\cdots,$$

then

$$cx=(1\ 2)\ (5)\ (16)\ (17)\ (4\ 23)\ (7\ 19)\cdots.$$

Hence cx is an involution and so c commutes with x. Since $I(x)\supset\{3, 6, 8, 9\}$, $I(x)\supset\{3, 6, 8, 9\}^{<c>}=\{3, 11, 6, 21, 8, 12, 9, 14\}$, contrary to (i) of (3.1). Thus

$$x=(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\ (16\ 17)\ (18\ 23)\ (19\ 22)\cdots .$$

Hence

$$cx=(1\ 2)\ (5)\ (16)\ (17)\ (4\ 22\ 19\ 7\ 18\ 23)\cdots ,$$

contrary to (v) of (3.1). Therefore case (ii.i) does not occur.

(ii.ii) Let $x=(1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8)\ (9)\cdots$.

Then

$$bx=(1\ 2)\ (4)\ (3\ 9)\ (8)\cdots .$$

Hence bx is of order two or four. If bx is of order four, then $(bx)^2$ is an involution such that $I((bx)^2)\supset\{1, 2\}$ and $\Delta(1, 2)\cap I((bx)^2)\supset\{3, 8, 9\}$, contrary to the assumption (7). Thus bx is of order two. Hence

$$x=(1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8)\ (9)\ (17\ 21)\ (16\ 20)\ (22)\ (23)\ (18\ 19)\cdots ,$$

and on $\{10, 11, \cdots, 15\}$ $x=(10\ 11)\ (12\ 15)\ (13\ 14)$, $(12\ 13)\ (10\ 14)\ (11\ 15)$ or $(14\ 15)\ (11\ 13)\ (10\ 12)$. If $x=(10\ 11)\ (12\ 15)\ (13\ 14)\cdots$, then $I(ax)=\{3, 4, 7, 18, 19, 10, 11\}$. Hence $I(ax)\cap I((ax)^c)=\{3, 11, 4, 19, 18, 10\}$, contrary to (vi) of (3.1). If $x=(14\ 15)\ (11\ 13)\ (10\ 12)\cdots$, then $I(ax)=\{3, 4, 7, 18, 19, 14, 15\}$. Hence $I(ax)\cap I((ax)^{bc})=\{3, 14, 15, 4, 18, 19\}$, which is also a contradiction. Thus

$$x=(1\ 2)\ (3)\ (8)\ (9)\ (12\ 13)\ (10\ 14)\ (11\ 15)\ (4)\ (5\ 6)\ (7)\ (16\ 20)\ (17\ 21)$$
$$(18\ 19)\ (22\ )(23)\,.$$

From now on the element x of this form is denoted by x_1. Then the $\langle a, b, c, x_1\rangle$-orbits are $\{1, 2\}$, Δ, Γ_1, $\Gamma_2\cup\Gamma_3$ and Γ_4.

By (3.5), G has an involution y fixing 16, 17 and commuting with a. Suppose that $y\in\langle a, b, c, x_1\rangle$. Since $\{a, b, c\}^{x_1}=\{a, b, a^{bc}\}$ and x_1 is of order two, the index of $\langle a, b, c\rangle$ in $\langle a, b, c, x_1\rangle$ is two. Hence $|\langle a, b, c, x_1\rangle|=2\cdot|\langle a, b, c\rangle|=2\cdot 9\cdot 2$. Thus both $\langle a, x_1\rangle$ and $\langle a, y\rangle$ are Sylow 2-subgroups of $\langle a, b, c, x_1\rangle$. On the other hand a is only one involution of $\langle a, x_1\rangle$ having fixed points in a $\langle a, b, c, x_1\rangle$-orbit $\Gamma_2\cup\Gamma_3$ and $\langle a, y\rangle$ has at least two involutions a and y having fixed points in $\Gamma_2\cup\Gamma_3$. This is a cotradiction since $\langle a, x_1\rangle$ and $\langle a, y\rangle$ are conjugate in $\langle a, b, c, x_1\rangle$. Thus y is not an element of $\langle a, b, c, x_1\rangle$. By the same argument as is used above for x, $y=(1)\ (2\ 3)\ (16)\ (17)\cdots$, $(2)\ (1\ 3)\ (16)\ (17)\cdots$ or $(1\ 2)\ (3)\ (16)\ (17)\cdots$. Set $H=\langle a, b, c, x_1, y\rangle$.

(ii.ii.i) Let $y=(1)\ (2\ 3)\ (16)\ (17)\cdots$. The proof in the case $y=(2)\ (1\ 3)\ (16)\ (17)\cdots$ is similar. Then $\{1, 2\}$ and Δ are contained in the same H-orbit. By (3.14), H is intransitive. Hence the H-orbit containing $\{1, 2\}\cup\Delta$ is $\{1, 2\}\cup\Delta$, $\{1, 2\}\cup\Delta\cup\Gamma_1$, $\{1, 2\}\cup\Delta\cup\Gamma_4$, $\{1, 2\}\cup\Delta\cup\Gamma_1\cup\Gamma_4$, $\{1, 2\}\cup\Delta\cup\Gamma_2\cup\Gamma_3$, $\{1, 2\}$

$\cup \Delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ or $\{1, 2\} \cup \Delta \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$. Since $|\{1, 2\} \cup \Delta \cup \Gamma_i| = 14$ ($i = 1, 4$), $\{1, 2\} \cup \Delta \cup \Gamma_i$ is not an H-orbit by (3.11). Since $|\{1, 2\} \cup \Delta \cup \Gamma_i \cup \Gamma_j| = 17$ ($\{i, j\} = \{1, 4\}$ or $\{2, 3\}$), $\{1, 2\} \cup \Delta \cup \Gamma_i \cup \Gamma_j$ is also not an H-orbit by the assumption (4). Next suppose that $\{1, 2\} \cup \Delta \cup \Gamma_i \cup \Gamma_2 \cup \Gamma_3$ ($i=1, 4$), is an H-orbit. Let $i=1$. Then Γ_4 is an H-orbit. Since x_1 fixes Γ_4 pointwise and four points of $\{1, 2\} \cup \Delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, there is an involution z which is conjugate to x_1 and fixes 1 and Γ_4 pointwise. Since ab^c fixes $\{1\} \cup \Gamma_4$ pointwise, ab^c fixes $I(z)$ pointwise by (3.4). This is a contradiction since $|I(ab^c)| = 5$. Hence $\{1, 2\} \cup \Delta \cup \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ is not an H-orbit. Since ax_1 is an involution fixing Γ_1 pointwise, in the same way as above $\{1, 2\} \cup \Delta \cup \Gamma_4 \cup \Gamma_2 \cup \Gamma_3$ is not an H-orbit. Therefore $\{1, 2\} \cup \Delta$ is an H-orbit. Since $|\{1, 2\} \cup \Delta| = 11$, H has an element of order eleven, which fixes exactly one point and has two 11-cycles. Hence $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is also an H-orbit.

We shall determine the form of the involution y. By the incidence structure of $P(a)$, $y = (1)(2\ 3)(4)(5)(6\ 7)(16)(17)\cdots$ or $y = (1)(2\ 3)(4\ 5)(6)(7)(16)(17)\cdots$. Suppose that y is of the first form. Since ab fixes $\{1, 4, 5, 16, 17\}$ which is contained in $I(y)$, ab fixes $I(y)$. This contradicts (3.4) since the order of ab is three and $|I(ab) \cap I(y)| \geq 2$. Hence y must be of the second form. Then y fixes $\Delta(1, 6)$ which is Γ_4 or $\Gamma_4 \cup \{20, 21\}$. Since Γ_4 is a $\langle a, b, c, x_1 \rangle$-orbit and $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is an H-orbit, $\Gamma_4^y \neq \Gamma_4$. Thus $\{20, 21\}^y = \{22, 23\}$ and so $y = (20\ 23)(21\ 22)\cdots$ or $(20\ 22)(21\ 23)\cdots$. If

$$y = (1)(2\ 3)(4\ 5)(6)(7)(16)(17)(20\ 23)(21\ 22)\cdots,$$

then

$$by = (1)(16)(4\ 5\ 17)(6\ 22\ 7\ 21)\cdots,$$

contrary to the assumption (4). Thus

$$y = (1)(2\ 3)(4\ 5)(6)(7)(16)(17)(20\ 22)(21\ 23)\cdots.$$

Hence

$$by = (1)(4\ 5\ 17)(16)(6\ 23\ 21)(7\ 20\ 22)(9\ 2\ 3\cdots)\cdots.$$

This shows that by is of order three. Hence y fixes 9 and so 8 because $ay = ya$. On the other hand $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is an H-orbit. Therefore $y = (18\ 19)\cdots$. Thus

$$y = (1)(2\ 3)(8)(9)(4\ 5)(6)(7)(16)(17)(18\ 19)(20\ 22)(21\ 23)\cdots.$$

Then

$$cy = (1)(4\ 18\ 19\ 5)(16\ 17)(6\ 23\ 7\ 21)(20\ 22)\cdots.$$

Hence cy is of order four. By (iii) of (3.1), $|I(cy)| = 3$. This implies that c and y have exactly one 2-cycle in common on $\{10, 11, \cdots, 15\}$. Thus

$$y = (1)(2\ 3)(8)(9)(13\ 15)(12\ 14)(10\ 11)(4\ 5)(6)(7)(16)(17)(18\ 19)(20\ 22)(21\ 23).$$

From now on the element y of this form is denoted by y_1. Since $H=\langle a,b,c,x_1,y_1\rangle$ is 3-fold transitive on $\{1, 2, 8, 9, \cdots, 15\}$, H is isomorphic to M_{11}, A_{11} or S_{11} by [5]. On the other hand $\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ is an H-orbit of length twelve. Since A_{11} and S_{11} have no permutation representation of degree twelve, H is isomorphic to M_{11}. Furthermore by (3.14), there is no group containing H as a proper subgroup.

(ii.ii.ii) Let $y=(1\ 2)(3)(16)(17)\cdots$. Then $y=(1\ 2)(3)(4)(5\ 6)(7)(16)(17)\cdots$ or $(1\ 2)(3)(5)(6)(4\ 7)(16)(17)\cdots$. Suppose that y is of the first form. Thern x_1y is not the identity element and fixes $\{1, 2, \cdots, 7\}$ pointwise. Hence x_1y is of order two and so y commutes with x_1. Since $x_1=(16\ 20)(17\ 21)\cdots$ and y fixes 16, 17, y fixes also 20, 21. On the other hand y fixes $I(x_1)$ and $\Delta(1, 2)$. Hence y fixes $I(x_1)\cap\Delta(1, 2)=\{3, 8, 9\}$ and so y has a 2-cycle $(8\ 9)$ becasue $|I(y)|=7$. Thus ax_1y is an involution fixing $\{1, 2, \cdots, 9\}$ pointwise, contrary to (i) of (3.1). Therefore y must be of the second form. Then

$$x_1y=(1)(2)(3)(4\ 7)(5\ 6)\cdots.$$

Hence x_1y is of order two or four. If x_1y is an involution, then $P(x_1y)$ is the plane containing 1, 2, 3. This is a contradiction since there is no plane containing 1, 2, 3 and different from $P(a)$. Thus x_1y is of order four. Since $\Delta(1, 7)=\Gamma_3$ or $\Gamma_3\cup\{22, 23\}$, $\Delta(2, 4)=\Gamma_3$ or $\Gamma_3\cup\{18, 19\}$ and $\Delta(1, 7)^y=\Delta(2, 4)$, $\Gamma_3^y=\Gamma_3$. By (iii) of (3.1), x_1y has exactly two 2-cycles. Hence y has a 2-cycle $(20\ 21)$ on Γ_3. Since $\Delta(1, 5)=\Gamma_1$ or $\Gamma_1\cup\{16, 17\}$, $\Delta(2, 5)=\Gamma_4$ or $\Gamma_4\cup\{16, 17\}$ and $\Delta(1, 5)^y=\Delta(2, 5)$, $\Gamma_1^y=\Gamma_4$. Thus $y=(18\ 22)(19\ 23)\cdots$ or $(18\ 23)(19\ 22)\cdots$. If y is of the first form, then $cy=(1\ 2)(4\ 23)(18\ 22)(6\ 20\ 21)\cdots$, which contradicts (v) of (3.1). Thus

$$y=(1\ 2)(3)(4\ 7)(5)(6)(16)(17)(20\ 21)(18\ 23)(19\ 22)\cdots.$$

This implies that $(bc)^y=bc$. Hence y fixes 14, 15, and so y has a 2-cycle $(10\ 11)$ because the order of x_1y is four. Then y has 2-cycles $(8\ 13)$ and $(9\ 12)$ since $bc=(8\ 12\ 10)(9\ 11\ 13)\cdots$ and $(bc)^y=bc$. Thus

$$y=(1\ 2)(3)(8\ 13)(9\ 12)(10\ 11)(14)(15)(4\ 7)(5)(6)(16)(17)(20\ 21)(18\ 23)(19\ 22).$$

From now on the element y of this form is denoted by y_2. Set $H=\langle a, b, c, x_1, y_2\rangle$. Then H is isomorphic to a subgroup of $N_{M_{11}}(M_9)$ of index 2, which we denote by $N(M_9)^*$, and the H-orbits are $\{1, 2\}$, Δ, $\Gamma_1\cup\Gamma_4$ and $\Gamma_2\cup\Gamma_3$.

Suppose that H is a proper subgroup of G. First assume that $\{1,2\}$ is a G-orbit. Then Δ is a G-orbit. By the assumption (4), the order of G_{123} is a divisor of $20\cdot48$. Since $|\Delta|=9$ and $3\in\Delta$, if G_{123} has an element of order three, then the order of G is divisible by 3^3. This is a contradiction since the order of M_{23} is not divisible by 3^3. If G_{123} has an element of order five, then this element fixes three points 1, 2, 3 and at least three points of $\Delta-\{3\}$ since $|\Delta-\{3\}|=8$, contrary to (iv) of (3.1). Thus G_{123} is a semiregular 2-group on $\Delta-\{3\}$ by the assumption (7). Since G contains H as a proper subgroup, $|G: G_{123}|=|H: H_{123}|=2\cdot 9$ and $|H_{123}|=4$, G_{123} is of order 8. Thus G is isomorphic to $N_{M_{11}}(M_9)$ by [5]. Therefore $G=\langle a, b, c, x_1, (x_1 y_2)^{y_1}\rangle$ and the G-orbits are $\{1, 2\}$, Δ, $\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$.

Next assume that $\{1, 2\}$ is contained in some G-orbit properly. If $\{1, 2\}\cup\Delta$ is a G-orbit, then G is isomorphic to M_{11} by (3.15). If $\{1, 2\}\cup\Gamma_1\cup\Gamma_4$ is a G-orbit, then the remaining G-orbits are Δ and $\Gamma_2\cup\Gamma_3$ or $\Delta\cup\Gamma_2\cup\Gamma_3$. Since $|\{1, 2\}\cup\Gamma_1\cup\Gamma_4|=8$ and x_1y_2 is of order four and contained in G_1, G is of order divisible by $8\cdot 4$. Suppose that $\Gamma_2\cup\Gamma_3$ is a G-orbit. Since $|\Gamma_2\cup\Gamma_3|=6$, $G_{\Gamma_2\cup\Gamma_3}$ has an involution. Hence for any point t of $\{1,2\}\cup\Gamma_1\cup\Gamma_4$ or Δ, there is an involution fixing t and $\Gamma_2\cup\Gamma_3$ pointwise. This is a contradiction since every involution fixing $\Gamma_2\cup\Gamma_3$ pointwise fixes the same seven points by (vi) of (3.1). Next suppose that $\Delta\cup\Gamma_2\cup\Gamma_3$ is a G-orbit. Since $|\Delta\cup\Gamma_2\cup\Gamma_3|=15$, G has an element of order five. Hence G_1 has an element u of order five. On the other hand we have already proved that the G_1-orbit containing 2 is $\{2\}$ or $\{2\}\cup\Delta$. Since $\{2\}$ and Δ are contained in the different G-orbit, $\{2\}$ and Δ are the G_1-orbit. Hence u fixes 1, 2 and four points of Δ, contrary to (3.1). In the same way it is impossible that $\{1, 2\}\cup\Gamma_2\cup\Gamma_3$ is a G-orbit.

Next since $|\{1, 2\}\cup\Delta\cup\Gamma_1\cup\Gamma_4|=17$ and M_{23} has no element of order seventeen, $\{1, 2\}\cup\Delta\cup\Gamma_1\cup\Gamma_4$ is not a G-orbit. In the same way $\{1, 2\}\cup\Delta\cup\Gamma_2\cup\Gamma_3$ is not a G-orbit.

Finally since $|\{1, 2\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|=14$, $\{1, 2\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ is not a G-orbit by (3.11).

Thus if G contains $x=(1\ 2)(3)(8)(9)\cdots$, then G is isomorphic to M_{11}, $N_{M_{11}}(M_9)$ or $N_{M_{11}}(M_9)^*$.

(iii) Assume that $x=(1)(2\ 3)(8)(9)\cdots$. In the case $x=(2)(1\ 3)(8)(9)\cdots$ the proof is similar. By the incidence structure of $P(a)$, $x=(1)(2\ 3)(4)(5)(6\ 7)(8)(9)\cdots$ or $(1)(2\ 3)(4\ 5)(6)(7)(8)(9)\cdots$. Suppose that x is of the first form. Since $\Delta(1, 4)=\Gamma_2$ or $\Gamma_2\cup\{18, 19\}$ and $\Delta(1, 4)^x=\Delta(1, 4)$, $\{16, 17\}^x=\{16, 17\}$ or $\{18, 19\}$. If $\{16, 17\}^x=\{16, 17\}$, then x or ax fixes $\{1, 4, 5, 16, 17\}$ pointwise. This is a contradiction since $\Delta(1, 4)\supseteq\{5, 16, 17\}$. Hence $\{16, 17\}^x=\{18, 19\}$, and so $x=(16\ 18)(17\ 19)\cdots$ or $(16\ 19)(17\ 18)\cdots$. Then $bx=(1)(2\ 3\ 9)(8)(4)(5\ 19\ 16\ 18\ 17)\cdots$ or $(1)(2\ 3\ 9)(8)(4)(5\ 18\ 16\ 19\ 17)\cdots$. Hence $|I((bx)^5)|\geq 8$, contrary to (ii) of (3.1). Thus x is of the second form. Since $\Delta(1, 6)=\Gamma_4$ or $\Gamma_4\cup\{20, 21\}$ and $\Delta(1, 6)^x=\Delta(1, 6)$, $\{22, 23\}^x=\{22, 23\}$ or $\{20, 21\}$. If $\{22, 23\}^x=\{22, 23\}$, then x or ax fixes $\{1, 6, 7, 22, 23\}$ pointwise. This is a contradiction since $\Delta(1, 6)\supseteq\{7, 22, 23\}$. If $x=(20\ 23)(21\ 22)\cdots$, then $bx=(1)(2\ 3\ 9)(8)(20\ 23)\cdots$, contrary to (v) of (3.1), Thus $x=(20\ 22)(21\ 23)\cdots$. Furthermore $\Delta(1, 5)=\Gamma_1$ or $\Gamma_1\cup\{16, 17\}$ and $\Delta(1, 4)^x=\Delta(1, 5)$. Hence similarly $x=(16)(17)(18\ 19)\cdots$. Thus

$$x=(1)\,(2\;3)\,(8)\,(9)\,(4\;5)\,(6)\,(7)\,(16)\,(17)\,(18\;19)\,(20\;22)\,(21\;23)\cdots.$$

Hence $x=y_1$. Set $H=\langle a,b,c,y_1\rangle$. Then the H-orbits are $\{1\}$, $\{2\}\cup\Delta$, $\Gamma_1\cup\Gamma_2$ and $\Gamma_3\cup\Gamma_4$. Since $H\leq\langle a,b,c,x_1,y_1\rangle_1$ and $\langle a,b,c,x_1,y_1\rangle\simeq M_{11}$, H is isomorphic to a subgroup of M_{10}. Furthermore since M_{10} has orbits of lengths 1, 10, 12 as a subgroup of M_{23} (see [1], p. 235), H is isomorphic to a proper subgroup of M_{10}. On the other hand H is doubly transitive on $\{2\}\cup\Delta$ and has an element $cy_1=(1)\,(13)\,(15)\,(2\;3\;10\;11)\,(8\;14\;9\;12)\cdots$, the order of H is a multiple of $10\cdot9\cdot4$. Thus H is isomorphic to a subgroup of M_{10} of index two, which is denoted by M_{10}^*.

Suppose that H is a proper subgroup of G. If $\{2\}\cup\Delta$ is a G-orbit, then G is isomoprhic to M_{10} by [5]. In this case $G=\langle a,b,c,y_1,x_1y_2\rangle$ and the G-orbits are $\{1\}$, $\{2\}\cup\Delta$ and $\Gamma_1\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$. Next assume that G has an orbit containing $\{2\}\cup\Delta$ properly. If $\{1\}\cup\{2\}\cup\Delta$ is a G-orbit, then G must be isomorphic to M_{11} by (3.15). If $\{2\}\cup\Delta\cup\Gamma_1\cup\Gamma_2$ is a G-orbit, then $|\{2\}\cup\Delta\cup\Gamma_1\cup\Gamma_2|=16$. Since G_2 is of even order, the order of G is divisible by $16\cdot2$. On the other hand $\{1\}\cup\Gamma_3\cup\Gamma_4$ is fixed by G as a set and $G^{\{1\}\cup\Gamma_3\cup\Gamma_4}\leq S_7$. Thus G has an involution u fixing $\{1\}\cup\Gamma_3\cup\Gamma_4$ pointwise. Hence $bc^{I(u)}=(1)\,(6)\,(20)\,(21)\,(7\;22\;23)$, contrary to (3.4). In the same way it is impossible that $\{2\}\cup\Delta\cup\Gamma_3\cup\Gamma_4$ is a G-orbit. If $\{1\}\cup\{2\}\cup\Delta\cup\Gamma_1\cup\Gamma_2$ is a G-orbit, Then $|\{1\}\cup\{2\}\cup\Delta\cup\Gamma_1\cup\Gamma_2|=17$, contrary to the assumption (4). In the sam eway it is impossible that $\{1\}\cup\{2\}\cup\Delta\cup\Gamma_3\cup\Gamma_4$ is a G-orbit. Finally by (3.14), Ω-$\{1\}$ is not a G-orbit.

Thus if G contains $x=(1)\,(2\;3)\,(8)\,(9)\cdots$ or $(2)\,(1\;3)\,(8)\,(9)\cdots$, then G is isomorphic to M_{10}^*, M_{10} or M_{11}.

When G is isomorphic to $N(M_9)$ or $N(M_9)^*$, we may assume that G is isomorphic to $\langle a,b,c,x_1,(x_1y_2)^{x_1}\rangle^{y_1x_1}=\langle a,b^{y_1x_1},c^{y_1x_1},y_1,x_1y_2\rangle$ or $\langle a,b,c,x_1,y_2\rangle^{y_1x_1}=\langle a,b^{y_1x_1},c^{y_1x_1},y_1,y_2^{y_1x_1}\rangle$ respectively, and these generators were used in the proof of Lemma 3.

3.17. *Assume that* $|\Delta(i,j)|\leq5$ *for any two points* i, j *of* Ω. *If there is two points* i', j' *of* Ω *such that* $|\Delta(i',j')|=5$, *then G is* (*vi*) *or* (*vii*) *of Lemma* 1, *and we have the following*:

(vi) *If G is isomorphic to* $PSL(2,11)$, *then G is isomorphic to* $\langle a,x_1,x_2\rangle$ *as a permutation group.*

(v) *If G is isomorphic to* S_5, *then G is isomorphic to* $\langle a,x_1,x_3\rangle$ *as a permutation group.*

Here

$$a=(1)\,(2)\,(3)\,(8\;9)\,(10\;11)\,(12\;13)\,(14\;15)\,(4)\,(5)\,(6)\,(7)\,(16\;17)\,(18\;19)\,(20\;21)\,(22\;23).$$

$$x_1=(1)\,(2\;3)\,(8)\,(9)\,(10\;11)\,(12\;14)\,(13\;15)\,(4\;5)\,(6)\,(7)\,(16)\,(17)\,(18\;19)\,(20\;22)\,(21\;23),$$

$x_2=$ (2) (8) (10) (1 12) (9 13) (3 15) (14 11) (4) (7) (17) (18) (5 23) (6 21) (16 22) (19 20),

$x_3=$ (3) (13) (10) (2 14) (1 8) (9 11) (12 15) (4) (5) (17) (22) (6 18) (7 20) (16 21) (19 23).

Proof. Let $|\Delta(1, 2)|=5$ and $\Delta(1, 2)=\{3, 8, 9, 10, 11\}$. Then we may assume that there is an involution a of the form

$$a = (1)\ (2) \cdots (7)\ (8\ 9)\ (10\ 11)\ (12\ 13)\ (14\ 15)\ (16\ 17)\ (18\ 19)\ (20\ 21)\ (22\ 23).$$

Let b be an involution fixing 1, 2, 8. By (3. 7), ab is of order three or five. If the order of ab is three, then the order of $(G_{1\,2})^{\Delta(1,2)}$ is a multiple of $5\cdot3\cdot2$. Hence $(G_{1\,2})^{\Delta(1,2)}$ contains A_5 by [5]. Thus G has an element b' such that b' is conjugate to b and ab' is of order five. Therefore we may assume that the order of ab is five and b is of the form

$$b = (1)\ (2)\ (4)\ (8)\ (3\ 10)\ (9\ 11)\ (12)\ (5\ 14)\ (13\ 15)\ (16)\ (6\ 18)\ (17\ 19)\ (20)\ (7\ 22)\ (21\ 23).$$

Hence

$$ab = (1)\ (2)\ (4)\ (3\ 10\ 9\ 8\ 11)\ (5\ 14\ 13\ 12\ 15)\ (6\ 18\ 17\ 16\ 19)\ (7\ 22\ 21\ 20\ 23).$$

Set $\Gamma_1=\{3, 8, 9, 10, 11\}$, $\Gamma_2=\{5, 12, 13, 14, 15\}$, $\Gamma_3=\{6, 16, 17, 18, 19\}$, and $\Gamma_4=\{7, 20, 21, 22, 23\}$. Similarly to (3.16) we may assume that $\Delta(1, 4)\ni 5$, $\Delta(1, 6)\ni 7$ and $\Delta(2, 4)\ni 6$. Then since $\{1, 2\}^{ab}=\{1, 2\}$, $\Delta(1, 2)\ni 3$ and $|\Delta(1,2)|\leq 5$, $\Delta(1, 2)=\Gamma_1$. Similarly $\Delta(1, 4)=\Gamma_2$ and $\Delta(2, 4)=\Gamma_3$. Since $G_{1\,2\,4}$ contains $\langle a, b\rangle$ and $\Delta(1, 2)$, $\Delta(1, 4)$ and $\Delta(2, 4)$ are orbits of $G_{1\,2}$, $G_{1\,4}$ and $G_{2\,4}$ respectively by (3.8), the $G_{1\,2\,4}$-orbits are $\{1\}$, $\{2\}$, $\{4\}$, Γ_1, Γ_2, Γ_3 and Γ_4.

Now we have one of the following cases and treat these cases separately.

(i) 1, 2, 4 belong to the same G-orbit.
(ii) Exactly two points of $\{1, 2, 4\}$ belong to the same G-orbit.
(iii) 1, 2, 4 belong to distinct G-orbits.

(i) Suppose that 1, 2, 4, belong to the same G-orbit. Then the G-orbit containing 1, 2, 4 is $\{1, 2, 4\}$, $\{1, 2, 4\}\cup\Gamma_i$, $\{1, 2, 4\}\cup\Gamma_i\cup\Gamma_j$ or $\{1, 2, 4\}\cup\Gamma_i\cup\Gamma_j\cup\Gamma_k$, where $i, j, k\in\{1, 2, 3, 4\}$.

First assume that $\{1, 2, 4\}$ is a G-orbit. Then G fixes and is transitive on $\Delta(1, 2)\cup\Delta(1, 4)\cup\Delta(2, 4)$. Thus the G-orbits are $\{1, 2, 4\}$, $\Gamma_1\cup\Gamma_2\cup\Gamma_3$ and Γ_4. By (3.5), there is an involution x fixing 20, 21 and commuting with a. If $x=$ (1) (2) (4)$\cdots$, then x fixes $I(a)$ pointwise by the incidence structure of $P(a)$. Thus $I(x)\supseteq I(a)\cup\{20, 21\}$, which is a contradiction. Hence we may assume that $x=$ (1) (2 4)$\cdots$. By the incidence structure of $P(a)$, $x=$(1) (2 4) (3 5) (6) (7) (20) (21)$\cdots$. Since Γ_4 is a G-orbit, $\{22, 23\}^x=\{22, 23\}$. If x fixes $\{22, 23\}$ pointwise, then since G is transitive on $\{1, 2, 4\}$ for any point i of $\{1, 2, 4\}$ some involution conjugate to x fixes $\Gamma_4 \cup \{i\}$ pointwise, contrary to (vi) of (3.1). Thus x has a 2-cycle (22 23). Let y be an involution fixing 7, 22 and commuting with b. In the same way $y=(7)\,(22)\,(20)\,(21\ 23)\cdots$. Thus $xy=(7)\,(20)\,(21\ 23\ 22)\cdots$ and so xy is of order three. Hence xy fixes $\{1, 2, 4\}$ pointwise or has a 3-cycle on $\{1, 2, 4\}$. If $xy=(1)\,(2)\,(4)\,(7)\,(20)\,(21\ 23\ 22)\cdots$, then $|I(xy) \cap I(a)| \geq 4$. Thus xy fixes noncollinear four points of $P(a)$ and hence it fixes $I(a)$ pointwise. This contradicts (ii) of (3.1). Hence xy has a 3-cycle on $\{1, 2, 4\}$. Then xy permutes $\Delta(1, 2)$ $\Delta(1, 4)$ and $\Delta(2, 4)$ cyclically. Since $\Delta(1, 2)=\Gamma_1$, $\Delta(1, 4)=\Gamma_2$ and $\Delta(2, 4)=\Gamma_3$, xy has no fixed point in $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Thus $|I(xy)|=2$, contrary to (ii) of (3.1).

Next assume that $\{1, 2, 4\} \cup \Gamma_i$ is a G-orbit, where $i \in \{1, 2, 3, 4\}$. Set $\Gamma= \{1,2,4\} \cup \Gamma_i$. Since $|\{2, 4\} \cup \Gamma_i|=7$, the G_1-orbit containing 2 is $\{2\}$, $\{2, 4\}$ or $\{2\} \cup \Gamma_i$ by (3.11). If $\{2, 4\}$ is a G_1-orbit, then $(G_1)^{\Gamma-\{1\}}$ has orbits of lengths two and five. Hence G^Γ is primitive. Since a primitive group of degree eight is doubly transitive by [5], G^Γ is doubly transitive. This is a contradiction. Next suppose that $\{2\}$ is a G_1-orbit. Then $(G^\Gamma)_1=(G^\Gamma)_2$. If $\{4\}$ and Γ_i are G_1-orbits, then $(G^\Gamma)_1=(G^\Gamma)_2=(G^\Gamma)_4$. Hence $|\Gamma|$ is divisible by 3, which is a contradiction. If $\{4\} \cup \Gamma_i$ is a G_1-orbit, then $(G^\Gamma)_1 \neq (G^\Gamma)_4$. Since G_4 is conjugate to G_1, $(G^\Gamma)_4$ has exactly one orbit $\{t\}$ of length one, where $t \in \Gamma - \{1, 2, 4\}$. On the other hand G_4 contains $\langle ab \rangle$ which fixes exactly three points 1, 2, 4 of Γ. Hence $t=1$ or 2, which is a contradiction. If $\{2\} \cup \Gamma_i$ is a G_1-orbit, then $\{4\}$ is a G_1-orbit. Hence in the same way we have a contradiction.

Next assume that $\{1, 2, 4\} \subset \Gamma_i \cup \Gamma_j$ is a G-orbit, where $i, j \in \{1, 2, 3, 4\}$. Then $|\{1, 2, 4\} \cup \Gamma_i \cup \Gamma_j|=13$ and so G has an element of order thirteen, contrary to the assumption (4).

Finally assume that the G-orbits are $\{1, 2, 4\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ and Γ_l, where $\{i, j, k, l\}=\{1,2,3,4\}$. Then G has orbits of lengths five and eighteen. This is a contradiction since M_{23} has no subgroup having orbits of lengths five and eighteen (see [1], p. 235).

Thus it is impossible that 1, 2, 4 belong to the same G-orbit.

(ii) Suppose that exactly two points of $\{1, 2, 4\}$ belong to the same G-orbit. We may assume that 1, 2 belong to the same G-orbit. Then the G-orbit containing $\{1, 2\}$ is $\{1, 2\}$, $\{1,2\} \cup \Gamma_i$, $\{1,2\} \cup \Gamma_i \cup \Gamma_j$, $\{1,2\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ or $\{1,2\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k \cup \Gamma_l$, where $\{i, j, k, l\}=\{1, 2, 3, 4\}$. By (3.11), $\{1, 2\} \cup \Gamma_i$ is not a G-orbit. Since M_{23} has no element of order seventeen, $\{1, 2\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k$ is not a G-orbit. Furthermore by (3.14), $\{1,2\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k \cup \Gamma_l$ is not a G-orbit. Thus the G-orbit containing $\{1, 2\}$ is $\{1, 2\}$ or $\{1,2\} \cup \Gamma_i \cup \Gamma_j$. We treat these cases separately.

(ii.i) Assume that $\{1, 2\}$ is a G-orbit. Then $\Delta(1, 2)=\Gamma_1$ is a G-orbit. By (3.5), there is an involution x fixing 8, 9 and commuting with a. By the assumption (7), $x \neq (1)\,(2)\,(8)\,(9)\cdots$. Thus x has a 2-cycle (1 2) and so x fixes a point 3 by the incidence structure of $P(a)$. Thus $x=(1\ 2)\ (3)\ (8)\ (9)\ (10)\ (11)\cdots$ or $(1\ 2)\ (3)\ (8)\ (9)\ (10\ 11)\cdots$. If x is of the first form, then $I(x)\supset\Gamma_1$. Now $I(x^{ab})=I(x)^{ab}\supset\Gamma_1{}^{ab}=\Gamma_1$ and hence by (vi) of (3.1) $I(x)^{ab}=I(x)$. This contradicts (3.4) since $PSL(3, 2)$ has no element of order five. Hence x is of the second form. By the incidence structure of $P(a)$, $x=(1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8)\ (9)\ (10\ 11)\cdots$ or $(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\ (10\ 11)\cdots$.

(ii.i.i) Let $x=(1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8)\ (9)\ (10\ 11)\cdots$. Since $\Delta(1, 4)^x$ $\Delta(2,4)$, $\Gamma_2{}^x=\Gamma_3$ and so $\Gamma_4{}^x=\Gamma_4$. Since $|I(x)|=7$, $|I(x)\cap\Gamma_4|=3$. Hence if $x=(20\ 21)\ (22)\ (23)\cdots$, then $bx=(1\ 2)\ (4)\ (8)\ (20\ 21\ 23)\cdots$, which is a contradiction by (v) of (3.1). Next if $x=(20)\ (21)\ (22\ 23)\cdots$, then $a^bx=(1\ 2)\ (4)\ (3\ 9)\ (8\ 10\ 11)$ $(7\ 21)\cdots$, which is also a contradiction. Thus the case (ii.i.i) does not occur.

(ii.i.ii) Let $x=(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\ (10\ 11)\cdots$. Set $H=\langle a, b, x\rangle$. Then $\{1, 2\}$ and Γ_1 are H-orbits and the remaining H-orbits consist of unions of Γ_2, Γ_3 and $\{4\}\cup\Gamma_4$. Suppose that Γ_2, Γ_3 and $\{4\}\cup\Gamma_4$ are H-orbits. Since $|\{4\}\cup\Gamma_4|=6$, H has an element u of order three. Then u has at least two fixed points in each orbits of length five. Thus $|I(u)|\geq 6$, contrary to (ii) of (3.1). Next suppose that $\{4\}\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4$ is an H-orbit. Then $|\{4\}\cup\Gamma_2\cup\Gamma_3\cup\Gamma_4|$ $=16$. Since H_4 has an involution a, H is of order divisible by $16\cdot 2$. On the other hand since $|\{1, 2\}\cup\Gamma_1|=7$, $|H:H_{\{1,2\}\cup\Gamma_1}|$ is a divisor of $|S_7|=7\cdot 5\cdot 9\cdot 16$. Hence $H_{\{1,2\}\cup\Gamma_1}$ has an involution, which fixes $\{1,2\}$ and $\Delta(1,2)=\Gamma_1$ pointwise, contrary to the assumption (7). Next suppose that Γ_i, $\Gamma_j\cup\{4\}\cup\Gamma_4$ are H-orbits, where $\{i, j\}=\{2, 3\}$. Then $|\Gamma_j\cup\{4\}\cup\Gamma_4|=11$. Hence H has an element of order eleven. Since an element of order eleven has two 11-cycles and one fixed point, it is impossible that $\{1, 2\}$ is an H-orbit of length two. Thus we have a contradiction. Therefore the H-orbits must be $\{1, 2\}$, Γ_1, $\Gamma_2\cup\Gamma_3$ and $\{4\}\cup\Gamma_4$. Using this result we shall determine the form of x. Since

$$bx=(1\ 2)\ (8)\ (3\ 11\ 9\ 10)\ (22\ 4\ 7\cdots)\cdots,$$

bx is of order four. Moreover $\{4\}\cup\Gamma_4$ is an H-orbit. Hence

$$x=(1\ 2)\ (3)\ (4\ 7)\ (5)\ (6)\ (8)\ (9)\ (10\ 11)\ (20\ 22)\ (21\ 23)\cdots$$

and so

$$bx=(1\ 2)\ (8)\ (3\ 11\ 9\ 10)\ (22\ 4\ 7\ 20)\ (21)\ (23)\cdots.$$

Hence bx has exactly two 2-cycles $(1\ 2)$ and $(i\ j)$ by (iii) of (3.1), where $i, j\in\Gamma_2\cup\Gamma_3$. Thus $I((bx)^2)=\{1, 2, 8, 21, 23, i, j\}$. On the other hand $(bx)^2$ is a central involution of a dihedral group $\langle b, x\rangle$. Hence $b^{I((bx)^2)}=(1)\ (2)\ (8)\ (21\ 23)\ (i\ j)$ and $x^{I((bx)^2)}=$ $(1\ 2)\ (8)\ (21\ 23)\ (i)\ (j)$. Thus $\{i, j\}=\{5, 14,\}$ $\{13, 15\}$, $\{6, 18\}$ or $\{17, 19\}$.

Suppose that $\{i, j\}=\{13, 15\}$ or $\{17, 19\}$. Then $I(x)\supset\{13, 15\}$ or $\{17, 19\}$. Since x commutes with a and $a^{\Gamma_2\cup\Gamma_3}=(5)\ (12\ 13)\ (14\ 15)\ (6)\ (16\ 17)\ (18\ 19)$, x fixes $\{3, 8, 9\}\cup\Gamma_2$ or $\{3, 8, 9\}\cup\Gamma_3$ pointwise. Thus $|I(x)|\geq 8$, contrary to (i) of (3.1). Hence $\{i,j\}=\{5, 14\}$ or $\{6, 18\}$ and so x fixes $\{5, 14, 15\}$ or $\{6, 18, 19\}$

pointwise because x commutes with a.

Suppose that x fixes 5, 14, 15. Since $\langle b, x\rangle$ is a dihedral group of order eight and not semiregular on $\Gamma_2 \cup \Gamma_3 - \{5, 14\}$ of length eight, $\langle b, x\rangle$ is intransitive on $\Gamma_2 \cup \Gamma_3 - \{5, 14\}$. Furthermore bx has two 4-cycles on $\Gamma_2 \cup \Gamma_3 - \{5, 14\}$. Hence $\langle b, x\rangle$ has two orbits of length four on $\Gamma_2 \cup \Gamma_3 - \{5, 14\}$. On the other hand x has exactly two fixed points 15, 6 and three 2-cycles on $\Gamma_2 \cup \Gamma_3 - \{5, 14\}$. Hence 15, 6 are contained in the same $\langle b, x\rangle$-orbit. Thus $\{15, 6, 13, 18\}$ and $\{12, 16, 17, 19\}$ are $\langle b, x\rangle$-orbits on $\Gamma_2 \cup \Gamma_3 - \{5, 14\}$. Hence $x=(15)(6)(13\ 18)\cdots$ and so $x=(15)(6)(13\ 18)(12\ 19)(16\ 17)\cdots$ because x commutes with a. Thus

$$x = (1\ 2)(3)(8)(9)(10\ 11)(4\ 7)(5)(6)(20\ 22)(21\ 23)(14)(15)(12\ 19)(16\ 17)(13\ 18)\,.$$

From now on the element x of this form is denoted by x_1.

Similarly if x fixes 6, 18, 19, then

$$x = (1\ 2)(3)(8)(9)(10\ 11)(4\ 7)(5)(6)(20\ 22)(21\ 23)(18)(19)(14\ 17)(15\ 16)(12\ 13)\,.$$

From now on the element x of this form is denoted by x_1'. Then both $\langle a, b, x_1\rangle$ and $\langle a, b, x_1'\rangle$ are isomorphic to S_5.

Set $H=\langle a, b, x_1\rangle$ and suppose that H is a proper subgroup of G. The proof in the case $H=\langle a, b, x_1'\rangle$ is similar. Since $H^{\Gamma_1}=S_5$ and Γ_1 is a G-orbit, $H^{\Gamma_1}=G^{\Gamma_1}$. Moreover $H_{\Gamma_1}=1$ and G contains H properly. Hence $G_{\Gamma_1}\neq 1$ and so G has a nonidentity element u fixing Γ_1 pointwise. Then $|I(u)|\geq 5$. Hence u is of order three or two. If u is of order three, then u fixes the G-orbit $\{1, 2\}$ pointwise. Thus $|I(u)|\geq 7$, contrary to (ii) of (3.1). Thus u is of order two. Then since ab fixes Γ_1, ab fixes $I(u)$. Thus $(ab)^{I(u)}$ is of order five, contrary to (3.4). Thus there is no group which contains H properly and satisfies the assumption of Lemma 1.

(ii.ii) Assume that $\{1,2\} \cup \Gamma_i \cup \Gamma_j$ is a G-orbit, where $i, j \in \{1, 2, 3, 4\}$. Then the remaining G-orbits are unions of $\{4\}$, Γ_k and Γ_l, where $k, l \in \{1, 2, 3, 4\} - \{i, j\}$. Hence the G-orbit containing $\{4\}$ is of length one, six or eleven. Since $|\{1, 2\} \cup \Gamma_i \cup \Gamma_j|=12$, if G has an orbit of length six or eleven, then G is a subgroup of a group which is isomorphic to M_{11} (see [1], p. 235) and if G has an orbit of length one, then G is a subgroup of M_{22}. However it is impossible that G has orbits of lengths six, five and twelve and isomorphic to a subgroup of M_{11} (see [1], p. 235). Next suppose that G has an orbit of length eleven. Then by (3.15), G is isomorphic to M_{11}. Let i_1, i_2 be two points of the G-orbit of length eleven. Then $G_{i_1 i_2}$ has orbits of lengths nine and twelve on $\Omega - \{i_1, i_2\}$. By (3.8), $|\Delta(i_1, i_2)|\geq 9$, contrary to the assumption. Hence G is a proper subgroup of M_{22}. Thus the lengths of the G-orbits are 1, 12, 5, 5 or 1, 12, 10. Hence G is a subgroup of M_{10} (see [1], p. 235) and has orbits $\{4\}$, $\{1, 2\} \cup \Gamma_i \cup \Gamma_j$, Γ_k, Γ_l or $\{4\}$, $\{1, 2\} \cup \Gamma_i \cup \Gamma_j$, $\Gamma_k \cup \Gamma_l$. Since $|\{1, 2\} \cup \Gamma_i \cup \Gamma_j|=12$, G has an element of order three. On the other hand since $G^{\Gamma_k \cup \Gamma_l}$ is a subgroup of M_{10} of degree ten, an element of order three fixes exactly one point of $\Gamma_k \cup \Gamma_l$. Hence it is impossible that G has two orbits Γ_k and Γ_l of length five. Thus $\Gamma_k \cup \Gamma_l$ is a G-orbit. Since G fixes a point 4 and two points 1, 2 belong to the same G-orbit, $\Delta(4, 1)=\Gamma_2$ and $\Delta(4, 2)=\Gamma_3$ are contained in the same G-orbit. Thus the G-orbits are $\{4\}$, $\{1, 2\} \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_1 \cup \Gamma_4$ or $\{4\}$, $\{1, 2\} \cup \Gamma_1 \cup \Gamma_4$ and $\Gamma_2 \cup \Gamma_3$. First assume that $\{4\}$, $\{1, 2\} \cup \Gamma_2 \cup \Gamma_3$ and $\Gamma_1 \cup \Gamma_4$ are G-orbits. Since $|\{1, 2\} \cup \Gamma_2 \cup \Gamma_3|=12$ and $G^{\Gamma_1 \cup \Gamma_4}$ is a subgroup of M_{10} of degree ten, G_{43} has an element of order three which has no fixed point on $\Gamma_1 \cup \Gamma_4-\{3\}$. Hence the lengths of the G_{43}-orbits in $\Gamma_1 \cup \Gamma_4-\{3\}$ are multiples of three. Thus $|\Delta(4, 3)|=3$ because $\Delta(4, 3)\ni 7$ and $|\Delta(4, 3)|\leq 5$. On the other hand $|\{1,2\} \cup \Gamma_2 \cup \Gamma_3|=12$ and G_1 is of even order. Hence G is of order divisible by $4\cdot 2$ Then since $|G_4: G_{43}|=10$ and $\Delta(4, 3)$ is a G_{43}-orbit of length three, G has an involution fixing $\{4, 3\} \cup \Delta(4, 3)$ pointwise, contrary to the assumption (7). Next assume that $\{4\}$, $\{1, 2\} \cup \Gamma_1 \cup \Gamma_4$ and $\Gamma_2 \cup \Gamma_3$ are G-orbits. Then for two points 1 and 3 of the G-orbit $\{1,2\} \cup \Gamma_1 \cup \Gamma_4$ $\Delta(4, 1)$ and $\Delta(4, 3)$ are contained in the same G-orbit. However this is a contradiction since $\Delta(4, 1)=\Gamma_2$ and $\Delta(4, 3)$ contains a point 7 of Γ_4.

(iii) Suppose that 1, 2, 4 belong to distinct G-orbits. Furst assume that G has no orbit of length one. Then the lengths of the G-orbits are 6, 6, 6, 5 or 6, 6, 11. By (3.15), the latter case dose not occur. Hence G has orbits of lengths 6, 6, 6, 5. But there is no maximal subgroup of M_{23} containing G (see [1], p. 235). Thus G has an orbit of length one.

Next assume that G has at least two orbits of length one. Then we may assume that $\{1\}$ and $\{2\}$ are G-orbits. By (3.5), there is an involution x fixing 8, 9 and commuting with a. Then x fixes 1, 2 and two points 8, 9 of $\Delta(1, 2)$, contrary to the assumption (7).

Thus G has exactly one orbit of length one. Hence we may asssume that $\{1\}$ is a G-orbit. Then the remaining G-orbits are unions of $\{2\} \cup \Gamma_i$, $\{4\} \cup \Gamma_j$, Γ_k and Γ_l, where $\{2\} \cup \Gamma_i$ and $\{4\} \cup \Gamma_j$ are not contained in the same G-orbit and $\{i, j, k, l\}=\{1, 2, 3, 4\}$. Thus the length of the G-orbit containing $\{2\}$ or $\{4\}$ is 16, 11 or 6. We treat these cases separately.

(iii.i) Suppose that there is a G-orbit of length sixteen. Then the G-orbits are $\{1\}$, $\{t_1\} \cup \Gamma_i \cup \Gamma_j$ and $\{t_2\} \cup \Gamma_l$, where $\{t_1, t_2\}=\{2, 4\}$ and $\{i, j, k, l\}=\{1,2,3,4,\}$. Since $|\{t_1\} \cup \Gamma_i \cup \Gamma_j \cup \Gamma_k|=16$ and the order of G_{t_1} is even, G is of order divisible by $16\cdot 2$. On the other hand since $|\{t_2\} \cup \Gamma_l|=6$, a Sylow 2-subgroup of $G^{\{t_2\} \cup \Gamma_l}$ is of order at most sixteen. Hence $G_{(\{t_2\} \cup \Gamma_l)}$ is of even order. Thus there is an involution x fixing $\{t_2\} \cup \Gamma_l$ pointwise. Since $|I(x)|=7$, $I(x)=\{1, t_1\} \cup \Gamma_l$. Then ab fixes $I(x)$ and $(ab)^{I(x)}$ is of order five, contrary to (3.4). Thus G has no orbit of length sixteen.

(iii.ii) Suppose that there is a G-orbit of length eleven. Since $\{1\}$ is a G-orbit of length one, G is isomorphic to $PSL(2, 11)$ and the G-orbits are $\{1\}$,

$\{2\} \cup \Gamma_i \cup \Gamma_j$ and $\{4\} \cup \Gamma_k \cup \Gamma_l$, where $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Now we determine the generators of G. Assume that $\{2\}$ and $\Gamma_1 = \Delta(1, 2)$ are contained in the same G-orbit. Then G has an orbit $\{2\} \cup \Gamma_1 \cup \Gamma_i$ where $i \in \{2, 3, 4\}$. Let $\{t\} = \Gamma_i \cap I(a)$. Then $\Delta(1, 2) = \Gamma_1$ and $\Delta(1, t)$ are contained in the same G-orbit. On the other hand by the incidence structure of $P(a)$, $I(a) \cap \Delta(1, t)$ is different from $\{2\}$, $\{3\}$ and $\{t\}$. Thus $\{\{2\} \cup \Gamma_1 \cup \Gamma_i\} \cap I(a) \supsetneqq \{2\ 3, t\}$, which is a contradiction. Similarly $\{4\}$ and $\Gamma_2 = \Delta(1, 2)$ are not contained in the same G-orbit. Thus the G-orbits are $\{1\}$, $\{2\} \cup \Gamma_2 \cup \Gamma_i$ and $\{4\} \cup \Gamma_1 \cup \Gamma_j$, where $\{i, j\} = \{3, 4\}$. Now we assume that the G-orbits are $\{1\}$, $\{2\} \cup \Gamma_2 \cup \Gamma_3$, and $\{4\} \cup \Gamma_1 \cup \Gamma_4$. When the G-orbits are $\{1\}$, $\{2\} \cup \Gamma_2 \cup \Gamma_4$ and $\{4\} \cup \Gamma_2 \cup \Gamma_3$, the generators are deternimed in the similar way and this group is isomorphic as a permutation group whose orbits are $\{1\}$, $\{2\} \cup \Gamma_2 \cup \Gamma_3$ and $\{4\} \cup \Gamma_1 \cup \Gamma_4$. Let x be an involution fixing 8, 9 and commuting with a. Since $\Delta(1, 2) \ni 8, 9$ and $I(x) \ni 1$, x does not fix 2. Furthermore since $\{2\} \cup \Gamma_2 \cup \Gamma_3$ is a G-orbit, $x = (1)(2\ 5)(6)\cdots$ or $(1)(2\ 6)(5)\cdots$. Now we assume that $x = (1)(2\ 5)(6)(8)(9)\cdots$. Then by the incidence structure of $P(a)$, $x = (1)(2\ 5)(3\ 4)(6)(7)(8)(9)\cdots$. Hence

$$bx = (1)(8)(14\ 2\ 5\cdots)(10\ 4\ 3\cdots)\cdots .$$

Since b and x fix 1, 8, $\Delta(1, 8) \cap I(b) = \{2\}$ and $\Delta(1, 8) \cap I(x) \neq \{2\}$, the order of bx is odd by (3.7). Furthermore since $\Delta(1, 8)$ is a $G_{1\,8}$-orbit and $|\Delta(1, 8)| \leq 5$, the order of bx is three or five. If bx is of order three, then x fixes 14, 10 and so fixes 15, 11 because $ax = xa$. Thus $|I(x)| > 7$, contrary to (i) of (3.1). Therefore bx is of order five and so $|I(bx)| = 3$. Hence b and x fix exactly one more point other than 1, 8 in common. Therefore x fixes exactly one point of $\{12, 16, 20\}$. Assume that x fixes 16. Then x fixes 17 because $ax = xa$. Thus $I(x) = \{1, 6, 7, 8, 9, 16, 17\}$. Since $\Delta(1, 8) \ni 2$ and $\{2\} \cup \Gamma_2 \cup \Gamma_3$ is a G-orbit, $\Delta(1, 8) \subseteq \{2\} \cup \Gamma_2 \cup \Gamma_3$ by (3.8). Since three points 1, 6, 7 are collinear, $\Delta(1, 8) \not\ni 6$. Since three points 1, 2, 8 are collinear in $P(b)$, $\Delta(1, 8) \not\ni 16$. Thus $\Delta(1, 8) \ni 17$ in $P(x)$. Furthermore since $\Delta(1, 8)^{\langle b, x\rangle} = \Delta(1, 8)$ and $|\Delta(1, 8)| \leq 5$, $\Delta(1, 8) = \{2, 5, 14, 17, 19\}$. Therefore

$$x = (1)(2\ 5)(3\ 4)(6)(7)(8)(9)(16)(17)(14\ 19)(15\ 18)\cdots .$$

Hence

$$ax = (1)(2\ 5)(3\ 4)(6)(7)(8\ 9)(16\ 17)(14\ 18)(15\ 19)\cdots$$

and

$$bax = (1)(2\ 5\ 18\ 6\ 14)(13\ 19\ 16\ 17\ 15\cdots)\cdots .$$

This shows that bax is of order five. Hence ax fixes 13 and so fixes 12. Furthermore since $|I(bax)| = 3$, b and ax fix exactly one more point other than 1, 12 in common. Hence ax fixes 20 and so fixes 22. Thus

$$x = (1)(25)(3\ 4)(6)(7)(8)(9)(16)(17)(14\ 19)(15\ 18)(12\ 13)(20\ 21)\cdots,$$

Therefore $x=(10\ 23)\ (11\ 22)$ or $(10\ 22)\ (11\ 23)$ on $\{10, 11, 22, 23\}$. If x is of the first form, then $bx=(3\ 23\ 20\ 21\ 10\ 4)\cdots$. This is a contradiction since bx is of order five. Thus

$$x = (1)\ (2\ 5)\ (3\ 4)\ (6)\ (7)\ (8)\ (9)\ (16)\ (17)\ (14\ 19)\ (15\ 18)\ (12\ 13)\ (20\ 21)\ (10\ 22)\ (11\ 23).$$

The element x of this form is denoted by x_2. Then $\langle a, b, x_2\rangle$ is isomorphic to $PSL(2, 11)$. Since any subgroup of M_{23} which is isomorphic to $PSL(2, 11)$ and has orbits of lengths 1, 11, 11 is isomorphic as a permutation group to $\langle a, b, x_2\rangle$, we need not show the form of x when $x=(1)\ (2\ 6)\ (5)\ (8)\ (9)\cdots$ or $(1)\ (2\ 5)\ (6)\ (8)\ (9)\cdots$ and x fixes 16 or 20.

(iii. iii) Suppose that there is a G-orbit of length six. Then by (iii. i) and (iii. ii), the lengths of the G-obits are 1, 6, 6, 5, 5 or 1, 6, 6, 10.

Assume that $\{t\}\cup\Gamma_i$ is a G-orbit, where $t\in\{2, 4\}$ and $i\in\{3, 4\}$. Let $t=2$ and $i=3$. The proofs in the remaining cases are similar. For two points 2 and 6 of $\{2\}\cup\Gamma_3$, $\Delta(1, 2)$ and $\Delta(1, 6)$ are contained in the same G-orbit. Since $\Delta(1, 2)=\Gamma_1$ and $\Delta(1, 6)\ni 7$, Γ_3 and Γ_1 are contained in the same G-orbit. Thus G-orbits are $\{1\}$, $\{2\}\cup\Gamma_3$, $\Gamma_1\cup\Gamma_4$ and $\{4\}\cup\Gamma_2$. Let x be an involution fixing 12, 13 and commuting with a. Since $\Delta(1, 4)\ni 12, 13$, x does not fix the point 4 by the assumption (7). Furthermore $\{4\}\cup\Gamma_2$ is a G-orbit. Hence by the incidence structure of $P(a)$ $x=(1)\ (4\ 5)\ (2\ 3)\ (6)\ (7)\cdots$ or $(1)\ (4\ 5)\ (2)\ (3)\ (6\ 7)\cdots$. If x is of the first form, then 2 and 3 belong to the same G-orbit, which is a contradiction. If x is of the second form, then 6 and 7 belong to the same G-orbit, which is also a contradiction.

Thus the G-orbit containing $\{2\}$ is $\{2\}\cup\Gamma_1$ or $\{2\}\cup\Gamma_2$ and the G-orbit containing $\{4\}$ is $\{4\}\cup\Gamma_1$ or $\{4\}\cup\Gamma_2$. Suppose that the stabilizer of two points in $G^{\{t\}\cup\Gamma_i}$ contains a four group, where $\{t\}\cup\Gamma_i$ is a G-orbit, $t\in\{2, 4\}$ and $i\in\{1, 2\}$. Since $G^{\{t\}\cup\Gamma_i}$ is doubly transitive on $\{t\}\cup\Gamma_i$, $G^{\{t\}\cup\Gamma_i}$ contains A_6 by [5]. Then since A_6 has no permutation representation of degree five, $\Gamma_3\cup\Gamma_4$ is a G-orbit. Furthermore since A_6 is isomorphic to $M_{10}{}^*$ of degree ten, $G^{\Gamma_3\cup\Gamma_4}$ is doubly transitive on $\Gamma_3\cup\Gamma_4$. On the other hand since $\Delta(1, 6)\ni 7$, $\Delta(1, 6)$ is a G_{16}-orbit containing 7 by (3. 8). Thus $\Delta(1, 6)=\Gamma_3\cup\Gamma_4-\{6\}$ and so $|(\Delta 1, 6)|=9$. This is a contradiction. Thus the stabilizer of two points in $G^{\{t\}\cup\Gamma_i}$ does not contain a four group. We use this result in the following proof.

First assume that $\{2\}\cup\Gamma_1$ and $\{4\}\cup\Gamma_2$ are G-orbits. Let x be an involution fixing 16, 17 and commuting with a. Since $\{2\}\cup\Gamma_1$ is a G-orbit, $\{2, 3\}^x=\{2, 3\}$. Suppose that x fixes $\{2, 3\}$ pointwise. Since $\Delta(1, 2)=\Gamma_1$, $I(x)\supset\{1, 2, 3\}$ and $I(ax)\supset\{1, 2, 3\}$, x and ax have no fixed point in $\Gamma_1-\{3\}$ by the assumption (7). Hence $x=(1)\ (2)\ (3)\ (8\ 10)\ (9\ 11)\cdots$ or $(1)\ (2)\ (3)\ (8\ 11)\ (9\ 10)\cdots$. Thus $\langle a, x\rangle^{\{2\}\cup\Gamma_1}$ is a four group contained in $(G^{\{2\}\cup\Gamma_1})_{23}$, contrary to the remark above. Similarly x does not fix $\{4, 5\}$ pointwise. Hence $x=(1)\ (2\ 3)\ (4\ 5)\ (6)\ (7)\ (16)$

(17)$\cdots$. Then x fixes two more points other than 1, 6, 7, 16 and 17. Since $\{2\}\cup\Gamma_1$ and $\{4\}\cup\Gamma_2$ are G-orbits of even length, it is impossible that x has a fixed point in both $\{2\}\cup\Gamma_1$ and $\{4\}\cup\Gamma_2$. Hence now we assume that x has no fixed fixed point in $\{2\}\cup\Gamma_1$. The proof in the case $|I(x)\cap\{\{4\}\cup\Gamma_2\}|=0$ is similar. Then $x=(1)(2\ 3)(4\ 5)(6)(7)(8\ 9)(10\ 11)\cdots$, $(1)(2\ 3)(4\ 5)(6)(7)(8\ 10)(9\ 11)\cdots$ or $(1)(2\ 3)(4\ 5)(6)(7)(8\ 11)(9\ 10)\cdots$. If x is of the first form, then $axb=(1)(2\ 10\ 3)(8)(9\ 11)\cdots$, contrary to (v) or (3. 1). Hence x is of the second or third form. If x is of the third form, then $ax=(1)(2\ 3)(4\ 5)(6)(7)(8\ 10)(9\ 11)\cdots$. Hence set $x'=x$ or ax in the second or third case respectilvely. Then $x'b=(1)(2\ 10\ 8\ 3)(9)(11)(5\ 4\ 14\cdots)\cdots$. Hence $x'b$ is of order four and so $x'b$ has exactly two 2-cycles and three fixed points by (ii) of (3.1). Hence $x'b=(1)(2\ 10\ 8\ 3)(9)(11)(4\ 5\ 14\ 12)(13\ 15)\cdots$. Then $\langle x', (x'b)^2\rangle^{\{4\}\cap\Gamma_2}$ is a four group contained in $(G^{\{4\}\cap\Gamma_2})_{13\ 15}$, contrary to the remark above.

Next assume that $\{2\}\cup\Gamma_2$ and $\{4\}\cup\Gamma_4$ are G-orbits. Let x be an involution fixing 16, 17 and commuting with a. Since $\{2\}\cup\Gamma_2$ is a G-orbit, $\{2, 5\}^x=\{2, 5\}$. If x fixes $\{2, 5\}$ pointwise, then by the incidence structure of $P(a)$ x fixes $I(a)$ pointwise. Thus $I(x)\supseteq I(a)\cup\{16, 17\}$, contrary to (i) of (3, 1). Hence x has a 2-cycle (2 5). Then by the incidence structure of $P(a)$, $x=(1)(2\ 5)(3\ 4)(6)(7)(16)(17)\cdots$. Then by the same argument as above we have a contradiction. Thus we complete the proof of (3.17).

Let

$$\sigma_1=(1\ 4\ 3)(2\ 5\ 6\ 7)(8\ 10)(9\ 11)(12\ 22\ 14\ 18\ 20\ 13\ 23\ 15\ 19\ 21)(16\ 17)$$

and

$$\sigma_2=(1\ 7\ 6)(2)(3\ 5)(4)(18\ 17\ 9\ 16)(10\ 23\ 20\ 18\ 12)(11\ 22\ 21\ 19\ 16)(14\ 15).$$

Then

$$a=a^{\sigma_1}=a^{\sigma_2},$$

$$(ax_1)^{\sigma_1}=x_2{}^{\sigma_2}=(1)(2\ 3)(8)(9)(10\ 11)(12\ 14)(13\ 15)(4\ 5)(6)(7)(16)(17)(18\ 19)(20\ 22)(21\ 23),$$

$$b^{\sigma_1}=(3)(13)(10)(2\ 14)(1\ 8)(9\ 11)(12\ 15)(4)(5)(17)(22)(6\ 18)(7\ 20)(16\ 21)(19\ 23),$$

$$b^{\sigma_2}=(2)(8)(10)(1\ 12)(9\ 13)(3\ 15)(14\ 11)(4)(7)(17)(18)(5\ 23)(6\ 21)(16\ 22)(19\ 20).$$

Then these elements belong to the group of (i) of (3.16), $\langle a, (ax_1)^{\sigma_1}, b^{\sigma_1}\rangle\simeq S_5$ and $\langle a, x_2{}^{\sigma_2}, b^{\sigma_2}\rangle\simeq PSL(2, 11)$. We used in the proof of Lemma 3 these generators.

Next we need the follwing (3.18) and (3.19), which are frequently used in the proof in (3.20).

3.18. *Let*

$$a=(i_1)(i_2)(j_1 j_2 j_3)\cdots.$$

and let b be an involution of the form

$$b=(i_1)(i_2)(j_1)(j_2 j_3)\cdots.$$

If $\Delta(i_1, i_2)\ni j_t$ *or* $\Delta(i_1, j_t)\ni i_2$ $(t=1, 2$ *or* $3)$, *then* $(ab)^2=1$ *and* $a^b=a^{-1}$.

Proof. If $\Delta(i_1, j_t)\ni i_2$, then $\Delta(i_1, i_2)\ni j_t$ by the assumption (3), $t=1, 2$ or 3. Hence we may assume that $\Delta(i_1, i_2)\ni j_t$, $t=1, 2$ or 3. Then since $\Delta(i_1, i_2)^a=\Delta(i_1, i_2)$, $\Delta(i_1, i_2)\supseteq\{j_1, j_2, j_3\}$. Since

$$ab=(i_1)(i_2)(j_1 j_3)(j_2)\cdots,$$

$|I((ab)^2)|\geq 5$. Hence the order of $(ab)^2$ is 1, 2 or 3 by the assumption (4). If $(ab)^2$ is of order two, then $I((ab)^2)\supset\{i_1, i_2\}$ and $I((ab)^2)\cap\Delta(i_1, i_2)\supseteq\{j_1, j_2, j_3\}$, contrary to the assumption (7). If $(ab)^2$ is of order three, then ab is of order six. However $|I(ab)|\geq 3$, contrary to (v) of (3.1). Hence $(ab)^2=1$ and so $bab=a^{-1}$. Since b is an involution, $a^b=a^{-1}$.

3.19. *Assume that* $|\Delta(i, j)|\leq 3$ *for any two points* i, j *of* Ω. *Let* x, y *be involutions such that* $xy=yx$ *and* $I(x)\neq I(y)$. *Then* $I(x)\cap I(y)$ *consists of three points, say* i_1, i_2, i_3, *and* $\Delta(i_1, i_2)=\{i_3\}$.

Proof. Since $I(x)\neq I(y)$, $|I(x)\cap I(y)|=3$. Set $I(x)\cap I(y)=\{i_1, i_2, i_3\}$. Then $I(\langle x, y\rangle)=\{i_1, i_2, i_3\}$. Thus $\langle x, y\rangle$ is of order four and has orbits of lengths one, two and four. Since $\langle x, y\rangle$ fixes $\{i_1, i_2\}$, $\Delta(i_1, i_2)$ consists of unions of $\langle x, y\rangle$-orbits. Since $|\Delta(i_1, i_2)|\leq 3$, $\Delta(i_1, i_2)$ does not contain the $\langle x, y\rangle$-orbit of length four. Furthermore for any $\langle x, y\rangle$-orbit of length two there is an involution in $\langle x, y\rangle$ fixing the $\langle x, y\rangle$-orbit of length two pointwise. Hence by the assumption (7), $\Delta(i_1, i_2)$ does not contain the $\langle x, y\rangle$-orbit of length two. Thus $\Delta(i_1, i_2)$ is the $\langle a, x\rangle$-orbit $\{i_3\}$ of length one.

3.20. *It is impossible that* $|\Delta(i, j)|\leq 3$ *for any two points* i, j *of* Ω.

Proof. Suppose by way of contradictin that $|\Delta(i, j)|\leq 3$ for any two points i, j of Ω. By (3. 9), we may assume that $\Delta(1, 2)=\{3, 8, 9\}$. Then by the assumption (6), there are two involutions a and b fixing 1, 2, 3 and 1, 2, 8 respectively. Then by (3. 7), ab is of odd order. Since $|\Delta(1, 2)|=3$, ab is of order three. Thus we may assume that

$$a=(1)(2)(4)(3)(8\ 9)(5)(10\ 11)(6)(12\ 13)(7)(14\ 15)(16\ 17)(18\ 19)(20\ 21)(22\ 23)$$

and

$$b=(1)(2)(4)(8)(3\ 9)(10)(5\ 11)(12)(6\ 13)(14)(7\ 15)(16\ 17)(18\ 20)(19\ 22)(21\ 23).$$

Then

$$ab=(1)(2)(4)(3\ 9\ 8)(5\ 11\ 10)(6\ 13\ 12)(7\ 15\ 14)(16)(17)(18\ 22\ 21)(19\ 20\ 23).$$

Set $\Gamma_1=\{3, 8, 9\}$, $\Gamma_2=\{5, 10, 11\}$, $\Gamma_3=\{6, 12, 13\}$, $\Gamma_4=\{7, 14, 15\}$, $\Lambda_1=\{18, 21, 22\}$, and $\Lambda_2=\{19, 20, 23\}$. Then $\langle a, b\rangle$-orbits are $\{1\}$, $\{2\}$, $\{4\}$, Γ_1, Γ_2, Γ_3, Γ_4, $\Lambda_1\cup\Lambda_2$ and $\{16, 17\}$. Similarly to (3. 16) we may assume that $\Delta(1, 4)\ni 5$, $\Delta(1, 6)\ni 7$ and $\Delta(2, 4)\ni 6$. Then $\Delta(1\ 2)=\Gamma_1$, $\Delta(1, 4)=\Gamma_2$ and $\Delta(2, 4)=\Gamma_3$.

Since $\Delta(16, 1)^{ab}=\Delta(16, 1)$, $\Delta(16, 1)=\{2\}$, $\{4\}$, $\{17\}$, $\{2, 4, 17\}$, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$, Λ_1 or Λ_2. If $\Delta(16, 1)\ni 2$, then $\Delta(1, 2)\ni 16$, which is a contradiction. Similarly $\Delta(16, 1)\not\ni 4$. Suppose that $\Delta(16, 1)=\Gamma_3$. Then $\Delta(17, 1)=\Delta(16, 1)^a=\Gamma_3$. Hence $\Delta(1, 6)=\{7, 16, 17\}$. This implies that there is an involution x fixing 1, 6, 16. By the assumption (7),

$$x=(1)(6)(16)(7\ 17)\cdots.$$

Hence

$$abx=(1)(16)(14\ 17\ 7\cdots)\cdots.$$

Thus abx is not an involution. On the other hand since $\Delta(16, 1)=\{6, 12, 13\}$ and x fixes 16, 1, 6,

$$x=(16)(1)(6)(12\ 13)\cdots.$$

Since $ab=(16)(1)(6\ 13\ 12)\cdots$, abx is of order two by (3. 18), which is a contradiction. Thus $\Delta(16, 1)\neq\Gamma_3$. Similarly $\Delta(16, 1)\neq\Gamma_4$. Hence $\Delta(16, 1)=\{17\}$, Γ_1, Γ_2, Λ_1 or Λ_2. Since $\Delta(17, 1)=\Delta(16, 1)^a$, $\Delta(17, 1)=\{16,\}$, Γ_1, Γ_2, Λ_1 or Λ_2. In the same way $\Delta(16, 2)=\{17\}$, Γ_1, Γ_3, Λ_1 or Λ_2, $\Delta(17, 2)=\{16\}$. Γ_1, Γ_3, Λ_1 or Λ_2, $\Delta(16, 4)=\{17\}$, Γ_2, Γ_3, Λ_1 or Λ_2 and $\Delta(17, 4)=\{16\}$, Γ_2, Γ_3, Λ_1 or Λ_2.

Now $\Delta(16, 17)^{\langle a,b\rangle}=\Delta(16, 17)$. Hence $\Delta(16, 17)=\{1\}$, $\{2\}$, $\{4\}$, $\{1, 2, 4\}$, Γ_1, Γ_2, Γ_3 or Γ_4. In the following we treat these cases for $\Delta(16, 17)$ separately.

(i) Assume that $\Delta(16, 17)=\Gamma_1$. The proof in the case $\Delta(16, 17)=\Gamma_2$ or Γ_3 is similar. Then by the assumption (6), there is an involution x of the form

$$x=(16)(17)(3)(8\ 9)\cdots.$$

Since $ab=(16)(17)(3\ 9\ 8)\cdots$, $(ab)^x=(ab)^{-1}$ by (3. 18). Since $I(ab)=\{1, 2, 4, 16, 17\}$, x fixes $\{1, 2, 4\}$ and so $x=(1)(2)(4)\cdots$, $(1)(2\ 4)\cdots$, $(2)(1\ 4)\cdots$ or $(4)(1\ 2)\cdots$. Suppose that $x=(1)(2)(4)\cdots$. Then $I(a)\cap I(x)\supseteq\{1, 2, 3, 4\}$ and $I(a)\neq I(x)$, contrary to (vi) of (3. 1). Next suppose that $x=(1)(2\ 4)\cdots$. Then $\Delta(1, 3)^x=\Delta(1, 3)$. Since $\Delta(1, 3)\ni 2$, $\Delta(1, 3)\ni 2^x=4$. Thus $\Delta(1, 3)\cap(I(a)\supseteq\{2, 4\}$, which is a contradiction since a is an involution fixing 1, 3. Similarly $x\neq(2)(1\ 4)\cdots$. Thus

$$x = (1\ 2)\ (3)\ (4)\ (8\ 9)\ (16)\ (17)\cdots .$$

Since

$$ax = (1\ 2)\ (3)\ (4)\ (8)\ (9)\ (16\ 17)\cdots ,$$

x commutes with a. Therefore by the incidence structure of $P(a)$,

$$x = (1\ 2)\ (3)\ (4)\ (5\ 6)\ (7)\ (8\ 9)\ (16)\ (17)\cdots .$$

Then because $(ab)^x=(ab)^{-1}$

$$x = (1\ 2)\ (4)\ (3)\ (8\ 9)\ (5\ 6)\ (10\ 13)\ (11\ 12)\ (7)\ (14\ 15)\ (16)\ (17)\cdots .$$

Since $|I(x)|=7$, x fixes exactly two points of $\{18, 19, 20, 21, 22, 23\}$. Without loss we may assume that x fixes 18, 19. Then

$$x = (1\ 2)\ (4)\ (3)\ (8\ 9)\ (5\ 6)\ (11\ 12)\ (10\ 13)\ (7)\ (14\ 15)\ (16)\ (17)$$
$$(18)\ (19)\ (21\ 22)\ (20\ 23).$$

Since $I(\langle a, x\rangle)=\{3, 7, 4\}$, $\Delta(3, 4)=\{7\}$, $\Delta(3, 7)=\{4\}$ and $\Delta(4, 7)=\{3\}$ by (3. 19).

Now $\Delta(16, 4)=\{17\}$, Γ_2, Γ_3, Λ_1 or Λ_2 and $\Delta(16, 1)=\{17\}$, Γ_1, Γ_2, Λ_1 or Λ_2. Since $(16, 17)=\Gamma_1$, $\Delta(16, 4)\neq\{17\}$ and $\Delta(16, 1)\neq\{17\}$. Since $\Delta(16, 4)^x=\Delta(16, 4)$ and $\Gamma_2{}^x=\Gamma_3$, $\Delta(16, 4)\neq\Gamma_2, \Gamma_3$. Thus $\Gamma(16, 4)=\Lambda_1$ or Λ_2. On the other hand $\Delta(16, 1)=\{17\}$, Γ_1, Γ_2, Λ_1 or Λ_2. Since $\Delta(16, 17)=\Gamma_1$, $\Delta(16, 1)\neq\{17\}$. Suppose that $\Delta(16, 1)=\Gamma_1$. Then $\Delta(17, 1)=\Delta(16, 1)^a=\Gamma_1$. Hence $\Delta(1, 3)=\{2, 16, 17\}$. Then by the assumption (6), there is an involution

$$u = (1)\ (16)\ (3)\ (8\ 9)\ (2\ 17)\cdots .$$

Since $\Delta(16, 1)=\Gamma_1$ and $ab=(16)\ (1)\ (3\ 9\ 8)\cdots$, $(ab)^u=(ab)^{-1}$ by (3. 18). Hence

$$u = (1)\ (16)\ (2\ 17)\ (3)\ (4)\ (8\ 9)\cdots .$$

Since $\Delta(3, 4)=\{7\}$, u fixes 7 and so

$$au = (1)\ (3)\ (4)\ (2\ 17\ 16)\ (7)\ (8)\ (9)\cdots ,$$

contrary to (ii) of (3. 1). Thus $\Delta(16, 1)\neq\Gamma_1$. Next suppose that $\Delta(16, 1)=\Gamma_2$. Since $\Delta(17, 1)=\Delta(16, 1)^a=\Gamma_2$, $\Delta(1, 5)=\{4, 16, 17\}$. Then by the assumption (6) there is an involution

$$u = (1)\ (16)\ (5)\ (10\ 11)\ (4\ 17)\cdots .$$

By (3. 18), $(ab)^u=(ab)^{-1}$. Hence

$$u = (1)\ (16)\ (4\ 17)\ (2)\ (5)\ (10\ 11)\cdots .$$

This implies that $\Delta(16, 4)=\Delta(16, 17)^u=\Gamma_1{}^u$. However u fixes 1, 2. Hence $\Gamma_1{}^u=\Gamma_1$, which is a contradiction since $\Delta(16, 4)=\Lambda_1$ or Λ_2. Thus $\Delta(16, 1)=\Lambda_1$ or Λ_2. Since $\Delta(16, 2)=\Delta(16, 1)^x$, $\Delta(16, 2)=\Lambda_1$ or Λ_2.

By (3. 5), there is an involution y fixing 10, 11 and commuting with a. Since y fixes $I(a)$, $y=(1)\ (4)\cdots$, $(1\ 4)\cdots$, $(1)\ (4\ i)\cdots$, $(4)\ (1\ i)\cdots$ or $(1\ i)\ (4\ j)\cdots$,

$i, j \in \{2, 3, 5, 6, 7\}$. Since $\Delta(1, 4) \supseteq \{10, 11\}$, the first case does not occur. In the remaining cases $y^{I(a)}$ is determined by the incidence structure of $P(a)$.

(i.i) Assume that $y=(1\ 4)(10)(11)\cdots$. Then $y=(1\ 4)(5)(2)(7)(3\ 6)(10)(11)\cdots$ or $(1\ 4)(5)(2\ 7)(3)(6)(10)(11)\cdots$.

(i.i.i) Suppose that $y=(1\ 4)(5)(2)(7)(3\ 6)(10)(11)\cdots$. Then

$$by = (1\ 4)(2)(5\ 11)(10)\cdots.$$

This shows that by is of order two or four. Since $(by)^2$ fixes $\{1, 4\} \cup \Delta(1, 4)$ pointwise, the order of by is not four. Hence by is of order two and so y commutes with b and so ab. Then y fixes $\{16, 17\}$. However $\Delta(16, 17)=\Gamma_1$ and $\Gamma_1^y \neq \Gamma_1$. Thus we have a contradiction.

(i.i.ii) Suppose that $y=(1\ 4)(5)(2\ 7)(3)(6)(10)(11)\cdots$. Then $\Delta(4, 7) = \Delta(1, 2)^y = \Gamma_1^y$. This is a contradiction since $|\Delta(4, 7)|=1$.

(i.ii) Assume that $y=(1)(4\ i)(10\ 11)\cdots$, $i \in \{2, 3, 5, 6, 7\}$.

(i.ii.i) Let $i=2$. Then $\Delta(1, 4)^y=\Delta(1, 2)$. Since $\Delta(1, 4)=\Gamma_2 \ni 10$, $\Delta(1, 2) \ni 10^y=10$, which is a contradiction.

(i.ii.ii) Let $i=3$. Then $y=(1)(4\ 3)(2\ 5)(6)(7)(10)(11)\cdots$. Since $\Delta(1, 10)^{\langle b, y\rangle}=\Delta(1, 10)$ and $\Delta(1, 10) \ni 4$, $\Delta(1, 10)=\{4, 3, 9\}$. Hence y fixes 9 and so 8. This implies that y fixes the subset $\{3, 4, 7, 8, 9\}$ of $I(ax)$. Hence by (vi) of (3. 1), y fixes $I(ax)$ and consequently y has a 2-cycle $(14\ 15)$. Thus

$$by = (1)(2\ 5\ 11)(3\ 9\ 4)(8)(10)(7\ 14\ 15)(13\ 6\cdots)\cdots.$$

This shows that by is of order three. Hence y has a 2-cycle $(12\ 13)$ and by fixes exactly two points of $\{16, 17, \cdots, 23\}$. Therefore y has a common 2-cycle $(i\ j)$ in $\{16, 17, \cdots, 23\}$ with b. If $(i\ j)=(16\ 17)$, then $|I(ay)| \geq 9$, which is a contradiction. Next suppose that $(i\ j)=(21\ 23)$. Since by is of order three and ay is of order two, $y=(21\ 23)(20\ 22)(16\ 18)(17\ 19)\cdots$ or $(21\ 23)(20\ 22)(16\ 19)(17\ 18)\cdots$ and so $xy=(1\ 5\ 6\ 2)(3\ 4)(8\ 9)(16\ 18)\cdots$ or $(1\ 5\ 6\ 2)(3\ 4)(8\ 9)(16\ 19)\cdots$ respectively, contrary to (iii) of (3.1). Thus $(i\ j)=(18\ 20)$ or $(19\ 22)$. Since by is of order three and ay is of order two, y is one of the following form:

$$y_1 = (1)(4\ 3)(2\ 5)(6)(7)(8)(9)(10)(11)(12\ 13)(14\ 15)(18\ 20)(19\ 21)(16\ 22)(17\ 23),$$

$$y_2 = (1)(4\ 3)(2\ 5)(6)(7)(8)(9)(10)(11)(12\ 13)(14\ 15)(19\ 22)(18\ 23)(16\ 20)(17\ 21)$$

or $y_i'=y_i^\sigma$, where $i=1$ or 2 and $\sigma=(16\ 17)$. We assume that $y=y_1$ or y_2. The proof in the case $y=y_1'$ or y_2' is similar. Then

$$aby_1 = (1)(2\ 5\ 11\ 10)(3\ 9\ 8\ 4)(6\ 12)(13)(7\ 14)(15)(18\ 16\ 22\ 19)(17\ 23\ 21\ 20),$$

$aby_2 = (1)\ (2\ 5\ 11\ 10)\ (3\ 9\ 8\ 4)\ (6\ 12)\ (13)\ (7\ 14)\ (15)\ (18\ 19\ 16\ 20)$
$(17\ 21\ 23\ 22)$.

Set $u_1 = x^{aby_1}$ and $u_2 = x^{aby_2}$. Then

$$u_1 = (1\ 5)\ (3)\ (9)\ (4\ 8)\ (11\ 12)\ (10\ 6)\ (2\ 13)\ (14)\ (7\ 15)\ (22)\ (23)\ (16)\ (20\ 19)\ (18)\ (17\ 21),$$

$$u_2 = (1\ 5)\ (3)\ (9)\ (4\ 8)\ (11\ 12)\ (10\ 6)\ (2\ 13)\ (14)\ (7\ 15)\ (20)\ (21)\ (19)\ (23\ 17)\ (16)\ (18\ 22).$$

Now $\Delta(16, 1) = \Lambda_1$ or Λ_2. Assume that $\Delta(16, 1) = \Lambda_1$. The proof in the case $\Delta(16, 1) = \Lambda_2$ is similar. Then $\Delta(16, 5) = \Delta(16, 1)^{u_1} = \{18, 17, 22\}$. Hence $\Delta(16, 17) \ni 5$. However $\Delta(16, 17) = \Gamma_1$. Thus we have a contradiction. Next $\Delta(16, 5) = \Delta(16, 1)^{u_2} = \Lambda_1$. Hence $\Delta(16, 11) = \Delta(16, 5)^{ab} = \Lambda_1{}^{ab} = \Lambda_1$ and $\Delta(16, 10) = \Delta(16, 11)^{ab} = \Lambda_1{}^{ab} = \Lambda_1$. Thus $\Delta(16, 1) = \Delta(16, 5) = \Delta(16, 11) = \Delta(16, 10) = \Lambda_1$. This implies that $\Delta(16, 18) \supseteq \{1, 5, 10, 11\}$, which is a contradiction. Thus $y \neq (1)\ (4\ 3)\cdots$.

(i.ii.iii) Let $i=5$. Then $y = (1)\ (4\ 5)\ (2)\ (3)\ (6\ 7)\ (10)\ (11)\cdots$ or $(1)\ (4\ 5)\ (2\ 3)\ (6)\ (7)\ (10)\ (11)\cdots$. If y is of the first form, then $I(\langle a, y\rangle) = \{1, 2, 3\}$. Hence by (3.19) $\Delta(1, 2) = \{3\}$, which is a contradiction. Hence y is of the second form. Then

$$by = (1)\ (4\ 5\ 11)\ (10)\ (9\ 2\ 3\cdots)\ (13\ 6\cdots)\ (15\ 7\cdots)\cdots.$$

This shows that by is of order three. Hence y fixes 9 and so 8. Furthermore $y = (12\ 13)\ (14\ 15)\cdots$ or $(12\ 15)\ (13\ 14)\cdots$. If $y = (12\ 13)\ (14\ 15)\cdots$, then $xy = (13\ 2)\ (8\ 9)\ (14)\ (15)\cdots$, contrary to (v) of (3.1). Next suppose that $y = (12\ 15)\ (13\ 14)\cdots$. Since $|I(ay)| = 7$ and $|I(by)| = 5$, ay and by fix exactly four and two points of $\{16, 17, \cdots, 23\}$ respectively. Thus $y = (16\ 17)\ (18\ 19)\cdots$, $(16\ 17)\ (20\ 21)\cdots$ or $(16\ 17)\ (22\ 23)\cdots$. However in any case $\{16, 17\}^y = \{16, 17\}$ and $\Gamma_1{}^y \neq \Gamma_1$, which is a contradiction.

(i.ii.iv) Let $i=6$ or 7. Then $y = (1)\ (4\ 6)\ (5\ 7)\ (2)\ (3)\ (10)\ (11)\cdots$ or $(1)\ (4\ 7)\ (5\ 6)\ (2)\ (3)\ (10)\ (11)\cdots$. Then $I(\langle a, y\rangle) = \{1, 2, 3\}$. Hence by (3.19) $\Delta(1, 2) = \{3\}$, which is a contradiction.

(i.iii) Assume that $y = (4)\ (1\ i)\ (10)\ (11)\cdots, i \in \{2, 3, 5, 6, 7\}$.

(i.iii.i) Let $i=2$. Then $\Delta(1, 4)^y = \Delta(1, 2)$. Since $\Delta(1, 4) = \Gamma_2 \ni 10$, $\Delta(1, 2) \ni 10^y = 10$. This is a contradiction.

(i.iii.ii) Let $i=3$. Then $y = (4)\ (1\ 3)\ (10)\ (11)\cdots$. Hence $\Delta(4, 3) = \Delta(4, 1)^y$. However $|\Delta(4, 3)| = 1$ and $|\Delta(4, 1)| = 3$. Thus we have a contradiction.

(i.iii.iii) Let $i=5$. Then $y = (4)\ (1\ 5)\ (2)\ (6)\ (3\ 7)\ (10)\ (11)\cdots$ or $(4)\ (1\ 5)\ (2\ 6)\ (3)\ (7)\ (10)\ (11)\cdots$. If y is of the first form, then $I(\langle a, y\rangle) = \{2, 4, 6.\}$. Hence by (3.19) $\Delta(2, 4) = \{6\}$, which is a contradiction. Hence y is of the second form. Then

$$by=(4)\,(1\ 5\ 11)\,(10)\,(13\ 2\ 6\cdots)\,(9\ 3\cdots)\,(15\ 7\cdots)\cdots.$$

This shows that by is of order three. Hence $y=(12)\,(13)\,(8\ 15)\,(9\ 14)\cdots$ or $(12)\,(13)\,(8\ 9)\,(14\ 15)\cdots$. Assume that $y=(12)\,(13)\,(8\ 15)\,(9\ 14)\cdots$. Then ay fixes exactly three points 3, 4, 7 of $\{1, 2, \ldots, 15\}$ and by fixes exactly three points 4, 10, 12 of $\{1, 2, \cdots, 15\}$. Since $|I(ay)|=7$ and $|I(by)|=5$, ay and by fix exactly four and two points of $\{16, 17, \cdots, 23\}$ respectively. Thus $y=(16\ 17)\,(18\ 19)\cdots$, $(16\ 17)\,(20\ 21)\cdots$ or $(16\ 17)\,(22\ 23)\cdots$. However in any case $\{16, 17\}^y=\{16, 17\}$ and $\Gamma_1^y\neq\Gamma_1$, which is a contradiction since $\Delta(16, 17)=\Gamma_1$. Hence $y=(12)\,(13)\,(8\ 9)\,(14\ 15)\cdots$.
Then

$$xy=(1\ 6)\,(2\ 5)(4)\,(3)\,(8)\,(9)\,(10\ 13)\,(11\ 12)\,(7)\,(14)\,(15)\cdots.$$

This shows that xy is of order two. Hence

$$y=(4)\,(1\ 5)\,(2\ 6)\,(3)\,(7)\,(10)\,(11)\,(12)\,(13)\,(8\ 9)\,(14\ 15)\,(16\ 18)\,(17\ 19)\,(20\ 22)\,(21\ 23)$$

or the same permutation with 16 and 17 interchanged. We assume that y is of the first form. The proof in the second case is similar. Since $\Delta(16, 1)=\Lambda_1$ or Λ_2, first assume that $\Delta(16, 1)=\Lambda_1$. Then $\Delta(16, 18)\ni 1$. Since $\langle x, y\rangle$ fixes $\{16, 18\}$, $\Delta(16, 18)\supseteq\{1\}^{\langle x,y\rangle}=\{1, 5, 6, 2\}$, which is a contradiction. Next assume that $\Delta(16, 1)=\Lambda_2$. Then $\Delta(16, 19)\ni 1$. Since $\langle x, ay\rangle$ fixes $\{16, 19\}$, $\Delta(16, 19)=\{1\}^{\langle x,ay\rangle}=\{2, 1, 5, 6\}$, which is also a contradiction.

(i.iii.iv) Let $i=6$. Then $y=(4)\,(1\ 6)\,(2\ 5)\,(3)\,(7)\,(10)\,(11)\cdots$. Then y is of the same form as $y=(4)\,(1\ 5)\,(2\ 6)\,(3)\,(7)\,(10)\,(11)\cdots$ in the case (i.iii.iii) with 1 and 2 interchanged. Hence in the same way we have a contradiction.

(i.iii.v) Let $i=7$. Then $y=(4)\,(1\ 7)\,(5\ 3)\,(2)\,(6)\,(10)\,(11)\cdots$. Then $I(\langle a, x\rangle)=\{2, 4, 6\}$. Hence by (3.19) $\Delta(2, 4)=\{6\}$, which is a contradiction.

(i.iv) Assume that $y=(1\ i)\,(4\ j)\cdots$, $i, j\in\{2, 3, 5, 6, 7\}$.

(i.iv.i) Let $i=2$. Then $y=(1\ 2)\,(4\ 7)\,(3)\,(5)\,(6)\,(10)\,(11)\cdots$. This shows that $\Delta(1, 2)^y=\Delta(1, 2)$. Hence $y=(8)\,(9)\cdots$ or $(8\ 9)\cdots$. If y is of the first form, then

$$by=(1\ 2)\,(3\ 9)\,(8)\,(5\ 11)\,(15\ 4\ 7\cdots)\cdots,$$

Then $|I((by)^2)|\geq 7$ and $(by)^2\neq 1$. Hence $(by)^2$ is of order two and fixes $\{1, 2\}\cup\Delta(1, 2)$ pointwise, contrary to the assumption (7). Next if y is of the second form, then

$$xy=(1)\,(2)\,(4\ 7)\,(8)\,(9)\,(3)\cdots.$$

Hence xy is an involution fixing $\{1, 2\}\cup\Delta(1, 2)$ pointwise, contrary to the assumption (7).

(i.iv.ii) Let $i=3$. Then $y=(1\ 3)\,(4\ 6)\,(2)\,(5)\,(7)\,(10)\,(11)\cdots$. Since b fixes a subset $\{2, 5, 10, 11\}$ of $I(y)$, b fixes $I(y)$ by (vi) of (3.1). Hence $I(y)\ni 7^b=15$ and so $I(y)\ni 14$ because $ay=ya$. Thus

$$y=(1\ 3)\ (4\ 6)\ (2)\ (5)\ (7)\ (10)\ (11)\ (14)\ (15)\cdots.$$

Since $I(ax)=\{4, 3, 8, 9, 7, 14, 15\}$, $I(ax)^y=\{6, 1, 8^y, 9^y, 7, 14, 15\}$. Then $\langle ab\rangle$ fixes the aubset $\{1, 7, 14, 15\}$ of $I(ax)^y$. Hence $\langle ab\rangle$ fixes $I(ax)^y$. Since $\{6\}^{\langle ab\rangle}=\{6, 12, 13\}$, $\{12, 13\}\subset I(ax)^y$. Thus $\{8, 9\}^y=\{12, 13\}$. Furthermore since $\Delta(16, 17)=\Gamma_1$ and $\Gamma_1^y\neq\Gamma_1$, y does not fix $\{16, 17\}$. Thus the $\langle a, b, x, y\rangle$ -orbits are $\{1, 2, 4\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3$, Γ_4 and $\{16, 17\}\cup\Lambda_1\cup\Lambda_2$. Since $|\{1, 2, 4\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3|=12$ and $\langle a, b, x, y\rangle_1$ has an element ab of order three, the order of $\langle a, b, x, y\rangle$ is a multiple of 3^2. By the assumption (4), a Sylow 3-subgroup of $\langle a, b, x, y\rangle$ is an elementary abelian group of order 3^2. Since Γ_4 is a $\langle a, b, x, y\rangle$ -orbit of length three, the order of $\langle a, b, x, y\rangle_{\Gamma_4}$ is a multiple of three. Furthermore $\langle a, b, x, y\rangle_{\Gamma_4}$ is normalized by ab. Hence $\langle a, b, x, y\rangle_{\Gamma_4}$ has an element u of order three and commuting with ab. Since $\Lambda_1\cup\Lambda_2\cup\{16, 17\}$ is a $\langle a, b, x, y\rangle$ -orbit of length eight and $|I(u)|=5$, u fixes exactly two points of $\Lambda_1\cup\Lambda_2\cup\{16, 17\}$. Thus u fixes 16, 17 because $(ab)u=u(ab)$. On the other hand by the assumption (4), $\langle ab, u\rangle$ has an orbit Γ of length nine. Since $\langle 1, 2, 4\rangle\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3$ is the only one $\langle a, b, x, y\rangle$-orbit of length at least nine, Γ is contained in $\{1, 2, 4\}\cup\Gamma_1\cup\Gamma_2\cup\Gamma_3$. Since $\langle ab, u\rangle$ is of order nine, $\langle ab, u\rangle$ is regular on Γ. Hence $\Gamma=\Gamma_1\cup\Gamma_2\cup\Gamma_3$. Thus $\langle ab, u\rangle$ fixes $\{16, 17\}$ and does not fix $\Gamma_1=\Delta(16, 17)$, which is a contradiction.

(i.iv.iii) Let $i=5$. Then $y=(1\ 5)\ (4\ j)\ (10)\ (11)\cdots$. Since 1, 5, 4 are collinear in $P(a)$, this is a contradiction.

(i.iv.iv) Let $i=6$. Then $y=(1\ 6)\ (4\ 3)\ (2)\ (5)\ (7)\ (10)\ (11)\cdots$. Since b fixes a subset $\{2, 5, 10, 11\}$ of $I(y)$, b fixes $I(y)$ by (vi) of (3.1). Hence $I(y)\ni 7^b=15$, and so $I(y)\ni 14$ because $ay=ya$. Then y fixes a subset $\{3, 4, 7, 14, 15\}$ of $I(ax)=\{3, 4, 7, 14, 15, 8, 9\}$. Hence by (vi) of (3.1), y fixes $I(ax)$ and so y has a 2-cycle (8 9). Then

$$by=(2)\ (10)\ (5\ 11)\ (14)\ (3\ 8\ 9\ 4)\ (13\ 1\ 6\cdots)\cdots.$$

This shows that by is of order four. Hence y has a 2-cycle (12 13). Thus

$$\begin{aligned} y\ &= (1\ 6)\ (4\ 3)\ (2)\ (5)\ (7)\ (10)\ (11)\ (14)\ (15)\ (8\ 9)\ (12\ 13)\cdots, \\ y^x &= (2\ 5)\ (4\ 3)\ (1)\ (6)\ (7)\ (12)\ (13)\ (14)\ (15)\ (8\ 9)\ (10\ 11)\cdots, \\ ay^x &= (2\ 5)\ (4\ 3)\ (1)\ (6)\ (7)\ (12\ 13)\ (14\ 15)\ (8)\ (9)\ (10)\ (11)\cdots. \end{aligned}$$

Then ay^x is y in (i.ii.ii). Hence y is not of this form.

(i.iv.v) Let $i=7$. Then $y=(1\ 7)\ (4\ 2)\ (3)\ (5)\ (6)\ (10)\ (11)\cdots$. Then $\Delta(1, 2)^y=\Delta(4, 7)$, which is a contradiction since $|\Delta(1, 2)|=3$ and $|\Delta(4, 7)|=1$.

(ii) Assume that $\Delta(16, 17)=\{1, 2, 4\}$. Then there is an involution $x=(16)\ (17)\ (1)\ (2\ 4)\cdots$. Then by (3.18), $(ab)^x=(ab)^{-1}$. Furthermore $\Delta(2. 4)^x=\Delta(2, 4)$. Hence $x=(6)\ (12\ 13)\cdots$, $(12)\ (6\ 13)\cdots$ or $(13)\ (6\ 12)\cdots$. Since in the second and third cases we have the first form by transforming x by $(ab)^2$ and ab respectively, we may assume that

$$x=(16)\,(17)(1)\,(2\;4)\,(6)\,(12\;13)\cdots.$$

Then

$$ax=(1)\,(2\;4)\,(6)\,(12)\,(13)\,(16\;17)\cdots.$$

Thus ax is of order two. Hence from $(ab)^x=(ab)^{-1}$, we have that

$$x=(1)\,(2\;4)\,(3\;5)\,(6)\,(7)\,(8\;11)\,(9\;10)\,(12\;13)\,(14\;15)\,(16)\,(17)\cdots.$$

Since $|I(x)|=7$, x fixes two points of $\{18, 19, \cdots, 23\}$. Without loss we may assume that x fixes 18, 19. Thus

$$x=(1)\,(2\;4)\,(3\;5)\,(6)\,(7)\,(8\;11)\,(9\;10)\,(12\;13)\,(14\;15)\,(16)\,(17)\,(18)\,(19)\,(21\;22)\,(20\;23),$$

Next there is an involution y fixing 16, 17, 2. Then by the same argument as is used above for x

$$y=(1\;4)\,(2)\,(5)\,(3\;6)\,(7)\,(8\;13)\,(9\;12)\,(10\;11)\,(14\;15)\,(16)\,(17)\cdots.$$

Then

$$xy=(1\;4\;2)\,(3\;5\;6)\,(7)\,(8\;10\;12)\,(9\;11\;13)\,(14)\,(15)\,(16)\,(17)\cdots.$$

Thus xy is of order three. Hence y is one of the following elements:

$$y_1=(1\;4)\,(2)\,(5)\,(3\;6)\,(7)\,(8\;13)\,(9\;12)\,(10\;11)\,(14\;15)\,(16)\,(17)\,(20)\,(21)\,(18\;22)\,(19\;23),$$
$$y_2=(1\;4)\,(2)\,(5)\,(3\;6)\,(7)\,(8\;13)\,(9\;12)\,(10\;11)\,(14\;15)\,(16)\,(17)\,(22)\,(23)\,(18\;21)\,(19\;20).$$

Suppose that $y=y_1$. The proof in the case $y=y_2$ is similar. Then

$$y_1^{ab}=(1\;4)\,(2)\,(3\;12)\,(5\;10)\,(9\;13)\,(11)\,(8\;6)\,(15)\,(7\;14)\,(16)\,(17)\,(18)\,(21\;22)\,(23)\,(19\;20),$$
$$xy_1^{ab}=(1\;4\;2)\,(3\;10\;13)\,(5\;12\;9)\,(6\;8\;11)\,(7\;14\;15)\,(16)\,(17)\,(18)\,(21)\,(22)\,(19\;20\;23).$$

Since $\Delta(1, 6)\ni 7$ and $\Delta(1, 16)\ni 17$, $\Delta(6, 18)\ni 16$ or 17 in the plane $P(x)$. Without loss we may assume that $\Delta(6, 18)\ni 16$. Then $\Delta(7, 18)\ni 17$. Thus $\Delta(16, 18)\ni 6$ and $\Delta(17, 18)\ni 7$. Since $\Delta(16, 18)^{xy_1^{ab}}=\Delta(16, 18)$ and $\Delta(17, 18)^{xy_1^{ab}}=\Delta(17, 18)$, $\Delta(16, 18)=\{6, 8, 11\}$ and $\Delta(17, 18)=\{7, 14, 15\}$. Furthermore since $\Delta(21, 22)^{\langle x, y_1^{ab}\rangle}=\Delta(21, 22)$, $\Delta(21, 22)=\{16\}$, $\{17\}$, $\{18\}$, $\{16, 17, 18\}$, $\{1, 2, 4\}$, $\{6, 8, 11\}$, $\{7, 14, 15\}$ or $\{19, 20, 23\}$. On the other hand we have shown that for $I(\langle a, b\rangle)=\{1, 2, 4\}$ and $I(ab)-I(\langle a, b\rangle)=\{16, 17\}$, $\Delta(16, 1)=\{17\}$, $\Delta(1, 2)$ $\Delta(1, 4)$, Λ_1 or Λ_2, where $\Lambda_1\cup\Lambda_2$ is a $\langle a, b\rangle$-orbit of length six. Furthermore in the case (i) we have proved that $\Delta(16, 17)\neq\Delta(1, 2)$, $\Delta(1, 4)$ and $\Delta(2, 4)$. Since $I(\langle x, y_1^{ab}\rangle)=\{16, 17, 18\}$ and $I(xy_1^{ab})-I(\langle x, y_1^{ab}\rangle)=\{21, 22\}$, by the same argument for $\langle x, y_1^{ab}\rangle$ as is used for $\langle a, b\rangle$, $\Delta(21, 18)=\{22\}$, $\Delta(18, 16)$, $\Delta(18, 17)$, $\{3, 10, 13\}$ or $\{5, 12, 9\}$ and $\Delta(21, 22)\neq\Delta(16, 17)$, $\Delta(16, 18)$,

$\Delta(17, 18)$. Thus $\Delta(21, 18)=\{22\}$, $\{6, 8, 11\}$, $\{7, 14, 15\}$, $\{3, 10, 13\}$ or $\{5, 12, 9\}$, and $\Delta(20, 21)=\{16\}$, $\{17\}$, $\{18\}$, $\{16, 17, 18\}$ or $\{19, 20, 23\}$. On the other hand since $\Delta(21, 22)=\Delta(21, 28)^{y_1}=\{18\}$, $\{3, 13, 10\}$, $\{7, 14, 15\}$, $\{6, 11, 8\}$ or $\{5, 9, 12\}$, $\Delta(21, 22)=\{18\}$. Hence there is an involution u fixing 21, 22 and commuting with x. Thus $u=(21)(22)(18)(16)\cdots$ or $(21)(22)(18)(16\ i)\cdots$, $i\in\{1, 6, 7, 17, 19\}$. We consider $u^{I(x)}$ in terms of the incidence structure of $P(x)$.

(ii.i) Assume that $u=(21)(22)(18)(16)\cdots$. Then $I(\langle x, u\rangle)\supset\{16, 18\}$. Hence by (3.19) $|\Delta(16, 18)|=1$, which is a contradiction since $\Delta(16, 18)=\{6, 8, 11\}$.

(ii.ii) Assume that $u=(21)(22)(18)(16\ i)\cdots$, $i\in\{1, 6, 7, 17, 19\}$.

(ii.ii.i) Let $i=1$. Then $u=(21)(22)(18)(16\ 1)(6\ 19)(7)(17)\cdots$, Then $I(\langle x, u\rangle)=\{17, 18, 7\}$. Hence by (3.19) $\Delta(17, 18)=\{7\}$, which is a contradiction since $\Delta(17, 18)=\{7, 14, 15\}$.

(ii.ii.ii) Let $i=6$. Then $u=(21)(22)(18)(16\ 6)(17)(7)(1\ 19)\cdots$ or $(21)(22)(18)(16\ 6)(17\ 7)(1)(19)\cdots$. If u is of the first form, then $I(\langle x, u\rangle)=\{17, 18, 7\}$. Hence by (3.19) $\Delta(17, 18)=\{7\}$, which is a contradiction. Next suppose that u is of the second form. Then $\Delta(16, 17)^u=\Delta(6, 7)$. Hence $|\Delta(6, 7)|=3$. On the other hand since $I(\langle a, x\rangle)=\{1, 6, 7\}$, $\Delta(6, 7)=\{1\}$ by (3, 19). Thus we have a contradiction.

(ii.ii.iii) Let $i=7$. Then $u=(21)(22)(18)(16\ 7)(17\ 6)(1)(19)\cdots$. Then $\Delta(16, 17)^u=\Delta(6, 7)$. Thus $|\Delta(6, 7)|=3$, which is a contradiction since $\Delta(6, 7)=\{1\}$.

(ii.ii.iv) Let $i=17$. Then $u=(21)(22)(18)(16\ 17)(6\ 7)(1)(19)\cdots$. Thus u fixes $\Delta(16, 17)=\{1, 2, 4\}$. Hence $\{2, 4\}^u=\{2, 4\}$. However this is a contradiction since $\Delta(2, 4)=\{6, 12, 13\}$ and $6^u=7\notin\Delta(2, 4)$

(ii.ii.v) Let $i=19$. Then $u=(21)(22)(18)(16\ 19)(1\ 6)(7)(17)\cdots$. Then $I(\langle x, u\rangle)=\{17, 18, 7\}$. Hence by (3.19) $\Delta(17, 18)=\{7\}$, which is a contradiction.

(iii) Assume that $\Delta(16, 17)=\Gamma_4$. By (3.5), there is an involution x fixing 16, 17 and commuting with a. Since $\Delta(16,17)=\Gamma_4$, we may assume that $x=(16)(17)(7)(14\ 15)\cdots$. Then by (3.18), $(ab)^x=(ab)^{-1}$. Hence x fixes $\{1, 2, 4\}$. If x fixes $\{1, 2, 4\}$ pointwise, then $I(a)\cap I(x)\supseteq\{1, 2, 4, 7\}$ and $I(a)\neq I(x)$, which is a contradiction by (vi) of (3.1). Thus $x=(1\ 2)(4)\cdots$, $(1\ 4)(2)\cdots$ or $(2\ 4)(1)\cdots$. Assume that $x=(1\ 2)(4)\cdots$. The proof in the case $x=(1\ 4)(2)\cdots$ or $(2\ 4)(1)\cdots$ is similar. Since $ax=xa$ and $(ab)^x=(ab)^{-1}$,

$$x=(1\ 2)(4)(3)(5\ 6)(7)(8\ 9)(10\ 13)(11\ 12)(14\ 15)(16)(17)\cdots.$$

This shows that x fixes exactly two points of $\{18, 19, \cdots, 23\}$. Hence we may assume that

$$x=(1\ 2)(3)(4)(5\ 6)(7)(8\ 9)(10\ 13)(11\ 12)(14\ 15)(16)(17)(18)(19)(21\ 22)(20\ 23).$$

Thus this element x is the element x in the proof of the case (i). Moreover since $I(\langle a, x\rangle)=\{3, 4, 7\}$, by (3.19) $\Delta(i, j)=\{k\}$, where $\{i, j, k\}=\{3, 4, 7\}$.

Now $\Delta(16, 1)=\{17\}$, Γ_1, Γ_2, Λ_1 or Λ_2 and $\Delta(16, 4)=\{17\}$, Γ_2, Γ_3, Λ_1 or Λ_2. Since $\Delta(16, 17)=\Gamma_4$, $\Delta(16, 1)\neq\{17\}$ and $\Delta(16, 4)\neq\{17\}$. Furthermore in the proof of the case (i) to show that $\Delta(16, 4)\neq\Gamma_2$, Γ_3 and $\Delta(16, 1),\neq\Gamma_1$ we did not require that $\Delta(16, 17)=\Gamma_1$. Hence by the same argument as in the case (i) $\Delta(16, 4)=\Lambda_1$ or Λ_2 and $\Delta(16, 1)=\Gamma_2$, Λ_1 or Λ_2. We shall treat these cases separately.

(iii.i) Assume that $\Delta(16, 1)=\Gamma_2$ and $\Delta(16, 4)=\Lambda_1$. The proof in the case $\Delta(16, 1)=\Gamma_2$ and $\Delta(16, 4)=\Lambda_2$ is similar. Since $\Delta(17, 1)=\Delta(16, 1)^a=\Gamma_2{}^a=\Gamma_2$, $\Delta(1, 5)=\{4, 16, 17\}$. Thus there is an involution y fixing 1, 5, 16. Then since $\Delta(16, 1)=\{5, 10, 11\}$,

$$y=(1)(5)(16)(4\ 17)(10\ 11)\cdots.$$

Since $\Delta(16, 1)=\{5, 10, 11\}$, $(ab)^y=(ab)^{-1}$ by (3.18). Thus

$$y=(1)(2)(5)(10\ 11)(16)(4\ 17)\cdots.$$

Hence $\Delta(16, 4)^y=\Delta(16, 17)$ and so $\Lambda_1{}^y=\Gamma_4$. Furthermore y fixes $\{1, 2\}$ pointwise. Since $\Delta(1, 2)=\Gamma_1$, $y=(3)(8\ 9)\cdots$, $(8)(3\ 9)\cdots$ or $(9)(3\ 8)\cdots$.

(iii.i.i) Suppose that $y=(3)(8\ 9)(1)(2)(5)(10\ 11)(16)(4\ 17)\cdots$. Then

$$ay=(1)(2)(3)(8)(9)(5)(10)(11)(4\ 17\ 16)\cdots.$$

which is a contradiction by (ii) of (3.1).

(iii.i.ii) Suppose that $y=(8)(3\ 9)(1)(2)(5)(10\ 11)(16)(4\ 17)\cdots$ or $(9)(3\ 8)(1)(2)(5)(10\ 11)(16)\cdots$.

First assume that $y=(8)(3\ 9)(1)(2)(5)(10\ 11)(16)(4\ 17)\cdots$. Since

$$x^{(ab)^{-1}}=(1\ 2)(4)(8)(3\ 9)(10\ 12)(5\ 13)(11\ 6)(14)(7\ 15)(16)(17)(21)(18\ 22)(23)(19\ 20),$$
$$x^{(ab)^{-1}}y=(1\ 2)(4\ 17)(16)(8)(3)(9)\cdots.$$

Then $x^{(ab)^{-1}}y$ is of order two. Hence form $(ab)^y=(ab)^{-1}$ and $\Lambda_1{}^y=\Gamma_4$, it follows that

$$y=(1)(2)(5)(10\ 11)(16)(4\ 17)(8)(3\ 9)(13)(6\ 12)(14\ 21)(7\ 22)(15\ 18)(23)(19\ 20)$$

which is denoted by y_1.

Next assume that $y=(9)(3\ 8)(1)(2)(5)(10\ 11)(16)\cdots$. Since

$$x^{ab}=(1\ 2)(4)(9)(3\ 8)(11\ 13)(6\ 10)(5\ 12)(15)(7\ 14)(16)(17)(22)(21\ 18)(20)(23\ 19),$$
$$x^{ab}y=(1\ 2)(4\ 17)(16)(9)(3)(8)\cdots.$$

Then $x^{ab}y$ is of order two. Hence form $\Lambda_1{}^y=\Gamma_4$ and $(ab)^y=(\mathrm{ab})^{-1}$, we have that

$$y=(1)(2)(5)(10\ 11)(9)(3\ 8)(16)(4\ 17)(20)(7\ 21)(12)(6\ 13)(15\ 22)(19\ 23)(14\ 18),$$

which is denoted by y_2.

Then

$$ay_1=(1)(2)(5)(10)(11)(3\ 9\ 8)(4\ 17\ 16)(7\ 22\ 23)(6\ 12\ 13)(18\ 20\ 14)(19\ 15\ 21),$$
$$ay_2=(1)(2)(5)(10)(11)(3\ 8\ 9)(4\ 17\ 16)(7\ 21\ 20)(6\ 13\ 12)(18\ 23\ 15)(19\ 14\ 22).$$

First suppose that $y=y_1$. Since $\Delta(10, 11)^{\langle a, y_1\rangle}=\Delta(10, 11)$ and $|\Delta(10, 11)|=1$ or 3, $\Delta(10, 11)=\{1\}, \{2\}, \{5\}, \{1, 2, 5\}, \{3, 8, 9\}, \{4, 16, 17\}, \{7, 22, 23\}$ or $\{6, 12, 13\}$.

In the cases (i) and (ii) we have proved that for $I(\langle a, b\rangle)=\{1, 2, 4\}$ and $I(ab)-I(\langle a, b\rangle)=\{16, 17\}$, $\Delta(16, 17)\neq\Delta(1, 2)$, $\Delta(1, 4)$, $\Delta(2, 4)$ and $\{1, 2, 4\}$. Since $I(\langle a, y_1\rangle)=\{1, 2, 5\}$ and $I(ay_1)-I(\langle a, y_1\rangle)=\{10, 11\}$, by the same argument for $\langle a, y_1\rangle$ as is used for $\langle a, b\rangle$ in the cases (i) and (ii), $\Delta(10, 11)\neq\{1, 2, 5\}$, $\Delta(1, 2)$, $\Delta(1, 5)$, $\Delta(2, 5)$ where $\Delta(1, 2)=\{3, 8, 9\}$ and $\Delta(1, 5)=\{4, 16, 17\}$. Since $\Delta(2, 5)\ni 7$ and $\Delta(2, 5)^{ay_1}=\Delta(2, 5)$, $\Delta(2, 5)=\{7, 22, 23\}$. Hence $\Delta(10, 11)=\{1\}$, $\{2\}$, $\{5\}$ or $\{6, 12, 13\}$. Futhermore since $\Delta(10, 1)=\Delta(1, 5)^{(ab)^2}=\{4, 16, 17\}$, $\Delta(10, 11)\neq\{1\}$. Similarly $\Delta(10, 11)\neq\{2\}$. Thus $\Delta(10, 11)=\{5\}$ or $\{6, 12, 13\}$. Suppose that $\Delta(10, 11)=\{6, 12, 13\}$. Then $\Delta(10, 5)=\Delta(10, 11)^b=\{6, 12, 13\}$. Thus $\Delta(10, 12)\supset\{11, 5\}$. Furthermore in the plane $P(b)$ $\Delta(12, 10)\ni 8$. Thus $\Delta(12, 10)=\{8, 11, 15\}$. Hence $\Delta(11, 13)=\Delta(12, 10)^x=\{9, 12, 6\}$. However since $\Delta(10, 11)=\{6, 12, 13\}$, $\Delta(10, 13)\ni 11$. Thus we have a contradiction. Hence $\Delta(10, 11)=\{5\}$.

Next suppose that $y=y_2$. Since $\Delta(10, 11)^{\langle a, y_2\rangle}=\Delta(10, 11)$, $\Delta(10, 11)=\{1\}, \{2\}, \{5\}, \{1, 2, 5\}, \{3, 8, 9\}, \{4, 16, 17\}, \{7, 20, 21\}$ or $\{6, 12, 13\}$. Since $\Delta(2, 5)\ni 7$ and $(2, 5)^{ay_2}=\Delta(2, 5)$, $\Delta(2, 5)=\{7, 12, 20\}$. Hence by the same argument as is used above, $\Delta(10, 11)=\{5\}$.

Now we assume that $\Delta(10, 11)=\{5\}$ and $y=y_1$. In the following if we interchang 8 and 9, then we have the proof for the case $\Delta(10,11)=\{5\}$ and $y=y_2$. By (3.5), there is an involution u fixing 10, 11 and commuting with a. Since $\Delta(10, 11)=\{5\}$, u fixes 5. If u fixes 1 or 2, then ay_1 fixes $I(u)$, contrary to (3.4). Hence $u=(10)(11)(5)(1\ 2)\cdots$ or $(10)(11)(5)(1\ i)(2\ j)\cdots$, $i, j\in\{3, 4, 6, 7\}$. If $u=(10)(11)(5)(1\ 2)\cdots$, then by the incidence structure of $P(a)$,

$$u=(10)(11)(5)(1\ 2)(4\ 7)(3)(6)\cdots.$$

Since u fixes $\Delta(1, 2)$, $u=(8)(9)\cdots$ or $(8\ 9)\cdots$. If $u=(8)(9)\cdots$, then

$$ay_1u=(1\ 2)(5)(10)(11)(3\ 9\ 8)\cdots,$$

contrary to (v) of (3.1). Hence $u=(8\ 9)\cdots$ and

$$ay_1u = (1\ 2)\ (5)\ (10)\ (11)\ (3\ 8)\ (9)\cdots .$$

Then ay_1u is of order two and so $(ay_1)^u=(ay_1)^{-1}$. Hence

$$u = (10)\ (11)\ (5)\ (1\ 2)\ (4\ 7)\ (3)\ (6)\ (8\ 9)\ (12\ 13)\cdots .$$

Then

$$xu = (1)\ (2)\ (3)\ (8)\ (9)\ (10\ 12\ 11\ 13)\cdots ,$$

contrary to (iii) of (3.1), Next suppose that $u=(10)\ (11)\ (5)\ (1\ i)\ (2\ j)\cdots,\ i, j\in \{3, 4, 6, 7\}$. Then by the incidence structure of $P(a)$, $u=(10)\ (11)\ (5)\ (1\ 4)\ (2\ 7)\ (3)\ (6)\cdots$ or $(10)\ (11)\ (5)\ (1\ 7)\ (2\ 4)\ (3)\ (6)\cdots$. Then $\Delta(1, 2)^u=\Delta(4, 7)$ and so $|\Delta(4, 7)|=3$. This is a contradiction since $\Delta(4, 7)=\{3\}$.

(iii.ii) Assume that $\Delta(16, 1)=\Lambda_1$ and $\Delta(16, 4)=\Lambda_2$. The proof in the case $\Delta(16, 1)=\Lambda_2$ and $\Delta(16, 4)=\Lambda_1$ is similar. Since $\Delta(16, 2)=\Delta(16, 1)^x=\Lambda_1^x =\Lambda_1$, $\Delta(16, 18)\ni 1, 2$. Moreover by the incidence structure of $P(x)$, $\Delta(16, 18)\ni 3$. Thus $\Delta(16, 18)=\{1, 2, 3\}$. Hence $\Delta(16, 22)=\Delta(16, 18)^{ab}=\{1, 2, 9\}$ and $\Delta(16, 21) =\Delta(16, 22)^{ab}=\{1, 2, 8\}$. On the other hand since $\Delta(16, 1)=\Lambda_1$ and $\Delta(16, 18)= \{1, 2, 3\}$, there is an involution

$$u = (16)\ (18)\ (1)\ (2\ 3)\ (21\ 22)\cdots .$$

Then $\Delta(16, 22)=\Delta(16, 21)^u\ni 2^u=3$, which is a contradiction.

(iii.iii) Assume that $\Delta(16, 1)=\Delta(16, 4)=\Lambda_1$. The proof in the case $\Delta(16, 1)=\Delta(16, 4)=\Lambda_2$ is similar. Since $\Delta(16, 2)=\Delta(16, 1)^x=\Lambda_1$, $\Delta(16, 18)= \{1, 2, 4\}$. Hence there is an involution y fixing 16, 1, 18. Then since $\Delta(16, 1) =\Lambda_1$,

$$y = (1)\ (16)\ (18)\ (21\ 22)\ (2\ 4)\cdots .$$

Hence by (3.18), $(ab)^y=(ab)^{-1}$. Hence

$$y = (1)\ (2\ 4)\ (16)\ (17)\ (18)\ (21\ 22)\cdots .$$

Thus y fixes 16, 17. Hence y fixes $\Delta(16, 17)$ and so $y=(7)\ (14\ 15)\cdots$, $(14)\ (7\ 15)\cdots$ or $(15)\ (7\ 14)\cdots$. If $y=(7)\ (14\ 15)\cdots$, then $I(xy)\supseteq\{16, 17, 18, 21, 22, 7, 14, 15\}$ and $x\neq y$, which is a contradiction. Hence $y=(14)\ (7\ 15)\cdots$ or $(15)\ (7\ 14)\cdots$.

First assume that $y=(14)\ (7\ 15)\cdots$. Then

$$by = (1)\ (2\ 4)\ (16\ 17)\ (14)\ (7)\ (15)\cdots .$$

Thus by is of order two. Furthermore $\Delta(1, 2)^y=\Delta(1, 4)$ and $\Delta(2, 4)^y=\Delta(2, 4)$. Hence

$$y = (1)\ (2\ 4)\ (8\ 10)\ (3\ 11)\ (5\ 9)\ (12)\ (6\ 13)\ (14)\ (7\ 15)\ (16)\ (17)\ (18)\ (21\ 22)\ (20)\ (19\ 23) .$$

Then

$$xy = (16)\ (17)\ (18)\ (21)\ (22)\ (7\ 15\ 14)\ (1\ 4\ 2)\ (3\ 11\ 12)\ (19\ 23\ 20)\ (5\ 13\ 8)\ (6\ 9\ 10).$$

Next assume that $y=(15)\,(7\ 14)\cdots$. Since

$$b^a = (1)\,(2)\,(4)\,(9)\,(3\ 8)\,(11)\,(5\ 10)\,(13)\,(6\ 12)\,(15)\,(7\ 14)\,(19\ 21)\,(18\ 23)\,(20\ 22)\,(16\ 17)\,,$$
$$b^a y = (1)\,(2\ 4)\,(16\ 17)\,(15)\,(7)\,(14)\cdots .$$

Thus $b^a y$ is of order two. Hence from the same reason as above,

$$y = (1)\,(2\ 4)\,(9\ 11)\,(3\ 10)\,(5\ 8)\,(13)\,(6\ 12)\,(15)\,(7\ 14)\,(16)\,(17)\,(18)\,(21\ 22)\,(23)\,(19\ 20)\,.$$

Then

$$xy = (16)\,(17)\,(18)\,(21)\,(22)\,(7\ 14\ 15)\,(1\ 4\ 2)\,(3\ 10\ 13)\,(19\ 20\ 23)\,(5\ 12\ 9)\,(6\ 8\ 11)\,.$$

Now we show that $\Delta(21, 22)=\{18\}$. Assume that y is of the first form. We have already proved that for $I(\langle a, b\rangle)=\{1, 2, 4\}$ and $I(ab)-I(\langle a, b\rangle)=\{16, 17\}$, $\Delta(16, 17)=\{1\}$, $\{2\}$, $\{4\}$ or Γ_4 and $\Delta(16, 1)=\{17\}$, $\Delta(1, 2)$, $\Delta(1, 4)$, Λ_1 or Λ_2, where Γ_4 is the $\langle a, b\rangle$-orbit of length three and different from $\Delta(1, 2)$, $\Delta(1, 4)$ and $\Delta(2, 4)$ and $\Lambda_1 \cup \Lambda_2$ is the $\langle a, b\rangle$-orbit of length six. Since $I(\langle x, y\rangle)=\{16, 17, 18\}$ and $I(xy)-I(\langle x, y\rangle)=\{21, 22\}$, by the same argument for $\langle x, y\rangle$ as is udsed for $\langle a, b\rangle$, $\Delta(21, 22)$ is $\{16\}$, $\{17\}$, $\{18\}$ or the $\langle x, y\rangle$-orbit wihch is of length three and different from $\Delta(16, 17)$, $\Delta(16, 18)$ and $\Delta(17, 18)$, and $\Delta(21, 18)=\{22\}$, $\Delta(18, 16)$, $\Delta(18, 17)$, $\{5, 13, 6\}$ or $\{6, 9, 10\}$, where $\Delta(16, 17)=\{7, 14, 15\}$, $\Delta(16, 18)=\{1, 2, 4\}$ and $\{5, 13, 6\} \cup \{6, 9, 10\}$ is the $\langle x, y\rangle$-orbit of length six. In the plane $P(x)$ $\Delta(16, 17)\ni 7$, $\Delta(16, 18)\ni 4$ and $\Delta(7, 4)\ni 3$. Hence $\Delta(17, 18)\ni 3$. Since $(17, 18)^{xy}=\Delta(17, 18)$, $\Delta(17, 18)=\{3, 11, 12\}$. Thus $\Delta(21, 22)=\{16\}$, $\{17\}$, $\{18\}$ or $\{19, 23, 20\}$ and $\Delta(21, 18)=\{22\}$, $\{1, 2, 4\}$, $\{3, 11, 12\}$, $\{5, 8, 13\}$ or $\{6, 9, 10\}$. On the other hand $\Delta(21, 22)=\Delta(21, 18)^{(ab)^2}=\{18\}$, $\{1, 2, 4\}$, $\{8, 5, 13\}$, $\{10, 9, 6\}$ or $\{12, 3, 11\}$. Thus $\Delta(21, 22)=\{18\}$. Next assume that y is of the second form. Then $\Delta(16, 17)$ and $\Delta(16, 18)$ are the same set as above and $\Delta(17, 18)=\{3, 10, 13\}$. Hence similarly $\Delta(21, 22)=\{18\}$.

Assume that y is of the first form. The proof for the second form of y is similar. Since $\Delta(21, 22)=\{18\}$, there is an involution u fixing 21, 22, 18 and commuting with x. Then u fixes $I(x)$. If u fixes 16 or 17, then xy fixes $I(u)$, contrary to (3.4) Thus $u=(16\ 17)\cdots$ or $(16\ i)\,(17\ j)\cdots$, $i, j\in\{3, 4, 7, 19\}$.

First assume that $u=(21)\,(22)\,(18)\,(16\ 17)\cdots$. Then by the incidence structure of $P(x)$,

$$u = (21)\,(22)\,(18)\,(16\ 17)\,(3\ 4)\,(17)\,(19)\cdots .$$

Furthermore $\Delta(16, 17)^u=\Delta(16, 17)$. Hence $\{14, 15\}^u=\{14, 15\}$. If u fixes $\{14, 15\}$ pointwise, then $I(u)\cap I(u^{ab})=\{21, 22, 18, 7, 14, 15\}$, contrary to (vi) of (3.1). If u has a 2-cycle $(14\ 15)$, then

$$au = (16)\,(17)\,(7)\,(14)\,(15)\,(18\ 19)\cdots ,$$

Thus au is an involution. Then ab fixes the subset $\{16, 17, 7, 14, 15\}$ of $I(au)$. Hence ab fixes $I(au)$, contrary to (3.4).

Next suppose that $u=(21)\,(22)\,(18)\,(16\ i)\,(17\ j)\cdots$, $i, j\in\{3, 4, 7, 19\}$. Then by the incidence structure of $P(x)$, $u=(21)\,(22)\,(18)\,(16\ 3)\,(17\ 4)\,(7)\,(19)\cdots$ or $(21)\,(22)\,(18)\,(16\ 4)\,(17\ 3)\,(7)\,(19)\cdots$. Then $\Delta(16, 17)^u=\Delta(3, 4)$. This is a contradiction since $|\Delta(16, 17)|=3$ and $|\Delta(3, 4)|=1$.

(iv) Assume that $\Delta(16, 17)=\{1\}$. The proof in the case $\Delta(16, 17)=\{2\}$ or $\{4\}$ is similar. We have proved that $\Delta(16, 1)=\{17\}$, Γ_1, Γ_2, Λ_1 or Λ_2, $\Delta(16, 2)=\{17\}$, Γ_1, Γ_3, Λ_1 or Λ_2 and $\Delta(16, 4)=\{17\}$, Γ_2, Γ_3, Λ_1 or Λ_2 Hence by assumption, $\Delta(16, 1)=\{17\}$, $\Delta(16, 2)\neq\{17\}$ and $(16, 4)\neq\{17\}$. Furthermore since $\Delta(17, 1)=\Delta(16, 1)^a$, $\Delta(17, 1)=\{16\}$. Suppose that $\Delta(16, 2)=\Gamma_3$. Since $\Delta(17, 2)=\Delta(16, 2)^a$, $\Delta(17, 2)=\Gamma_3$. Thus $\Delta(2, 6)=\{4, 16, 17\}$. Hence there is an involution u fixing 2, 6, 16. Then since $\Delta(16, 2)=\Gamma_3$,

$$u=(16)\,(2)\,(6)\,(12\ 13)\,(4\ 17)\cdots.$$

Then by (3.18), $(ab)^u=(ab)^{-1}$. Hence u fixes 1. Thus u fixes $\Delta(16, 1)$. This is a contradiction since $\Delta(16, 1)=\{17\}$ and $17^u\in\Delta(16, 1)$. Thus $\Delta(16, 2)\neq\Gamma_3$. In the same way $\Delta(16, 4)\neq\Gamma_2$. Therefore $\Delta(16, 2)=\Gamma_1$, Λ_1 or Λ_2 and $\Delta(16, 4)=\Gamma_2$, Λ_1 or Λ_2.

Now let x be an involution fixing 16, 17 and commuting with a. Since $\Delta(16, 17)=\{1\}$, x fixes 1. If x fixes 2 or 4, then ab fixes at least four points of $I(x)$ and so fixes $I(x)$, contrary to (3.4). Next if x fixes 3 or 5, then by the incidence structure of $P(a)$, x fixes 2 or 4. This is a contradiction. Thus x fixes three points 1, 6, 7 of $I(a)$. Hence $x=(2\ 4)\,(3\ 5)$, $(2\ 3)\,(4\ 5)$ or $(2\ 5)\,(3\ 4)$ on $I(a)-\{1, 6, 7\}$. First we determine the form of x in each cases.

(iv.i) Assume that $x=(16)\,(17)\,(1)\,(6)\,(7)\,(2\ 4)\,(3\ 5)\cdots$. Then $\Delta(1, 2)^x=\Delta(1, 4)$. Hence $\{8, 9\}^x=\{10, 11\}$. If $x=(8\ 10)\,(9\ 11)\cdots$, then

$$abx=(1)\,(2\ 4)\,(16)\,(17)\,(3\ 11\ 8\ 5\ 9\ 10)\cdots,$$

contrary to (vi) of (3.1). Hence $x=(8\ 11)\,(9\ 10)\cdots$. Then

$$abx=(1)\,(2\ 4)\,(16)\,(17)\,(3\ 10)\,(8\ 5)\,(9\ 11)\cdots.$$

This shows that abx is of order two and so $(ab)^x=(ab)^{-1}$. Hence

$$x=(1)\,(2\ 4)\,(3\ 5)\,(8\ 11)\,(9\ 10)\,(6)\,(12\ 13)\,(7)\,(14\ 15)\,(16)\,(17)\cdots.$$

Since $|I(x)|=7$, x fixes exactly two points of $\{18, 19, \cdots, 23\}$. Without loss we may assume that x fixes 18, 19. Then

$$x=(1)\,(2\ 4)\,(3\ 5)\,(8\ 11)\,(9\ 10)\,(6)\,(12\ 13)\,(7)\,(14\ 15)\,(16)\,(17)\,(18)\,(19)\,(21\ 22)\,(20\ 23),$$

which is denoted by x_1.

(iv.ii) Assume that $x=(16)(17)(1)(6)(7)(2\ 3)(4\ 5)\cdots$. Then $\{1\}$ and $\{16, 17\}$ are $\langle a, b, x\rangle$-orbits, and the remaining orbits are unions of $\{2\}\cup\Gamma_1$, $\{4\}\cup\Gamma_2$, Γ_3, Γ_4 and $\Lambda_1\cup\Lambda_2$. If $\{8, 9\}^x=\{12, 13\}$ or $\{14, 15\}$ then $\{2\}\cup\Gamma_1\cup\Gamma_3$ or $\{2\}\cup\Gamma_1\cup\Gamma_4$ is a $\langle a, b, x\rangle$-orbit of length seven, contrary to (3.11). Thus $\{8, 9\}^x\neq\{12, 13\}$, $\{14, 15\}$. In the same way $\{10, 11\}^x\neq\{12, 13\}$, $\{14, 15\}$. Next if $\{8, 9, 10, 11\}^x\subseteq\Lambda_1\cup\Lambda_2$, then $\{2, 4\}\cup\Gamma_1\cup\Gamma_2\cup\Lambda_1\cup\Lambda_2$ or $\{2, 4\}\cup\Gamma_1\cup\Gamma_2\cup\Lambda_1\cup\Lambda_2\cup\Gamma_i$ $(i=3$ or $4)$ is a $\langle a, b, x\rangle$-orbit. Then $|\{2, 4\}\cup\Gamma_1\cup\Gamma_2\cup\Lambda_1\cup\Lambda_2|=14$ and $|\{2, 4\}\cup\Gamma_1\cup\Gamma_2\cup\Lambda_1\cup\Lambda_2\cup\Gamma_i|=17$, contrary to (3.11) and the assumption (4) respectively. Thus $\{8, 9\}^x=\{8, 9\}$, $\{8, 9\}^x=\{10, 11\}$ or $\{10, 11\}^x=\{10, 11\}$.

First assume that $\{8, 9\}^x=\{8, 9\}$. If x fixes $\{8, 9\}$ pointwise, then

$$bx=(1)(8)(2\ 3\ 9)(16\ 17)\cdots,$$

contrary to (vi) of (3.1). Hence x has a 2-cycle $(8\ 9)$. Then

$$ax=(1)(2\ 3)(4\ 5)(6)(7)(8)(9)(16\ 17)\cdots,$$
$$bax=(1)(2\ 3\ 9)(8)(16)(17)(11\ 4\ 5\cdots)(13\ 6\cdots)(15\ 7\cdots)\cdots.$$

Thus bax is of order three and so $ax=(11)(10)(12\ 13)(14\ 15)\cdots$ or $(11)(10)(12\ 15)(13\ 14)\cdots$. If ax is of the first form, then x fixes $\{1, 6, 7, 16, 17, 12, 13, 14, 15\}$ pointwise, contrary to (i) of (3.1). Thus ax is of the second form. Hence

$$x=(1)(2\ 3)(4\ 5)(6)(7)(8\ 9)(10\ 11)(12\ 14)(13\ 15)(16)(17)\cdots.$$

Then

$$bx=(1)(2\ 3\ 8\ 9)(4\ 5\ 10\ 11)(12\ 14)(6\ 15\ 7\ 13)(16\ 17)\cdots.$$

This shows that bx is of order four. Furthermore since x fixes exactly two points of $\{18, 19, \cdots, 23\}$, we may assume that x is of the from

$$x=(1)(2\ 3)(4\ 5)(6)(7)(8\ 9)(10\ 11)(12\ 14)(13\ 15)(16)(17)(18)(19)(20\ 22)(21\ 23),$$

which is denoted by x_2.

Next assume that $\{8, 9\}^x=\{10, 11\}$. Then $x=(8\ 11)(9\ 10)\cdots$ or $(8\ 10)(9\ 11)\cdots$. If x is of the first form, then

$$abx=(1)(2\ 3\ 4\ 5\ 8)(16)(17)\cdots,$$

contrary to (iv) of (3.1), Thus x is of the second form. Then

$$ax=(1)(2\ 3)(4\ 5)(6)(7)(16\ 17)(8\ 11)(9\ 10)\cdots,$$
$$bax=(1)(2\ 3\ 10\ 9)(4\ 5\ 8\ 11)(16)(17)\cdots,$$
$$(bax)^2=(1)(2\ 10)(3\ 9)(4\ 8)(5\ 11)(16)(17)\cdots.$$

Thus bax is of order four and $(bax)^2$ is an involution commuting with b and ax.

Since $I(b)=\{1, 2, 4, 8, 10, 12, 14\}$, $(bax)^2$ fixes $\{12, 14\}$ pointwise. Since $|I(bax)|=3$, $bax=(1\ 2\ 14)\cdots$ and so $ax=(12\ 14)\cdots$. Hence

$$x=(1)(2\ 3)(4\ 5)(6)(7)(16)(17)(8\ 10)(9\ 11)(12\ 15)(13\ 14)\cdots.$$

This implies that x and ax fix exactly two and four points of $\{18, 19, \cdots, 23\}$ respectively. Hence we may assume that x is of the form

$$x=(1)(2\ 3)(4\ 5)(6)(7)(16)(17)(8\ 10)(9\ 11)(12\ 15)(13\ 14)(18)(19)(20\ 21)(22\ 23),$$

which is denoted by x_3.

Finally assume that $\{10, 11\}^x=\{10, 11\}$. If x fixes $\{10, 11\}$ pointwise, then

$$bx=(1)(4\ 5\ 11)(10)(16\ 17)\cdots,$$

contrary to (v) of (3.1). Thus x has a 2-cycle (10 11). Then

$$ax=(1)(2\ 3)(4\ 5)(6)(7)(16\ 17)(10)(11)\cdots,$$
$$bax=(1)(4\ 5\ 11)(10)(16)(17)(9\ 2\ 3\cdots)\cdots.$$

This shows that bax is of order three. Hence ax fixes 9 and so 8. Thus $x=x_2$.

(iv.iii) Assume that $x=(16)(17)(1)(6)(7)(2\ 5)(3\ 4)\cdots$. Then $\{1\}$ and $\{16, 17\}$ are $\langle a, b, x\rangle$-orbits and the remaining orbits are unions of $\{2\}\cup\Gamma_2$, $\{4\}\cup\Gamma_1$, Γ_3, Γ_4 and $\Lambda_1\cup\Lambda_2$. Hence by the same argument as in the case (iv.ii), x is one of the following elements:

$$x_4=(1)(2\ 5)(3\ 4)(6)(7)(8\ 9)(10\ 11)(12\ 14)(13\ 15)(16)(17)(18)(19)(20\ 22)(21\ 23),$$
$$x_5=(1)(2\ 5)(3\ 4)(6)(7)(8\ 10)(9\ 11)(12\ 15)(13\ 14)(16)(17)(18)(19)(20\ 21)(22\ 23).$$

We have proved that $\Delta(1, 2)=\Gamma_1$, Λ_1 or Λ_2 and $\Delta(16, 4)=\Gamma_2$, Λ_1 or Λ_2. In the following we treat these cases separatly and show that there is no group which satisfies the assumption of Lemma 1 and contains $\langle a, b, x_i\rangle$, $1<i<5$.

(iv.iv) Assume that $\Delta(16, 2)=\Lambda_1$. The proof in the case $\Delta(16, 2)=\Lambda_2$, $\Delta(16, 4)=\Lambda_1$ or $\Delta(16, 4)=\Lambda_2$ is similar.

(iv.iv.i) Assume that there is an involution x_1. Then $\Delta(16, 4)=\Delta(16, 2)^{x_1}=\Lambda_1$. Hence $\Delta(16, 18)\supseteq\{2, 4\}$. Moreover in the plane $P(x_1)$ $\Delta(1, 6)\ni 7$ and $\Delta(1, 16)\ni 17$. Hence $(16, 18)\ni 6$ or 7. Thus $\Delta(16, 18)=\{2, 4, 6\}$ or $\{2, 4, 7\}$. If $\Delta(16, 18)=\{2, 4, 6\}$, then there is an involution u fixing 16, 18, 2. Since $\Delta(16, 2)=\Lambda_1$,

$$u=(16)(18)(2)(21\ 22)(4\ 6)\cdots.$$

Then $\Delta(16, 13)=\Delta(16, 4)^{uab}=\Lambda_1^{uab}=\Lambda_1$. Hence $\Delta(16, 18)\ni 13$, which is a contradiction. Similarly $\Delta(16, 18)\neq\{2, 4, 7\}$. Thus the proof in the case (iv.iv.i) is complete.

(iv.iv.ii) First assume that there is an involution x_2. Then $\Delta(16, 3)=\Delta(16, 2)^{x_2}=\Lambda_1^{x_2}=\{18, 20, 23\}$. Hence $\Delta(16, 18)\supseteq\{2, 3\}$. Moreover in the plane $P(x_2)$ $\Delta(16, 18)\ni 6$ or 7. Hence $\Delta(16, 18)=\{2, 3, 6\}$ or $\{2, 3, 7\}$. If $\Delta(16, 18)=\{2, 3, 6\}$, then there is an involuion u fixing 16, 18, 2. Since $\Delta(16, 2)=\Lambda_1$

$$u=(16)\,(18)\,(2)\,(21\ 22)\,(3\ 6)\cdots .$$

On the other hand $\Delta(16, 22)=\Delta(16, 18)^{ab}=\{2, 9, 13\}$ and $\Delta(16, 21)=\Delta(16, 22)^{ab}=\{2, 8, 12\}$. Then since $\Delta(16, 22)^u=\Delta(16, 21)$, $\{9, 13\}^u=\{8, 12\}$. If $u=(8\ 9)\,(12\ 13)\cdots$, then

$$abu=(16)\,(2)\,(22)\,(9)\,(6\ 12\ 3\ 8)\cdots ,$$

contrary to (iii) of (3.1). Next if $u=(8\ 13)\,(9\ 12)\cdots$, then

$$au=(2)\,(3\ 6)\,(8\ 12)\,(9\ 13)\cdots .$$

Thus au is of order two. Hence $I(u)\supseteq\{2, 18, 16\}^{\langle a\rangle}=\{2, 18, 19, 16, 17\}$. Thus $I(u)\cap I(x_2)\supseteq\{16, 17, 18, 19\}$ and $I(u)\neq I(x_2)$, contrary to (vi) of (3.1). Similarly $\Delta(16, 18)\neq\{2, 3, 7\}$. Thus there is no group containing x_2.

Next assume that there is the involution x_4. Since x_4 is the same permutation as x_2 with 3 and 5 interchanged, by the same argument as is used for x_2 there is no group containing x_4.

(iv.iv.iii) First assume that there is the involution x_3. Since

$$abx_3=(1)\,(2\ 3\ 11\ 8)\,(4\ 5\ 9\ 10)\,(6\ 14\ 7\ 12)\,(13\ 15)\,(16)\,(17)\,(18\ 23\ 19\ 21)\,(20\ 22),$$

$\Delta(16, 11)=\Delta(16, 2)^{(abx_3)^2}=\{19, 22, 23\}$ and $\Delta(16, 8)=\Delta(16, 11)^{abx_3}=\{21, 10, 19\}$. Hence $\Delta(16, 19)\supset\{11, 18\}$. Since x_3 fixes $\Delta(16, 19)$, $\Delta(16, 19)\supseteq\{11, 8\}^{\langle x_3\rangle}=\{11, 9, 8, 10\}$, which is a contradiction.

Next assume that there is the involution x_5. Since x_5 is the same permutation as x_3 with 3 and 5 interchanged, by the same argument as is used for x_3 there is no group containing x_5.

(iv.v) Assume that $\Delta(16, 2)=\Gamma_1$ and $(16, 4)=\Gamma_2$. Then $\Delta(17, 2)=\Delta(16, 2)^a=\Gamma_1$. Hence $\Delta(2, 3)=\{1, 16, 17\}$ and consequently there is an involution u fixing 2, 3, 16. Since $\Delta(16, 2)=\Gamma_1$,

$$u=(16)\,(2)\,(3)\,(8\ 9)\,(1\ 17)\cdots .$$

Then since $\Delta(16, 2)=\Gamma_1$, $(ab)^u=(ab)^{-1}$ by (3.18). Hence u fixes 4. Then

$$au=(2)\,(4)\,(3)\,(9)\,(1\ 17\ 16).\cdots .$$

Thus au is of order three. In the cases (i), (ii) and (iii) we have proved that for $I(\langle a, b\rangle)=\{1, 2, 4\}$ and $I(ab)-I(\langle a, b\rangle)=\{16, 17\}$, $\Delta(16, 17)=\{1\}$, $\{2\}$ or $\{4\}$. Since $I(\langle a, u\rangle)=\{2, 3, 4\}$ and $I(ab)-I(\langle a, b\rangle)=\{8, 9\}$, by the same argument for $\langle a, u\rangle$ as is used for $\langle a, b\rangle$ $\Delta(8, 9)=\{2\}$, $\{3\}$ or $\{4\}$. If $\Delta(8, 9)=\{2\}$, then $\Delta(2, 8)\ni 9$. Since b fixes $\Delta(2, 8)$, $\Delta(2, 8)\ni 9^b=3$. Thus $\Delta(2, 8)\supset\{3, 9\}$ and so $|\Delta(2, 8)|=3$. However since au fixes $\Delta(2, 8)$ and $\Delta(2, 8)\cap I(au)=\{3, 9\}$, $|\Delta(2, 8)|\neq 3$. Thus we have a contradiction. In the same way $\Delta(8, 9)\neq\{4\}$. Thus $\Delta(8, 9)=\{3\}$.

Now there is an involution v fixing 8, 9 and commuting with a. Then v fixes 3 and $I(a)$. If v fixes 2 or 4, then au fixes at least four points of $I(v)$. Hence au fixes $I(v)$, contrary to (3.4). Hence $v=(8)(9)(3)(5)(6)\cdots$. Since $av=va$ and $I(\langle a, v\rangle)=\{3, 5, 6\}$, by (3.19) $\Delta(3, 5)=\{6\}$. If there is the involution x_i $(2\leq i\leq 5)$, then $\Delta(3, 5)^{x_i}=\Delta(2, 4)$. This is a contradiction since $|\Delta(2, 4)|=3$. Next suppose that there is the involution x_1. Since u fixes $\{16, 4\}$ pointwise and $\Delta(16, 4)=\Gamma_2$, $u=(5)(10\ 11)\cdots$, $(10)(5\ 11)\cdots$ or $(11)(5\ 10)\cdots$. If $u=(5)(10\ 11)\cdots$, then

$$au=(2)(4)(3)(8)(9)(5)(10)(1\ 17\ 16)\cdots,$$

contrary to (ii) of (3.1). Next suppose that $u=(10)(5\ 11)\cdots$. Then

$$bu=(2)(4)(1\ 17\ 16)(3\ 8\ 9)(5)(10)(11)\cdots.$$

Thus bu is of order three. Then since $\Delta(2, 4)^u=\Delta(2, 4)$ and $|I(au)|=|I(bu)|=5$,

$$u=(2)(4)(3)(8\ 9)(1\ 17)(16)(10)(5\ 11)(13)(6\ 12)\cdots.$$

Thus $\Delta(5, 8)=\Delta(3, 5)^{ux_1}=\{6\}^{ux_1}=\{13\}$. Since v fixes 5, 8, v fixes 13 and so 12. Thus $I(v)\cap I(v^{x_1})=\{3, 5, 6, 12, 13\}$, contrary to (vi) of (3.1). Finally suppose that $u=(11)(5\ 10)\cdots$. Then

$$b^a u=(2)(4)(1\ 17\ 16)(3\ 9\ 8)(5)(10)(11)\cdots.$$

Hence by the same argument as above we have a contradiction.

Thus the proof of Lemma 1 is complete.

REMARK. Let G be one of the groups of Lemma 1. Then it is not difficult to prove that G satisfies the assumptions of Lemma 1.

OSAKA KYOIKU UNIVERSITY

References

[1] J.H.Conway: *Three lectures on exceptional groups*, (in Finite Simple Groups), Academic Press, London and New York, 1971, 215–247.

[2] T.Oyama: *On multiply transitive groups* VII, Osaka J. Math. **5** (1968), 155–164.

[3] T.Oyama: *On multiply transitive groups* VIII, Osaka J. Math. **6** (1969), 315–319.

[4] T.Oyama: *On multiply transitive groups* X, Osaka J. Math. **8** (1971), 99–130.

[5] C.C. Sims: *Computational methods in the study of permutation groups*, (in Computational Problems in Abstract Algebra), Pergamon Press, London, 1970, 169–183.

[6] H.Wielandt: Finite Permutation Groups, Academic Press, New York, 1964.