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PERFECT CATEGORIES III

(HEREDITARY AND QF-3 CATEGORIES)

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Recently the author has defined perfect or semi-artinian Grothendieck categories with some assumptions [8], as a generalization of cagegories of modules in [1].

Further he has generalized essential results in [6] to such categories [9]. This note is a continuous work to give a generalizations of results in [3], [4] and [5].

Let R be a ring with identity. R.M. Thrall defined a QF-3 algebra in [3] and many authors defined QF-3 rings and studied them (cf. [10]).

R is called right QF-3 if R has a minimal a fithful right R-module and R is called right QF-3^+ if the injective hull $E(R_R)$ is projective, (see [2]).

We generalize those concepts to semi-perfect Grothendieck categories $\mathfrak{A}$ with generating set of finitely generated objects, (which are equivalent to group valued functor categories $(\mathfrak{C}^0, Ab)$ by [8], Theorem 3).
We shall completely determin structures of hereditary (more weakly locally PP) and perfect QF-3 (resp, QF-3^+) or semi-perfect and semi-artinian QF-3 (resp. QF-3^+, however this is a case of QF-3) categories $\mathfrak{A}$. Furthermore, we shall show that $\mathfrak{A}$ is equivalent to product of $\mathfrak{A}_\alpha$ and $\mathfrak{A}_\alpha$ is the full subcategory $\mathfrak{M}_S^+$ [1], where S is the ring of upper (resp. lower) tri-angular matrices of a division ring over a well ordered set I, almost all of whose entries are zero, such that if $\mathfrak{A}$ is QF-3 I has the last element (resp. if $\mathfrak{A}$, is semi-artinian QF-3^+, then I has the last element and hence, $\mathfrak{A}$ is QF-3) and vice versa with some restrictions. Those results are generalizations of [4] and [5].

1. Preliminary results

Let $\mathfrak{A}$ be a Grothendieck category with generating set of finitely generated objects. If every object (resp. finitely generated object) has a projective cover, then $\mathfrak{A}$ is called *perfect* (resp. *semi-perfect*). On the other hand, if every non-zero object has the non-zero socle, $\mathfrak{A}$ is called *semi-artinian*.

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1) see §1.

If $\mathfrak{A}$ is semi-perfect, then $\mathfrak{A}$ has a generating set of completely indecomposable projective $\{P_\alpha\}_I$. Let $(\{P_\alpha\}^\circ, Ab)$ be the additive contravariant functor category of the pre-additive category $\{P_\alpha\}$ to the category Ab of abelian groups. Put $R=\sum_{\alpha,\beta\in I}\oplus[P_\alpha, P_\beta]$. Then R is called the *induced ring* from $\mathfrak{A}$ by $\{P_\alpha\}$. By e_α we shall denote idempotents 1_{P_α} in R. Let $\mathfrak{M}_R$ be the category of all right R-modules. By $\mathfrak{M}_R^+$ we denote the full subcategory of $\mathfrak{M}_R$ whose objects consist of all M such that $MR=M$. Then

Theorem A ([8], Theorem 3). *Let $\mathfrak{A}$ be as above Then the following are equivalent.*

1) $\mathfrak{A}$ *is semi-perfect.*
2) $\mathfrak{A}\approx(\{P_\alpha\}^\circ, \mathrm{Ab})$.
3) $\mathfrak{A}\approx\mathfrak{M}_R^+$.

In this note, we only consider a semi-perfect category $\mathfrak{A}$ and hence, $\mathfrak{A}$ will be identified with $(\{P_\alpha\}^\circ, Ab)$ or $\mathfrak{M}_R^+$ in the following. We note in this case $e_\alpha R$ corresponds to P_α and $e_\alpha Re_\beta\approx[P_\beta, P_\alpha]$.

We shall make use of same notations in [8] and [9] without further comments and categorical terminologies in [11]. Rings in this note do not contain identities in general.

2. Locally PP-categories

Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with generating set of finitely generated. If $\{P_\alpha\}$ and $\{Q_\beta\}$ are generating sets of $\mathfrak{A}$ such that P_α and Q_β are completely indecomposable and projetve, then P_α is isomorphic to some Q_β and vice versa by Krull-Remak-Schmidt's theorem. Let R be the induced ring from $\mathfrak{A}$ by $\{P\}_\alpha$, $R=\sum\oplus[P_\alpha, P_\beta]$. If fR is projective in $\mathfrak{M}_R^+$ for any α and β any element f in $[P_\alpha, P_\beta]$, $\mathfrak{A}$ is called a *locally* (*right*) *PP-category*, (we called it "partially" in [3]).

This is equivalent to a fact that every functor T_f in $(\{P_\alpha\}^\circ, Ab)$ defined by $T_f(P_\gamma)=fRe_\gamma$ is representative for every $f\in[P_\alpha, P_\beta]$. We define similarly a left PP-category.

We can easily see from the following lemma that right PP-categories are also left PP-categories and that this defintion dose not depend on $\{P_\alpha\}$.

Lemma 1. *Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with a generating set $\{P_\alpha\}$ as above. Then $\mathfrak{A}$ is locally PP if and only if any $f\in[P_\alpha, P_\beta]$ is zero or monomorphic*, (cf. [9], Proposition 3).

Proof. We assume that $\mathfrak{A}$ is locally PP and $0\neq f\in[P_\alpha, P_\beta]$. Since $fe_\alpha=f$, $0\leftarrow fR\overset{\times f}{\leftarrow} e_\alpha R$ is exact. Further, $e_\alpha R$ is indecomposable, and hence, $fR\overset{\times f}{\approx} e_\alpha R$.

Put $K=\mathrm{Ker} f$ and $i: K\to P_\alpha$. If $i\neq 0$, there exists P_γ and $h\in[P_\gamma, K]$ such that $0\neq ih\in[P_\gamma, P_\alpha]\subseteq R$. Then $0=fih=fe_\alpha ih$ and $e_\alpha ih\in e_\alpha R$. Hence, $ih=e_\alpha ih=0$, which is a contradiction. Therefore, f is monomorphic. Conversely, if f is monomorphic, then a mapping $\psi: fR\to e_\alpha R(\psi(fr)=e_\alpha r)$ is isomorphic. Hence, fR is projective in $\mathfrak{M}_R^+$.

As an analogy of Theorem 4 in [9], we have

Theorem 1 ([9]). *Let $\mathfrak{A}$ be a semi-perfect Grothendieck category with generating set of finitely generated object. Then $\mathfrak{A}$ is locally PP and perfect (resp. semi-artinian) if and only if $\mathfrak{A}$ is equivalent to $[I, \mathfrak{A}_i]^r$ (resp. $[I, \mathfrak{A}_i]^l$)*[2] *with functors T_{ij} such that ψ_{kji}: $T_{kj}(B)\to T_{ki}(P)$ for $k>j>i$ (resp. $k<j<i$) is monomorphic, for any minimal object B in $T_{ji}(P)$ and $P\in\mathfrak{A}_i$, where $\mathfrak{A}_i$'s are semi-simple categories with generating sets.*

Proof. We assume that $\mathfrak{A}$ is locally PP and $\{P_\alpha\}$ is a generating set of completely indecomposable projectives. Making use of Lemma 1 and the proof of Theorem 4 in [9] we know that $\mathfrak{A}$ is equivalent to $[I, \mathfrak{A}_i]^r$ (resp. $[I, \mathfrak{A}_i]^l$) and that $\{\boldsymbol{P}_\alpha^{(i)}=\tilde{S}_i(P_{i\alpha})\}$[2] (resp. $\{S_i(P_{i\alpha})\}$) is a generating set in $[I, \mathfrak{A}_i]^r$ (resp. $[I, \mathfrak{A}_i]^l$), where $\{P_{i\alpha}\}$ is a generating set of $\mathfrak{A}_i$ and $P_{i\alpha}$ is minimal. Since $f\in[\boldsymbol{P}_\alpha^{(i)}, \boldsymbol{P}_\beta^{(j)}]$ is monmomorphic by Lemma 1, we have the conditions in the theorem. The converse is also clear from the structure of $[I, \mathfrak{A}_i]^r$ (resp. $[I, \mathfrak{A}_i]^l$) and Lemma 1.

REMARK. If we replace a minimal objects B in the above condition by any finite coproduct of B_{α_i}, it is equivalent to the condition $(*)-1)$ in Theorem 3 in [9]. Hence, this fact gives us the defference between semi-hereditaty and locally PP. We have immediately from Lemma 1. [9], Propositions 3 and 5 and their proofs

Theorem 2. *Let $\mathfrak{A}$ be as in Theorem 1 and $\{P_\alpha\}$ a generating set of completely indecomposable projectives. If $\mathfrak{A}$ is locally PP, then the following are equivalent.*

1) *All P_α are J-nilpotent.*
2) $1\mathrm{L}(P_\alpha)<\infty$ *for all α.*
3) *$\mathfrak{A}$ is semi-artinian.*

Futhermore, the following are equivalent.

1) $\mathrm{rL}(P_\alpha)<\infty$ *for all α.*
2) *$\mathfrak{A}$ is perfect,* (cf. [9], Theorem 6).

3. *QF*-3 categories

Let $\mathfrak{A}$ be a Grothendieck category with generating set of projectives $\{P_\alpha\}$. An object C in $\mathfrak{A}$ is called *faithful* if for any non-zero morphism $f: P_\alpha\to P_\beta$, there exists $g\in[P_\beta, C]$ such that $gf\neq 0$. Let $\{Q_\beta\}$ be another generating set of projec-

2) see [8], §3.

tives and $f'\neq 0\in[Q_\varepsilon, Q_\delta]$. Since $Q_\varepsilon\oplus Q_\varepsilon'=\sum_J\oplus P_\alpha$ and $Q_\delta\oplus Q_\delta'=\sum_{J'}\oplus P_\beta$, we have a non-zero morphim f: $\sum_J\oplus P_\alpha\to\sum_{J'}\oplus P_\beta$ such that $f\,|\,Q_\varepsilon=f'$ and $f\,|\,Q_\varepsilon'=0$. Hence, there exist α, β such that $(p_\beta f\,|\,P_\alpha)\neq 0$, where p_β is the projection of $\sum_{J'}\oplus P_\beta$ to P_β. Then we have $g'\in[P_\beta, C]$ such that $g'(p_\beta f\,|\,P_\alpha)\neq 0$. Hence, $g'p_\beta f\neq 0$. Let i_{Q_ε} and i_{Q_δ} be inclusions. Peut $g'p_\beta i_{Q_\delta}=g\in[Q_\delta, C]$. Then $g'p_\beta f i_{Q_\varepsilon}=g'p_\beta i_{Q_\delta}f'=gf'$ and $\mathrm{Ker}\,f=Q_\varepsilon'$. Therefore, $gf'\neq 0$. Thus, we have shown that the faithfulness of C dose not depend on generating sets of projectives.

Let $(\mathfrak{C}^0, Ab)$ be the contravariant additive functor category, where $\mathfrak{C}$ is the small pre-additive category $\{P_\alpha\}$. Then $\mathfrak{A}$ is equivalent to $(\mathfrak{A}^0, Ab)$. Hence C is faithful and only if the corresponding functor in the above is a faithful functor. Furthermore, $(\mathfrak{C}^0, Ab)$ is eqivalent to $\mathfrak{M}_R^+$, where R is the induced ring from $\{P_\alpha\}$. Then faithful functors correspond to faithful modules in $\mathfrak{M}_R^+$.

An object M is called a *minimal faithful* if M is faithful and every faithful object is a coretract of M. According to R.M. Thrall [13], we call $\mathfrak{A}$ QF-3 if $\mathfrak{A}$ contains a minimal faithful object M or equivalently, if $\mathfrak{M}_R^+$ has a minimal faithful module.

From now on we shall assume that $\mathfrak{A}$ is a Grothendieck category with generating set of small projectives P_α. Further, we shall assume that $\mathfrak{A}$ is a locally PP and semi-perfect category and hence, we may assume that all P_α are completely indecomposable and $P_\alpha\not\approx P_\beta$ for $\alpha\neq\beta$.

Every object A in $\mathfrak{A}$ has an injective hull of A in $\mathfrak{A}$ (see [11], p. 89, Theorem 3.2). We denote it by $E(A)$. If $E(\sum_I\oplus P_\alpha)$ is projective, $\mathfrak{A}$ is called QF-3^+ (see [2]).

Let Q be an injective envelope of R in $\mathfrak{M}_R^+$ and M a minimal faithful module in $\mathfrak{M}_R^+$. Then M is a retract of Q and hence, M is injective. Furthermore, since R is faithful, M is also a retract of R. Therefore, M is projective, and injective and we may assume that M is a right ideal of R.

Since R is semi-perfect, $R=\sum_I\oplus e_\alpha R$ and $e_\alpha Re_\alpha$'s are local rings. In the proof of theorem 4 in [9], we considered indecomposable projective objects P in $\mathfrak{M}_R^+$ such that $[P, e_\alpha R]=0$ for all $e_\alpha R\not\approx P$. We call such P *belonging to the first block*. Contrary, if $[e_\alpha R, P]=0$, P is called *belonging to the last bolck*.

Lemma 2. *Let $\mathfrak{A}$ be a locally PP and QF-3 semi-perfect Grothendieck category and R the induced ring. Then a minimal faithful object is a coproduct of $e_{\alpha_i}R$'s which belong to the first block.*

Proof. Since M is injective and a retract of $\sum_I\oplus e_\alpha R$, $M=\sum_J\oplus e_{\alpha_i}R$ by [14], Lemma 2. Further, since $e_{\alpha_i}R$ is injective $[e_{\alpha_i}R, eR]=0$ by Lemma 1 if $e_{\alpha_i}R\not\approx eR$. Hence, $e_{\alpha_i}R$ belongs to the first block.

Lemma 3. *Let $\mathfrak{A}$ be as above and $\sum_J \oplus e_i R$ a minimal faithful ideal. Then for any $\delta \in I$ there exist $\varphi(\delta)$ in J such that $e_{\varphi(\delta)} Re_\delta \neq 0$.*

Proof. Let x be a non-zero element in $e_\delta Re_\delta$. Since $\sum_I \oplus e_i R = \sum_{J, I \ni \alpha} \oplus e_i Re_\alpha$ is faithful, $e_{\varphi(\delta)} Re_\delta x \neq 0$ for some $\varphi(\delta)$.

Let e_i be as above. We put $\mathrm{R}(i) = \{\gamma \mid \in I,\ e_i Re_\gamma \neq 0\}$.

Lemma 4. *Let $\mathfrak{A}$ be as above and further perfect. Then* $\mathrm{R}(i)$ *contains the last element δ in* $\mathrm{R}(i)$ *namely, $e_i Re_\delta \neq 0$ and $e_\delta R$ belongs to the last block.*

Proof. We assume that R(1) does not contain the last element in R(1). Put $N = \sum_{\gamma \in \mathrm{R}(1)} \oplus e_1 R / (\sum_{\varepsilon \geqslant \gamma} e_1 Re_\varepsilon) \oplus \sum_{j \geqslant 2} \oplus e_j R$ and put $N_1 = \sum_{r \in \mathrm{R}(1)} \oplus e_1 R / (\sum_{\varepsilon \geqslant \gamma} e_1 Re_\varepsilon)$, and $N_2 = \sum_{j \geqslant 2} \oplus e_j R$. We shall show that N is faithful in $\mathfrak{M}_R^+$. Let $x = \sum x_{\alpha\beta}$, $x_{\alpha\beta} \in e_\alpha Re_\beta$ and $x_{\alpha\beta} \neq 0$. If $\varphi(\alpha) \neq 1$, we take $0 \neq y \in e_{\varphi(\alpha)} Re_\alpha \in N_2$. Then $yx = \sum yx_{\alpha\beta} \in \sum \oplus e_{\varphi(\alpha)} Re_\beta$ and $yx \neq 0$ by Theorem 1, since $e_\delta Re_\delta$ is a division ring by Lemma 1. We assume $\varphi(\alpha) = 1$. Then $\alpha \in \mathrm{R}(1)$ and there exists $y \in e_1 Re_\alpha$ and $0 \neq yx_{\alpha\beta} \in e_1 Re_\beta$. Hence, $\beta \in \mathrm{R}(1)$. Since R(1) does not have the last element, we obtain γ in R(1) such that $\beta < \gamma$. Hence $\{y + (\sum_{\varepsilon \geqslant \gamma} e_1 Re_\varepsilon)\} x \neq 0$. Therefore, N is faithful and N contains a submodule N_0 which is isomorphic to $e_1 R$. Then $N_0 = nR \approx e_1 R$ and $ne_1 = n$. Since $e_j Re_1 = 0$ for $j \geqslant 2$, $n \in N_1$. Let $n = \sum_{i=1}^n \bar{r}_{\gamma_i}$, $\bar{r}_{\gamma_i} \in e_1 R / (\sum_{\gamma_i \leq \varepsilon} e_1 Re_\varepsilon)$. Then $n(e_1 Re_\gamma) = 0$ for $\gamma = \max(\gamma_i)$. However, $e_1(e_1 Re_\gamma) \neq 0$. Which is a contradiction.

Theorem 3 ([4], Theorem 1). *Let $\mathfrak{A}$ be a perfect or semi-perfect and semi-artinian and locally PP-Grothendieck category with a generating set of small preojectives $\{G_\gamma\}_I$. If $\mathfrak{A}$ is QF-3, there exist non-isomoiphic indecomposable and projective objects $\{P_\alpha\}_J$ (resp. $\{Q_\beta\}_J$) such that*

1) *$\{P_\alpha\}$ (resp. $\{Q_\beta\}$) is an isomorphic representative class of the projectives in the first (resp. last) block,*

2) *$\sum_J \oplus P_\alpha$ is a minimal faithful and injective object and*

3) *each P_α contains the unique minimal subobject S_α which is isomorphic to Q_α. Hence $[S_\alpha : \Delta_\alpha] = 1$ and S_α is projective in $\mathfrak{M}_R^+$ where $\Delta_\alpha = [Q_\alpha, Q_\alpha]$ is a division ring. Furthermore, any indecomposable projective is isomorphic to a subobject in some P_α.*

Proof. We shall prove the theorem on the induced ring $R = \sum \oplus e_\alpha R$; $e_\alpha R \not\approx e_\beta R$ if $\alpha \neq \beta$. We know from Lemmas **2** and 3 that $\sum_J \oplus e_i R$ is a minimal faithful ideal, $e_i R$ belongs to the first block and $e_i R$ contains a submodule $e_i Re_{\gamma_i}$ where γ_i is the last element in R(i). Since $e_{\gamma_i} Re_\varepsilon = 0$ for $\varepsilon \neq \gamma_i$, $\mathfrak{r}_i = e_i Re_{\gamma_i}$ is a right ideal. Put $\Delta_i = e_{\gamma_i} Re_{\gamma_i}$, then Δ_i is a division ring by Lemma 1. $e_i R$ is indecomposable and injective. On the other hand, any Δ_i-submodule of $\mathfrak{r}_i$ is a R-module. Hence, $[\mathfrak{r}_i: \Delta_i]=1$ and $\mathfrak{r}_i$ is the unique minimal subideal in e_iR. Since $\mathfrak{r}_i\approx e_{\gamma_i}Re_{\gamma_i}=e_{\gamma_i}R$, $\mathfrak{r}_i$ is projective. Furthermore, $\mathfrak{r}_i\not\approx\mathfrak{r}_j$ if $i\neq j$, since $e_iR\not\approx e_iR_j$ and e_iR, e_jR are injective hull of $\mathfrak{r}_i$ and $\mathfrak{r}_j$, respectively. Let $e_\delta R$ be in the last block. Then $e_{\varphi(\delta)}Re_\delta\neq 0$ and $\varphi(\delta)\in J$. Hence, $e_{\varphi(\delta)}Re_\delta=\mathfrak{r}_{\varphi(\delta)}$. Therefore, $\{e_{\gamma_i}R\}$ is an isomorphic respresentative class of projectives in the last block. Let $\varepsilon\in I-J$. Then $e_{\varphi(\varepsilon)}Re_\varepsilon\neq 0$ by Lemma 3. Hence, $[e_\varepsilon R, e_{\varphi(\varepsilon)}R]\neq 0$, which means that $e_\varepsilon R$ does not belong to the first block. Furthermore, $e_\varepsilon R$ is ismorphic into $e_{\varphi(\varepsilon)}R$ by Lemma 1.

Lemma 5. *Let R be the induced ring from a locally PP-Grothendieck category with generating set $\{P_\alpha\}$ as above. We assume that $\{e_iR\}_J$ is a set of injective objects such that $E=\mathrm{E}(R)$ in $\mathfrak{M}_R^+$ is an essential extension of $\sum_J\oplus e_iR^{(K_i)}$. Then any $f\in[e_\beta R, E]$ is either zero or monomorphic, where $e_iR^{(K_i)}=\sum_{K_i}\oplus e_iR$ and e_β is any primitive idempotent.*

Proof. We assume $f\neq 0$. Then $\mathfrak{r}=f^{-1}(\sum_{t=1}^n e_{i_t}R)\neq 0$ for some e_{i_t}. Since $\sum_{t=1}^n e_{i_t}R$ is injective, $f\,|\,\mathfrak{r}$ is extended to $g\in[e_\beta R, \sum_{t=1}^n e_{i_t}R]$. Then g is monomorphic by Lemma 1. Therefore, f is monomorphic.

Theorem 4. *Let $\mathfrak{A}$ be a perfect, locally PP-Grothendieck category with generating set of small projectives. Then $\mathfrak{A}$ is QF-3^+ if and only if every projective P_γ in the first block are injective and for any indecomposable projective P, there exists P_α in $\{P_\gamma\}$ that $[P, P_\alpha]\neq 0$. Hence, $\{P_\tau\}$ is an isomorphic reprensentative class of all projective and injective indecomposable objects.*

Proof. Let R be the induced ring from completely indecomposable projectives P_α. We assume $\mathfrak{A}$ is QF-3^+. Then $E=\mathrm{E}(R)$ is isomorphic to $\sum_{J\ni j}\oplus e_{\alpha_j}R^{(K_j)}$, It is clear that $e_{\alpha_j}R$ belongs to the first block from Lemma 1. For any projective $e_\beta R$, $\mathrm{E}(e_\beta R)\subset E$. Hence, $[e_\beta R, e_{\alpha_j}R]\neq 0$ for some j, which implies $\{e_{\alpha_j}R\}$ consist of all projectives in the first block. Conversely, we assume that all projectives $\{e_iR\}_J$ in the first block are injective and have the property in the theorem. Since $[e_\beta R, e_iR]\neq 0$ for any $e_\beta R$, $E\supset\sum_{K_i,J}\oplus e_iR^{(K_i)}\supset R$ for suitable indices K_i. We assume $E\neq\sum_{K_j,J}\oplus e_jR^{(K_j)}$. Then there exists $g\in[e_kR, E]$ such that $\mathrm{Im}\,g\not\subset\sum\oplus e_jR^{(K_j)}$. On the other hand, we obtain $g'\in[e_kR, E_0]$ such that $g'\,|\,g^{-1}(E_0)=g$ from the proof of Lemma 5, where E_0 is a finite coproduct of e_jR's. Then $(g-g')\,|\,E_0=0$. Therefore, $g=g'$ by Lemma 5, which is a contradiction.

REMARK. The fact $[e_\beta R, e_{\alpha_j}R]\neq 0$ is equivalent to the validity of Lemma 3 for the above $\mathfrak{A}$.

Theorem 4′. *Let $\mathfrak{A}$ be a semi-perfect, semi-artinian and locally PP-Grothendieck category with generating set of small projectives. Then $\mathfrak{A}$ is QF-3^+ if and only if $\mathfrak{A}$ contains projectives P_α in the first block and all of such P_α are injective and for any indecomposable projective P, there exists P_α such that $[P, P_\alpha]\neq 0$. Hence, $\{P_\alpha\}$ consist of all projective and injective indecomposable objects. In this case $\mathfrak{A}$ is QF-3*, (cf. [2], Proposition 2 and [12], Proposition 3.1).

Proof. We assume $\mathfrak{A}$ is QF-3^+. Let S be the socle of $E=\mathrm{E}(R)$ and $S=\sum\oplus S_\gamma$,where S_γ's are minimal objects in E. Then $E=\mathrm{E}(S)$ and $E_\gamma=\mathrm{E}(S_\gamma)$ is imdecomposable and projective by the assumption. Hence, from [8], Corollary 1 to Lemma 2 $E_\gamma\approx e_\gamma R$, which belongs to the first block. Let $e_\beta R$ be any indecomposable ideal. Then $\mathrm{E}(e_\beta R)\subset E$. Hence, $[e_\beta R, e_\gamma R]\neq 0$ by Lemma 1 and the proof of Lemma 5. Since each $e_\gamma R$ has the non-zero socle, $\mathfrak{A}$ is QF-3 by the standard argument (cf. the proof of Lemma 7 below). The converse is similarly proved as in the proof of Theorem 4.

Lemma 6. *Let $\mathfrak{A}$ be as in Theorem 3 (resp. Theorem 4′) and e_1R in the first block. Let η be the last (resp. first) element in* R(1). *Then* $\mathrm{R}(1)=\mathrm{C}(\eta)$. *If $\mathfrak{A}$ is as Theorem 4,* $\mathrm{R}(1)^\gamma\supseteq\mathrm{C}(\gamma)$ *for any $\gamma\in\mathrm{R}(1)$ and for any δ and $\delta'\in(1)$ there exists ε in* R(1) *such that $e_\delta Re_\varepsilon\neq 0$ and $e_{\delta'}Re_\varepsilon\neq 0$, where* $\mathrm{R}(1)^\gamma=\{\alpha\mid\in\mathrm{R}(1), \alpha\leq\gamma\}$ *and* $\mathrm{C}(\eta)=\{\delta\mid\in I, e_\delta Re_\eta\neq 0\}$.

Proof. Let γ be in R(1) and δ be in $(I-\mathrm{R}(1))^\gamma$. Then $e_{\varphi(\delta)}Re_\delta\neq 0$ and $\varphi(\delta)\neq 1$. We assume $e_\delta Re_\gamma\neq 0$. Then $e_{\varphi(\delta)}Re_\gamma\supset(e_{\varphi(\delta)}Re_\delta)(e_\delta Re_\gamma)\neq 0$ by Theorem 1. We take non-zero element x, y in $e_{\varphi(\delta)}Re_\gamma$ and e_1Re_γ, respectively. Consider a mapping ψ: $xR\to yR$ such that $\psi(xr)=yr$. Then ψ is well defined and R-homomorphic by Theorem 1. Hence, $[e_{\varphi(\delta)}R, e_1R]\neq 0$, which is a contradiction. Therefore, $\mathrm{R}(1)^\gamma\supset\mathrm{C}(\gamma)$. Let x be a non-zero element in e_1Re_γ. Then xR is a projective and indecomposable ideal in e_1R by the assumption. Hence, $xR\overset{\psi}{\approx}e_qR$ for some q. Put $\psi(x)=e_qr$. Then $\psi(x)=\psi(xe_\gamma)=e_qre_\gamma$. This implies $q\leqslant\gamma$ (resp. $q\geqslant\gamma$). Similarly, we have $q\geqslant\gamma$ (resp. $q\leqslant\gamma$). We assume R(1) contains the last (resp. first) elemeny η. Then $e_\gamma Re_\eta\approx xRe_\eta=$(the socle of $e_1R)\neq 0$. Hence, $\mathrm{R}(1)=\mathrm{C}(\eta)$. Let $\gamma'\in\mathrm{R}(1)$. Then $e_\gamma R$ and $e_{\gamma'}R$ are monomorphic to e_1R. Since e_1R is injective, their images have a non-zero intersection $\mathfrak{r}$. Hence, $\mathfrak{r}e_\varepsilon\neq 0$ for some ε. Therefore, $e_\gamma Re_\varepsilon\neq 0$ and $e_{\gamma'}Re_\varepsilon\neq 0$.

Lemma 7 (cf. [12]). *Let Δ be a division ring and I a well ordered set. Let $\{e_{ij}\}_I$ be a set of matrix units. Put $R=\sum_{i\leq j\in I}\oplus e_{ij}\Delta$. Then $e_{11}R$ is injective and hence, R is hereditary and QF-3 in $\mathfrak{M}_R^+$. R is QF=3 if and only of I contains the last element.*

Proof. We first note that each $e_{ii}R$ contains only right ideals of form $e_{ij}R$ $i\leqslant j$ and $[e_{ii}R, e_{11}R]\approx\Delta$. Let

$$\begin{array}{ccccc} 0 & \longrightarrow & N & \longrightarrow & M \\ & & \downarrow{\scriptstyle f} & & \\ & & e_{11}R & & \end{array}$$

be a given exact diagram in $\mathfrak{M}_R^+$. We shall extend f to M by the standard argument. We obtain a maximal extension f_0: $N_0 \to e_{11}R$ such that $N_0 \supset N$ and $f_0|N=f$. If $M \neq N_0$, there exists m in M such that $me_{ii} \notin N_0$, since $\{e_{ii}R\}$ is a generating set. Put $M'=N_0+me_{ii}R$ and $\mathfrak{r}=\{x \mid \in e_{ii}R,\ mx \in N_0\}$. Then $\mathfrak{r}$ is a right ideal in $e_{ii}R$. Hence, $\mathfrak{r} \approx e_{jj}R$ for some $j>i$. We define g: $\mathfrak{r} \to e_{11}R$ by setting $g(x)=f_0(mx)$ for $x \in \mathfrak{r}$. Then $e_{1i}|\mathfrak{r}$ and g are in $[\mathfrak{r}, e_{11}R] \approx e_{j1}\Delta \approx \Delta$. Hence, $g=\delta(e_{1i}|\mathfrak{r})$ for some δ in Δ, namely $g(x)=\delta e_{1i}x$ for any x in $\mathfrak{r}$. Therefore, we have an extension f_0': $M' \to e_{11}R$ by $f_0'(n_0+mx)=f_0(n_0)+\delta e_{1i}x$. Hence, $N_0=M$. We know from [8], Lemma 7 and [9], Proposition 1 that R is perfect and $\mathrm{J}(R)=\sum_{i,j \geq i+1} \oplus e_{ij}\Delta$. Since $\mathrm{J}(R)$ is projective, R is hereditary by [9], Lemma 3. Therefore, R is QF-3^+ by Theorem 4. If R is QF-3, $e_{11}R$ is a minimal faithful module by Theorem 3. Hence, I has the last element by Theorem 3. Conversely, I has the last element, then $e_{11}R$ contains the unique submodule $e_{1\gamma}R$. It is clear that $e_{11}R$ is faithful module. Let M be a faithful module in $\mathfrak{M}_R^+$. Then there exists m in M such that $me_{1\gamma} \neq 0$. Hence, we have a monomorphism f of $e_{11}R$ to M by $f(e_{11}r)=me_{11}r$. Therefore, R is QF-3.

Lemma 8. *Let Δ be a division ring and $\{e_{ij}\}_I$ a set of matrix units. Put $S=\sum_I \oplus \Delta e_{ij}$ and $R=\sum_{i \geq j} \oplus \Delta e_{ij}$. Then*

1) *R is semi-hereditary.*

2) *R is semi-hereditary and QF-3 (or QF-3^+) if and only if I has the last element.*

3) *R is hereditary and QF-3^+ (or QF-3) if and only if I is finite,* (cf. [12]).

Proof. 1) Let $\mathfrak{r}$ be a right ideal generated by $\{x_1, x_2, \cdots, x_n\}$. Since $x_i=\sum_\alpha x_i e_\alpha$ and $x_i e_\alpha \in \mathfrak{r}$, we may assume that $x_i \in Re_{\alpha_i}$, where $e_{\alpha_i}=e_{\alpha_i \alpha_i}$. Let $\alpha_i=\max(\alpha_i)$. Considering Re_{α_i} as a Δ-vector space, we may assume $x_1, \cdots, x_t$ are linearly independent over Δ. If $\sum_{i=1}^{t} x_i r_i=0$ for $r_i \in R$ and $x_1 r_1 \neq 0$, then $r_1 e_\varepsilon \neq 0$ for $\varepsilon \leqslant \alpha_1$. Considering in S, we have $\sum_i x_i e_{\alpha_i} r_1 e_{\varepsilon\alpha_1}=0$ and $e_{\alpha_i} r_1 e_{\alpha_i} \neq 0$. Therefore, $\sum x_i R=\sum \oplus x_i R$. Put $\alpha_2=\max(\{\alpha_i\}-\alpha_1)$. We consider a vector space V_2 generated by $\{\sum_1^t \oplus x_i Re_{\alpha_2}, x_j e_{\alpha_2}\}$. We may assume $V_2=\sum \oplus x_i Re_{\alpha_2} \oplus y_1\Delta \oplus \cdots \oplus y_s\Delta$, where $y_j=x_k e_{\alpha_2}$ for some k. We shall show that $\sum \oplus x_i R+\sum y_j R=\sum \oplus x_i R \oplus \sum \oplus y_j R$. We have already shown that $\sum y_i R=\sum \oplus y_i R$. Let $\sum x_i r_i=\sum y_j r_j'$; r_i, $r_j' \in R$. If $r_1' \neq 0$, $r_1' e_{\varepsilon'} \neq 0$ for some ε'. Then multiplying $e_{\varepsilon'\alpha_2}$ in the above, we have $\sum x_i e_{\alpha_1} r_i e_{\varepsilon'\alpha_2}=\sum y_i e_{\alpha_2} r_i' e_{\varepsilon'\alpha_2}$ and

$e_{\alpha_1}r_ie_{\varepsilon'\alpha_2}\in Re_{\alpha_2}$, $\delta_1=e_{\alpha_2}r_1'e_{\varepsilon'\alpha_2}\neq 0$. Hence, $\sum y_i\delta_i=\sum x_ie_{\alpha_2}r_ie_{\varepsilon'\alpha_2}\in\sum x_iRe_{\alpha_2}$, which is a contradication. On the other hand, $x_iR\approx e_{\alpha_1}R$, $y_jR\approx e_{\alpha_2}R$. Repeating this argument, we show that $\mathfrak{r}$ is projective.

2) We assume that I has the last element α. We shall show that $e_{\alpha\alpha}R$ is injective as an analogy of Lemma 7. Let $\mathfrak{r}$ be a right ideal in some $e_{\beta\beta}R$. Put $\mathrm{R}(\mathfrak{r})=\{\gamma\,|\in I, \mathfrak{r}e_{\gamma\gamma}\neq 0\}$. If $\mathrm{R}(\mathfrak{r})$ contains the last element δ in $\mathrm{R}(\mathfrak{r})$, then $\mathfrak{r}_\delta=\sum_{\delta'\leq\delta} e_{\beta\beta\delta}Re_{\delta'\delta'}\approx e_{\delta\delta}R$. Let ε be the least element in $I-\mathrm{R}(\mathfrak{r})$. If ε is not a limit element, $\mathrm{R}(\mathfrak{r})$ contains the element. We assume ε is limit. Then $\mathfrak{r}=\bigcup_{\varepsilon'<\varepsilon}\mathfrak{r}_{\varepsilon'}$. We shall show $[\mathfrak{r}, e_{\alpha\alpha}R]\approx\Delta e_{\alpha\alpha}$. Let $f\in[\mathfrak{r}, e_{\alpha\beta}R]$ and put $f_{\varepsilon'}=f\,|\,\mathfrak{r}_{\varepsilon'}\in[\mathfrak{r}_{\varepsilon'}, e_{\alpha\alpha}R]$ $\approx[e_{\varepsilon'\varepsilon'}R, e_{\alpha\alpha}R]$. Then $f_{\varepsilon'}=\delta_{\varepsilon'}e_{\alpha\alpha}$ for some $\delta_{\varepsilon'}\in\Delta$. For ε' ε'' we have $\delta_{\varepsilon'}e_{\alpha\varepsilon'}$ $=f_{\varepsilon'}(e_{\alpha\varepsilon'})=f(e_{\alpha\varepsilon'})=f_{\varepsilon''}(e_{\beta\varepsilon'})=\delta_{\varepsilon''}e_{\alpha\varepsilon'}$. Hence, $\delta_{\varepsilon'}=\delta_{\varepsilon''}$. If we put $\delta=\delta_{\varepsilon'}$, $f=\delta e_{\alpha\beta}$. Thus, we have prepared necessary facts to use the proof of Lemma 7. Therefore, $e_{\alpha\alpha}R$ is injective in $\mathfrak{M}_R^+$ and R is QF-3^+ and QF-3 by Theorem 4′. The converse is clear from 1) and Theorems 3 and 4′.

3) If I is finite, R is a hereditary and QF-3 artinian ring by [4], Theorem 3. We assume that R is hereditary and QF-3 or QF-3^+. Then I has the last element by Theorem 4. We assume that I contains a limit number α. Consider $\mathrm{J}(e_\alpha R)=\sum_{\alpha<\gamma}\oplus e_{\alpha\gamma}\Delta$. Let $x=\sum_{i=1}^{n} e_{\alpha\gamma_i}\delta_i$. Then $x=\sum e_{\alpha\gamma_i+1}\delta_ie_{\gamma_i+1\gamma_i}\in\mathrm{J}(e_\alpha R)\mathrm{J}(R)$ $\subseteq\mathrm{J}^2(e_\alpha R)$. Hence, $\mathrm{J}(e_\alpha R)=\mathrm{J}^2(e_\alpha R)$, which implies $\mathrm{J}(e_\alpha R)$ is not projective by [8], Proposition 2. Therefore, I does not contain the limit number, but contain the last element, Hence, I is finite.

From the above proof and [9] Lemma 3 we have

Corollary. *Let R be as above. Then R is hereditary if and only if $|I|\leqslant\aleph_0$ and does not contain the last element.*

Theorem 5. *Let $\mathfrak{A}$ be a perfect or semi-perfect and semi-artiniam, and locally PP-Grothendieck category with generating set of small projectives. If $\mathfrak{A}$ is QF-3^+ or QF-3, then $\mathfrak{A}$ is equivalent to $\Pi\mathfrak{A}_\alpha$, where $\mathfrak{A}_\alpha$'s are of the same type as $\mathfrak{A}$ and $\mathfrak{A}_\alpha$ is not expressed as a product of full subcategories.*

Proof. Let R be the induced ring from $\mathfrak{A}$ and $\sum e_iR$ the coproduct of projectives in the first block. We shall show $e_\varepsilon Re_{\varepsilon'}=0$ for either $\varepsilon\in\mathrm{R}(i)$, $\varepsilon'\notin\mathrm{R}(i)$ or $\varepsilon\notin\mathrm{R}(i)$, $\varepsilon'\in\mathrm{R}(i)$. If $\varepsilon\in\mathrm{R}(i)$ $e_\varepsilon R$ is monomorphic to a submodule of e_iR. Hence, $e_\varepsilon Re_{\varepsilon'}=0$ if $\varepsilon'\notin\mathrm{R}(i)$. Next, we assume $\varepsilon'\in\mathrm{R}(i)$. If $e_\varepsilon Re_{\varepsilon'}\neq 0$ for $\varepsilon\notin\mathrm{R}(i)$, $0\neq e_\varepsilon Re_{\varepsilon'}e_\varepsilon Re_{\gamma_i}\subset e_\varepsilon Re_{\gamma_i}$ for some $\gamma_i\in\mathrm{R}(i)$ (or the last (resp. first) element in $\mathrm{R}(i)$) by Lemma 1, which contradicts to a fact $\mathrm{R}^{\gamma_i}(i)\supset\mathrm{C}(\gamma_i)$. Put $R_i=\sum_{\varepsilon,\varepsilon'\in R(i)^\varepsilon} e_\varepsilon Re_{\varepsilon'}$. Then $R=\sum\oplus R_i$ as a ring by Theorems 3, 4 and 4′. It is clear that each R_i is QF-3^+ or QF-3 and directly indecomposable. Hence, we have the theorem.

From the above theorem, we may restrict ourselves to a case of indecomposable categories if $\mathfrak{A}$ is as in the theorem.

Theorem 6. *Let $\mathfrak{A}$ be an indecomposable semi-perfect Grothendieck category with generating set of finitely generated objects. Then we have*

1) *$\mathfrak{A}$ is perfect, (semi-) hereditary and QF-3^+ (resp. QF-3) if and only if $\mathfrak{A}$ is equivalent to $[I, \mathfrak{M}_\Delta]^r$, where I is a well ordered set (resp. with last element).*

2) *$\mathfrak{A}$ is semi-artinan, hereditary and QF-3^+ (or QF-3) if and only if $\mathfrak{A}$ is equivalent to $[I, \mathfrak{M}_\Delta]^l$, where I is a finite set*

3) *$\mathfrak{A}$ is semi-artinian, semi-hereditary and QF-3^+ (or QF-3) if and only if $\mathfrak{A}$ is equivalent to $[I, M_\Delta]^l$, where I is a well ordered set with last element. Where Δ is a division ring and functors T_{ij} in $[I, \mathfrak{M}_\Delta]$ are equal to $1_{\mathfrak{M}_\Delta}$, (cf. [2′], Theorem 3.2).*

Proof. $[I, \mathfrak{M}_\Delta]^r$ is perfect, hereditary and QF-3^+ by Lemma 7 and [9], Theorem 3. We assume that I contains the last element. $[I, \mathfrak{M}_\Delta]^r$ is QF-3 by Lemma 7. If I is finite, $[I, \mathfrak{M}_\Delta]^l$ is semi-primary, hereditary and QF-3^+ (and QF-3) by Lemma 8. Finally, $[I, \mathfrak{M}_\Delta]^l$ is semi-artinian, semi-hereditary and QF-3^+ (QF-3) by Lemma 8 and [9], Proposition 1. Next, we assume that $\mathfrak{A}$ is one of the forms in the theorem. Let R be the induced ring: $R=\sum_I \oplus e_i R$. Then $e_1 R$ in the case 1) and $e_\alpha R$ in cases 2) and 3) are in the first block by Theorems 4 and 4′, respectively, where α is the last element in I. Since, $\mathfrak{A}$ is indecomposable, $e_1 R e_\gamma$ (resp. $e_\alpha R e_\gamma$) $\neq 0$ for any $\gamma \in I$ by Theorem 5, Lemma 3 and Remark. Let $\mathfrak{A}$ be herediary (cases 1) and 2)). If $[e_1 R e_\gamma : \Delta_\gamma] \geqslant 2$ (resp. $[e_\alpha R e_\gamma : \Delta_\gamma] \geqslant 2$) for any $\gamma \in I$, there exist linearly independent elements x, y over $\Delta_\gamma = e_\gamma R e_\gamma$. Then $xR+yR=xR\oplus yR$ by [9], Theorem 3, which contradicts to the indecomposability of $e_1 R$ and $e_\alpha R$. Let a, b be non-zero elements in $e_1 R e_\gamma$. As the proof of Lemma 6, a mapping $\psi: aR \to bR$ such that $\psi(a)=b$ gives a R-homomorphism. Furthermore, ψ is extended in $[e_1 R, e_1 R] = \Delta$, Hence $b=\delta a$ for some $\delta \in \Delta_1$. Therefore, $[e_1 R e_\gamma : \Delta_1]=1$. Similarly, we obtain $[e_\alpha R e_\gamma : \Delta_\alpha]=1$. Next, we assume $\mathfrak{A}$ is semi-hereditary and QF-3^+ (case 3)). Then $e_\alpha R$ is in the first block and injective. Let x, y be non-zero elements in $e_\alpha R e_\gamma$. Then $xR+yR$ is a projective right ideal in $e_\alpha R$. Since $e_\alpha R$ contains the unique minimal module and R is semi-perfect, $xR+yR \overset{\psi}{\approx} e_\delta R$ for some $\delta \in I$. Put $\psi^{-1}(e_\delta)=z$, then $z \in e_\alpha R e_\delta$ and $x=zr$, $y=zr'$ for $r, r' \in R$. Hence, $r=\delta$ and $x=ze_\delta r e_\delta$, $y=ze_\delta r' e_\delta$. Therefore $[e_\alpha R e_\gamma : \Delta_\gamma]=1$. Similarly to the above, we can show $[e_\alpha R e_\gamma : \Delta_\gamma]=1$. Thus, in any cases $e_1 R e_\varepsilon$ (resp. $e_\alpha R e_\varepsilon$) is a simple Δ_ε-module. Hence, if $e_\varepsilon R e_\gamma \neq 0$, $e_1 R e_\varepsilon \otimes_{\Delta_\varepsilon} e_\varepsilon R e_\gamma \subset e_1 R e_\gamma$ implies $[e_\varepsilon R e_\gamma : \Delta_\varepsilon] = [e_\varepsilon R e_\gamma : \Delta_\gamma]=1$ from Theorem 1. Let $x \neq 0 \in e_i R e_j$. Then Δ_i is isomorphic to Δ_j by ξ: $\delta_i x = x\xi(\delta_i)$. First we choose non-zero elements m_{1j} in $e_1 R e_j$. Then $e_j R$ is monomorphic to $\sum_{k \geq j} m_{1k}\Delta$ by the multiplication of m_{1j} from the left side. Hence, we can choose m_{jk} in $e_j R e_k$ such that $m_{1j} m_{jk} = m_{1k}$ (if $e_j R e_k \neq 0$). Then

$m_{1i}(m_{ij}m_{jk})=m_{1j}m_{jk}=m_{1k}=m_{1i}m_{ik}$. Therefore, $m_{ij}m_{jk}=m_{ik}$ if $m_{ij}\neq 0$ and $m_{jk}\neq 0$. Thus, R is a subring of $\sum_{i\leq j}\oplus e_{ij}\Delta$ (resp. $\sum_{i\geq j}\oplus e_{ij}\Delta$) such that all of elements of some (i, j)-entries may be equal to zero, where $\Delta\approx\Delta_i$. We assume $e_iRe_j=0$ (in cases 1) and 2)). Then $i\neq 1$ (resp. $i\neq\alpha$) and there exists γ from Lemma 6 such that $e_iRe_\gamma\neq 0$, $e_jRe_\gamma\neq 0$. Put $e=e_{11}+e_{ii}+e_{jj}+e_{\gamma\gamma}$ (resp. $e=e_{11}+e_{ii}+e_{jj}+e_{\alpha\alpha}$). Then $eRe=e_{11}\Delta\oplus e_{1i}\Delta\oplus e_{1j}\Delta\oplus e_{1\gamma}\Delta\oplus e_{ii}\Delta\oplus e_{i\gamma}\Delta\oplus e_{jj}\Delta\oplus e_{j\gamma}\Delta\oplus e_{\gamma\gamma}\Delta$ is hereditary by [9], Corolalry to Lemma 2 if R is hereditary. However, we can easily see that eRe is not hereditary (cf. [6], Theorem 1). Therefore, $R=\sum_{i\leq j}\oplus e_{ij}\Delta$, (resp. $R=\sum_{i\geq j}\oplus e_{ij}\Delta$). Finally, we assume that R is semi-hereditay (case 3)). Let $\gamma<\delta$ be in I. Then since $m_{\alpha\gamma}R+m_{\alpha\delta}R$ is projective, $m_{\alpha\gamma}R+m_{\alpha\delta}R=zR$ as before, where $z\in e_\alpha Re_\delta$. Hence, $zR=m_{\alpha\delta}R\supset m_{\alpha\gamma}R$. Therefore, $0\neq m_{\alpha\gamma}=m_{\alpha\delta}e_\delta er_\gamma$ implies $e_\delta Re_\gamma\neq 0$. Thus, $\mathfrak{A}$ is equivalent to $[I, \mathfrak{M}_\Delta]'$. The remainimg parts are clear from Theorems 3, 4 and 4′ and Lemma 8.

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