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ON KOLMOGOROV'S COMPLEXITY AND INFORMATION

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1. Summary. About Kolmogorov's complexity measure K, we prove the following theorem, which seems rather unexpected.

Theorem. *For any C, there exists a sentence y such that*

$$K(y)-K(y\,|\,x)>C$$

holds except for finitely many sentences x.

2. Preliminary. Let Σ be any non-empty finite set of symbols. For $n=0, 1, 2, \cdots$, Σ^n denotes the n products of Σ. Let $\Sigma^*=\bigcup_{n=0}^{\infty}\Sigma^n$. An element of Σ^* is considered as a finite sequence of symbols belonging to Σ, and will be called a *sentence*. Denote by Λ the unique element belonging to Σ^0. Let $N=\{1, 2, 3, \cdots\}$ be the set of positive integers. A partial recursive function from $\Sigma^*\times N$ to Σ^* is called an *algorithm*. Let f be an algorithm. For sentences x and y, define

$$K_f(y\,|\,x)=\begin{cases}\min\limits_{f(x,p)=y}\log_2 p & \cdots \text{ if such } p \text{ exists}\\ \infty & \cdots \text{ else.}\end{cases}$$

An algorithm A is called an *asymptotically optimal* algorithm if for any algorithm f, there exists a constant C such that

$$K_A(y\,|\,x)<\infty$$

and

$$K_A(y\,|\,x)-K_f(y\,|\,x)\leqq C$$

for any sentences x and y. It was proved in [1] that an asymptotically optimal algorithm does exist and is unique in the sense that if A and B are asymptotically optimal algorithms, then there exists a constant C such that $|K_A(y|x)-K_B(y|x)|\leqq C$ for any sentences x and y. Let A be any asymptotically optimal algorithm, which is fixed throughout this paper. Denote $K_A(.\,|\,.)$ simply by $K(.\,|\,.)$, which is called *Kolmogorov's complexity measure*. Denote $K(y)=K(y\,|\,\Lambda)$ for any $y\in\Sigma^*$.

3. Proof of Theorem. It is well known ([2]) that there exists a partial recursive function ψ from $\Sigma^* \times N \times N$ to Σ^* such that

(1) if $s \leqq t$, then $\psi(.,.,t)$ is an extension of $\psi(.,.,s)$,
(2) $A(x,p)=y$ if and only if there exists t such that $\psi(x,p,t)=y$, and
(3) the domain of ψ is a recursive set of $\Sigma^* \times N \times N$.

For any $p \in N$ and $t \in N$, define a subset $S_{p,t}$ of Σ^* by

$$S_{p,t} = \{\psi(\Lambda, i, t);\ i = 1, 2, \cdots, p^2\} .$$

From the property (3), it follows that the predicate about x, i and t that $\psi(\Lambda, i, t)$ is defined and equal to x is recursive.

Hence, $\{(x, p, t);\ x \notin S_{p,t}\}$ is a recursive set of $\Sigma^* \times N \times N$.
Theorefore,

$$L = \{(p, t, i);\ \psi(\Lambda, i, t) \ \textit{ is defined and } \psi(\Lambda, i, t) \notin S_{p,t}\}$$

is a recursive set of $N \times N \times N$. Define a partial recursive function j from $N \times N$ to N by

$$j(t, p) = \min \{i;\ (p, t, i) \in L\} .$$

Let σ be a one-to-one recursive function from Σ^* onto N. Finally, define an algorithm B by

$$B(x, p) = A(\Lambda, j(\sigma(x), p)) .$$

Since A is an asymptotically optimal algorithm,

$$\{A(\Lambda, i);\ i \in N\} = \Sigma^* .$$

Then, for any $p \in N$, there exists the minimum j, which will be denoted by $h(p)$, such that $A(\Lambda, j)$ is defined and does not belong to the set $\{A(\Lambda, i); i=1,2,\cdots,p^2\}$. Put $y(p)=A(\Lambda, h(p))$. From the property (1) and (2), it follows that for any $p \in N$, there exists the minimum t, which will be denoted by $\tau(p)$, such that $A(\Lambda, i)=\psi(\Lambda, i, t)$ for any $i \leqq h(p)$. This equality should be taken it for granted that if one hand side is defined, then the other hand side is also defined and they coincide. Then it is easy to verify that if $t \geqq \tau(p)$, then $j(t, p)$ is defined and equal to $h(p)$. Let $D_p = \{x \in \Sigma^*;\ \sigma(x) < \tau(p)\}$. Then D_p is a finite set for any $p \in N$. Form the above descriptions, if $x \in \Sigma^* - D_p$, then $B(x, p)$ is defined and equal to $y(p)$ for any $p \in N$. Since

$$y(p) \notin \{A(\Lambda, i);\ i = 1, 2, \cdots, p^2\} ,$$
$$K(y(p)) > 2 \log_2 p .$$

One the other hand, if $x \in \Sigma^* - D_p$, then

$$K_B(y(p)\,|\,x) \leqq \log_2 p\ ,$$

since $y(p)=B(x, p)$. Let C_0 be a constant sucn that

$$K(y\,|\,x) \leqq K_B(y\,|\,x)+C_0$$

for any $x, y \in \Sigma^*$. Let C be an arbitrary constant. Select p so large that $\log_2 p - C_0 \geqq C$. Then

$$K(y(p)) - K(y(p)\,|\,x) > C$$

for any $x \in \Sigma^* - D_p$, which completes the proof of Theorem.

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References

[1] A. N. Kolmogorov: *Three approaches to the concept of "the amount of information"*, Problems in the Transmission of Information, **1** (1965), 3–11.

[2] M. Davis: Computability and Unsolvability, McGraw-Hill, New York, 1958.