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COBORDISM OF REGULAR TORUS ACTIONS

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1. Introduction and definitions

Let S^1 be the unit sphere in the field of complex numbers $\boldsymbol{C}$. Let T^k be the k-dimensional torus, that is, $T^k=S^1\times\cdots\times S^1$ (k copies). We denote elements of T^k by coordinates $(\lambda_1, \cdots, \lambda_k)$, $\lambda_i\in S^1$. For any sequence $(i_1, \cdots, i_n)$ of integers with $0\leqq i_1<i_2<\cdots<i_n\leqq k$, we define a subgroup $T^k_{(i_1,\cdots,i_n)}$ of T^k by

$$T^k_{(i_1,\cdots,i_n)}=\{(\lambda_1, \cdots, \lambda_k)\in T^k\,|\,\lambda_j=1 \text{ for } j\notin\{i_1, \cdots, i_n\}\}\ .$$

In particular, $T^k_0=1\times\cdots\times 1$, $T^k_i=\underbrace{1\times\cdots\times 1\times S^1}_{i}\times 1\times\cdots\times 1$ for $0<i\leqq k$ and $T^k_{(1,2,\cdots,k)}=T^k$.

Let $\varphi\colon T^k\times M\to M$ be a differentiable action of T^k on a compact oriented differentiable manifold M. We denote such an action by a pair (M, φ). (M, φ) is called *regular*, when for any point x in M the isotropy group $I(x)=\{\lambda\in T^k\,|\,\varphi(\lambda, x)=x\}$ is of the form $T^k_{(i_1,\cdots,i_n)}$ for some sequence $(i_1, \cdots, i_n)$.

In particular, a regular T^1-action is called a *semi-free S^1-action*. If (M, φ) is a regular T^k-action, then the S^1-action $(M, \varphi\,|\,T^k_i\times M)$ is a semi-free S^1-action for $1\leqq i\leqq k$. But the reverse, in general, is not true. For example, given a free S^1-action (M, φ), the T^2-action (M, Φ) defined by $\Phi((\lambda_1, \lambda_2), x)=\varphi(\lambda_1\lambda_2, x)$ is not regular nevertheless the S^1-actions $(M, \Phi\,|\,T^2_i\times M)$ $(i=1, 2)$ are free.

In this paper, we study regular T^k-actions by the method of Stong [3]. In section 3, we obtain the result which asserts that a stationary point free regular T^k-action on a closed oriented differentiable manifold bounds as a regular T^k-action on a compact oriented differentiable manifold (Corollary 3-2). And in section 4, we also obtain the result which asserts that a cobordism class of a regular T^k-action (M, φ) on a closed oriented differentiable manifold M is determined by the normal bundle of the stationary point set in M (Theorem 4-1).

As a corollary to Theorem 4-1, we obtain that the cobordism group of regular T^k-actions on closed oriented manifolds is isomorphic to the tensor product of k copies of the cobordism group of semi-free S^1-actions on closed oriented manifolds (Corollary 4-3).

In the case of $k=1$, Corollary 3-2 and Theorem 4-1 have been obtained by Uchida [4].

2. Preliminaries

Let G be a compact Lie group. We consider a family $\mathfrak{F}$ of subgroups of G satisfying that if $H\in\mathfrak{F}$ then $gHg^{-1}\in\mathfrak{F}$ for all $g\in G$. All families considered will be assumed to satisfy this condition.

Given families $\mathfrak{F}\supset\mathfrak{F}'$ of subgroups of G, an $(\mathfrak{F}, \mathfrak{F}')$*-free G-action* is a pair (M, φ) consisting of a compact oriented differentiable manifold M and a differentiable action φ: $G\times M\to M$ such that

(1) if $x\in M$, then the isotropy group $I(x)\in\mathfrak{F}$, and
(2) if $x\in\partial M$, then $I(x)\in\mathfrak{F}'$.

If $\mathfrak{F}'$ is empty, then necessarily $\partial M=\phi$.

Given (M, φ), define $-(M, \varphi)=(-M, \varphi)$, $-M$ the maniflod with the opposite orientation to M. Also define $\partial(M, \varphi)=(\partial M, \varphi|G\times\partial M)$, ∂M oriented by inward normal vectors.

Two $(\mathfrak{F}, \mathfrak{F}')$-free G-actions (M, φ) and (M', φ') are *cobordant*, if there are an $(\mathfrak{F}', \mathfrak{F}')$-free G-action (V, ψ) and an $(\mathfrak{F}, \mathfrak{F})$-free G-action (W, ψ') such that

(1) $\partial(V, \psi)=\partial(M, \varphi)\cup(-\partial(M', \varphi'))$ (disjoint union)
(2) $\partial(W, \psi')=(M, \varphi)\cup(V, \psi)\cup(-(M', \varphi'))$ (glueing the boundaries).

This cobordism relation is an equivalence relation. We denote by $[M, \varphi]$ the cobordism class of (M, φ).

The set of cobordism classes of $(\mathfrak{F}, \mathfrak{F}')$-free G-actions forms an abelian group with the operation induced by disjoint union, and this group will be denoted by $\Omega_*(G; \mathfrak{F}, \mathfrak{F}')$. $\Omega_n(G; \mathfrak{F}, \mathfrak{F}')$ denotes the summand consisting of cobordism classes of $(\mathfrak{F}, \mathfrak{F}')$-free G-actions (M, φ) with $\dim M=n$.

By the cartesian product, $\Omega_*(G: \mathfrak{F}, \mathfrak{F}')$ is an Ω_*-module, Ω_* the oriented cobordism ring.

The following two theorems are essentially the results of Conner and Floyd.

Theorem 2-1 (see [2; (5.3)]). *Let G be a compact Lie group and $\mathfrak{F}\supset\mathfrak{F}'\supset\mathfrak{F}''$ families of subgroups of G. Then the sequence*

$$\cdots\to\Omega_n(G; \mathfrak{F}', \mathfrak{F}'')\xrightarrow{i}\Omega_n(G; \mathfrak{F}, \mathfrak{F}'')\xrightarrow{j}\Omega_n(G; \mathfrak{F}, \mathfrak{F}')\xrightarrow{\partial}\Omega_{n-1}(G; \mathfrak{F}', \mathfrak{F}'')\to\cdots$$

is exact, where i and j are induced by considering $(\mathfrak{F}', \mathfrak{F}'')$-free or $(\mathfrak{F}, \mathfrak{F}'')$-free as being $(\mathfrak{F}, \mathfrak{F}'')$-free or $(\mathfrak{F}, \mathfrak{F}')$-free respectively, and ∂ is induced by sending $[M, \varphi]$ to $[\partial(M, \varphi)]$.

Turning to actions of T^k, non-trivial, non-isomorphic, and regular irreducible orthogonal representations of T^k on the 2-dimensional real vector space $\boldsymbol{C}$ are given by the complex multiplication by i-th coordinate of T^k for $i=1, 2, \cdots, k$. So we can express a translation of [1; Theorem 38.3] into regular T^k-actions in the following fashion:

Theorem 2–2. *Suppose that ξ is an n-dimensional real vector bundle over a connected, locally connected, paracompact base, and that φ: $T^k \times \xi \to \xi$ is a regular T^k-action which carries each fibre orthogonally onto itself such that the stationary points are only zero vectors. There are then vector subbundles ξ_i of ξ, $i=1, \cdots, k$, with $\xi=\xi_1 \oplus \cdots \oplus \xi_k$, and there exists a complex vector bundle structure on each ξ_i such that $\varphi((\lambda_1, \cdots, \lambda_k), (v_1, \cdots, v_k))=(\lambda_1 \cdot v_1, \cdots, \lambda_k \cdot v_k)$ for $(\lambda_1, \cdots, \lambda_k) \in T^k$ and $v_i \in \xi_i$, $i=1, \cdots, k$, where $\cdot$ denotes the complex multiplication. In particular, each ξ_i is invariant under the action of T^k.*

3. Cobordism of regular T^k-actions

For $1 \leqq p \leqq k+1$ let $\mathfrak{F}_p$ denote the family of subgroups $T^k{}_{(i_1, \cdots, i_n)}$ of T^k such that $\{i_1, \cdots, i_n\}$ does not contain $\{1, 2, \cdots, p\}$, and let $\mathfrak{F}_0 = \phi$. Note that $\mathfrak{F}_{k+1}$ is the family of all subgroups of the type $T^k{}_{(i_1, \cdots, i_n)}$. We have inclusions

$$\phi = \mathfrak{F}_0 \subset \mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots \subset \mathfrak{F}_k \subset \mathfrak{F}_{k+1} .$$

We note that if (M, φ) is a regular T^k-action, then the isotropy groups all belong to $\mathfrak{F}_p$ if and only if $(M, \varphi | T^k{}_{(1, \cdots, p)} \times M)$ is stationary point free.

Theorem 3–1. *For $0 \leqq p < k$, the sequence of Theorem 2–1 for the families $\mathfrak{F}_{k+1} \supset \mathfrak{F}_{p+1} \supset \mathfrak{F}_p$ becomes a split exact sequence of Ω_*-modules:*

$$0 \to \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p) \xrightarrow{j} \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_{p+1}) \xrightarrow{\partial} \Omega_*(T^k; \mathfrak{F}_{p+1}, \mathfrak{F}_p) \to 0 .$$

Proof. It suffices to construct an Ω_*-module homomorphism γ: $\Omega_*(T^k; \mathfrak{F}_{p+1}, \mathfrak{F}_p) \to \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_{p+1})$ satisfying $\partial\gamma = 1$.

Given $[M, \varphi] \in \Omega_n(T^k; \mathfrak{F}_{p+1}, \mathfrak{F}_p)$, let N be the stationary point set of $T^k{}_{(1, \cdots, p)}$, i.e., $N = \{x \in M \mid \varphi(\lambda, x) = x$ for all $\lambda \in T^k{}_{(1, \cdots, p)}\}$. Then N is a T^k-invariant submanifold of M with $\partial N = N \cap \partial M$, and since all isotropy groups on ∂M belong to $\mathfrak{F}_p$, $N \cap \partial M = \phi$. Let ν be the normal bundle of N in M. The T^k-action φ induces a T^k-action $\bar{\varphi}$: $T^k \times \nu \to \nu$ by bundle maps covering φ. Since T^k_{p+1} freely acts on N by φ, T^k_{p+1} also freely acts on ν by $\bar{\varphi}$. Let D be the unit disc in $\boldsymbol{C}$, and let T^k_{p+1} act on D by the complex multiplication by $(p+1)$-th coordinate of T^k_{p+1}. Then we obtain the orbit manifold $D(\nu) \times D / T^k_{p+1}$ by the diagonal action, where $D(\nu)$ is the disc bundle of ν. The submanifold $D(\nu) \times S^1 / T^k_{p+1}$ of $D(\nu) \times D / T^k_{p+1}$ is equivariantly identified with $D(\nu)$ by an identification $[v, z] \mapsto \bar{\varphi}((1, \cdots, 1, z^{-1}, 1, \cdots, 1), v)$ where z^{-1} lies on $(p+1)$-th coordinate, and $D(\nu)$ is equivariantly identified with a tubular neighborhood $T(N)$ of N in M. So we can form an $(n+1)$-dimensional manifold W from $D(\nu)\times D/T^k_{p+1} \cup M\times[0, 1]$ by identifying $D(\nu)\times S^1/T^k_{p+1}$ with $T(N)\times 1$. We may orient W such that the inclusion $M \rightarrow M\times 0 \subset W$ is orientation-preserving.

Define a T^k-action ψ on $D(\nu)\times D/T^k_{p+1}$ by $\psi(\lambda, [v, z])=[\bar{\varphi}(\lambda, v), z]$ for $\lambda\in T^k$, $v\in D(\nu)$ and $z\in D$. Then ψ is compatible with $\varphi\times 1$ on the identified part, so we obtain an $(\mathfrak{F}_{k+1}, \mathfrak{F}_{p+1})$-free T^k-action (W, χ) where χ restricts to ψ on $D(\nu)\times D/T^k_{p+1}$ and to $\varphi\times 1$ on $M\times[0, 1]$.

Performing the same construction on a cobordism shows that $\gamma([M, \varphi]) =[W, \chi]$ defines an Ω_*-module homomorphism. We have $\partial\gamma([M, \varphi])= [\partial W, \chi \mid T^k\times\partial W]=[M, \varphi]$. The last equation follows by applying [2; (5.2)], in fact, $(\partial W, \chi\mid T^k\times\partial W)$ and (M, φ) are cobordant by a cobordism $(\partial W\times[0, 1], (\chi\mid T^k\times\partial W)\times 1)$. q.e.d.

Corollary 3–2. *If (M, φ) is a stationary point free regular T^k-action on a closed oriented manifold M, then there is a regular T^k-action $(\mathfrak{M}, \Phi)$ on a compact oriented manifold $\mathfrak{M}$ such that $\partial\mathfrak{M}=M$ and $\Phi\mid T^k\times\partial\mathfrak{M}=\varphi$.*

Proof. We have the exact sequence

$$\Omega_*(T^k; \mathfrak{F}_k, \mathfrak{F}_0) \xrightarrow{i} \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0) \xrightarrow{j} \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_k) \xrightarrow{\partial} \Omega_*(T^k; \mathfrak{F}_k, \mathfrak{F}_0)$$

for the families $\mathfrak{F}_{k+1}\supset\mathfrak{F}_k\supset\mathfrak{F}_0$. We note that $\Omega_*(T^k; \mathfrak{F}_k, \mathfrak{F}_0)$ is the cobordism group of stationary point free regular T^k-actions on closed oriented manifolds and $\Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$ is the cobordism group of all regular T^k-actions on closed oriented manifolds. Thus it is sufficient to show that j is monic. Let j_p: $\Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p)\rightarrow\Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_{p+1})$ be the canonical homomorphism, then j_p is monic by Theorem 3–1 if $0\leqq p<k$. So $j=j_{k-1}\circ\cdots\circ j_1\circ j_0$ is monic. q.e.d.

4. Cobordism of bundles with regular T^k-actions

In this section, we show that a cobordism class of a regular T^k-action on a closed oriented manifold is determined by the normal bundle of the stationary point set.

Given $[M, \varphi]\in\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$, let N be a connected component of the stationary point set of φ, and let $\nu\rightarrow N$ be the normal bundle of N in M. By Theorem 2–2, we have the decomposition

$$(*)\quad \nu = \nu_1\oplus\nu_2\oplus\cdots\oplus\nu_k$$

of complex vector bundles with the induced T^k-action $\bar{\varphi}$ satisfying

$\bar{\varphi}((\lambda_1, \cdots, \lambda_k), (v_1, \cdots, v_k)) = (\lambda_1 \cdot v_1, \cdots, \lambda_k \cdot v_k)$ for $(\lambda_1, \cdots, \lambda_k) \in T^k$ and $(v_1, \cdots, v_k) \in \nu$ $(v_i \in \nu_i)$. N may be oriented compatibly with the canonical orientation of ν and the given orientation of $\tau(M)$. Let $f_i \colon N \to BU(n_i)$ be the classifying map of ν_i, then define a homomorphism

$$\alpha \colon \Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0) \to \bigoplus_{m=r+2\sum n_i} \Omega_r(BU(n_1) \times \cdots \times BU(n_k))$$

by sending $[M, \varphi]$ to $\oplus[N, f_1 \times \cdots \times f_k]$ where the sum is taken over all connected components of the stationary point set.

Let $s \colon \oplus\Omega_r(BU(n_1) \times \cdots \times BU(n_k)) \to \oplus' \Omega_r(BU(n_1) \times \cdots \times BU(n_k))$ be the projection where the sum $\oplus$ is taken over all $(r, n_1, \cdots, n_k)$ with $m = r + 2\sum_{i=1}^k n_i$, and $\oplus'$ is taken over all $(r, n_1, \cdots, n_k)$ with $m = r + 2\sum_{i=1}^k n_i$, and $n_i \neq 1$ for $i = 1, \cdots, k$.

Then we have

Theorem 4-1. *By the composition* $s\alpha$

$$\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0) \cong \oplus\Omega_r(BU(n_1) \times \cdots \times BU(n_k)),$$

where the sum is taken over all $(r, n_1, \cdots, n_k)$ *with* $m = r + 2\sum_{i=1}^k n_i$ *and* $n_i \neq 1$ *for* $i = 1, \cdots, k$.

As immediate corollaries we have the following Corollaries 4-2 and 4-3. (Of course Corollary 3-2 can also be obtained as a corollary to Theorem 4-1.)

Corollary 4-2. *Given a regular* T^k*-action* (M, φ) *on an* m*-dimensional closed oriented manifold, let* ν *be the normal bundle of a connected component of the stationary point set in* M. *If* ν *has at least one* (*complex*) 1*-dimensional summand in the decomposition* (*) *for all connected components of the stationary point set, then* $[M, \varphi] = 0$ *in* $\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$. *In particular, if the stationary point set is 2-codimensional, then* $[M, \varphi] = 0$ *in* $\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$.

$\Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0)$ is the cobordism group of semi-free S^1-actions on oriented closed manifolds. We consider the tensor product $\Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0) \otimes \cdots \otimes \Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0)$, (over Ω_*), of k copies of $\Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0)$ and define a homomorphism $\beta \colon \Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0) \otimes \cdots \otimes \Omega_*(T^1; \mathfrak{F}_2, \mathfrak{F}_0) \to \Omega_*(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$ by sending $[M_1, \varphi_1] \otimes \cdots \otimes [M_k, \varphi_k]$ to $[M_1 \times \cdots \times M_k, \varphi_1 \times \cdots \times \varphi_k]$ where $\varphi_1 \times \cdots \times \varphi_k$ is a regular T^k-action defined by $\varphi_1 \times \cdots \times \varphi_k((\lambda_1, \cdots, \lambda_k), (x_1 \cdots, x_k)) = (\varphi_1(\lambda_1, x_1), \cdots, \varphi_k(\lambda_k, x_k))$ for $(\lambda_1, \cdots, \lambda_k) \in T^k$ and $x_i \in M_i$. We then have

Corollary 4-3. β *is an isomorphism of* Ω_**-modules.*

Proof. By applying the Künneth formula to the spaces $BU(n_1) \times \cdots \times BU(n_k)$, the Corollary follows from Theorem 4-1. q.e.d.

In order to prove Theorem 4–1, we introduce cobordism groups of complex vector bundles with regular T^k-actions.

We consider a *(T^k, p)-manifold-bundle* given by a collection $((M, \varphi), (\xi_1, \varphi_1), \cdots, (\xi_p, \varphi_p))$ where (M, φ) is a regular T^k-action on an oriented closed manifold, and ξ_i is a complex vector bundle over M with a regular T^k-action φ_i by complex vector bundle maps covering φ.

Let $\Omega_m(T^k; n_1, \cdots, n_p)$ denote the group of cobordism classes of (T^k, p)-manifold-bundles with $\dim M = m$ and $\dim_C \xi_i = n_i$. Similarly, let $\hat{\Omega}_m(T^k; n_1, \cdots, n_p)$ denote the group obtained under the assumption that T^k_1 freely acts on M by φ (i.e. (M, φ) is $(\mathfrak{F}_1, \mathfrak{F}_0)$-free).

Lemma 4–4. $\Omega_n(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p) \cong \bigoplus_{n=m+2\sum n_i} \Omega_m(T^{k-p}; n_1, \cdots, n_p)$.

Proof. We define homomorphisms $\sigma\colon \Omega_n(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p) \to \bigoplus_{n=m+2\sum n_i} \Omega_m(T^{k-p}; n_1, \cdots, n_p)$ and $\rho\colon \bigoplus_{n=m+2\sum n_i} \Omega_m(T^{k-p}; n_1, \cdots, n_p) \to \Omega_n(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p)$ satisfying $\sigma\rho = 1$ and $\rho\sigma = 1$ as follows:

Given $[M, \varphi] \in \Omega_n(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_p)$, let N be a connected component of the stationary point set of $T^k{}_{(1,\cdots,p)}$. Then N is an oriented closed submanifold which is invariant under the action of T^k. Let ν be the normal bundle of N in M. The T^k-action φ induces the T^k-action $\bar{\varphi}\colon T^k \times \nu \to \nu$ by bundle maps covering φ. Restrict $\bar{\varphi}$ to the action of $T^p = T^k{}_{(1,\cdots,p)}$, then the restricted T^p-action on ν is satisfied with the hypothesis of Theorem 2–2, so we have the decomposition $\nu = \nu_1 \oplus \cdots \oplus \nu_p$ of complex vector bundles with the T^p-action $\bar{\varphi}$ satisfying $\bar{\varphi}((\lambda_1, \cdots, \lambda_p), (v_1, \cdots, v_p)) = (\lambda_1 \cdot v_1, \cdots, \lambda_p \cdot v_p)$ for $(\lambda_1, \cdots, \lambda_p) \in T^p$ and $(v_1, \cdots, v_p) \in \nu$. By the commutativity of the action of T^k and the non-equivalence of T^p-representations on fibres in disticnct summands of ν, we see that the decomposition of ν is compatible with the T^k-action $\bar{\varphi}$. Then we define σ by sending $[M, \varphi]$ to $\oplus[(N, \varphi'), (\nu_1, \bar{\varphi}_1), \cdots, (\nu_p, \bar{\varphi}_p)]$ where, considering $T^{k-p} = T^k{}_{(p+1,\cdots,k)}$, $\varphi' = \varphi \mid T^{k-p} \times N$ and $\bar{\varphi}_i = \bar{\varphi} \mid T^{k-p} \times \nu_i$, and where the sum is taken over all connencted components of the stationary point set of $T^k{}_{(1,\cdots,p)}$.

Next we define ρ by sending $[(M, \varphi), (\xi_1, \varphi_1), \cdots, (\xi_p, \varphi_p)] \in \oplus\Omega_m(T^{k-p}; n_1 \cdots, n_p)$ to $[D(\xi), \mu]$, where $D(\xi)$ is the disc bundle of $\xi = \oplus_{i=1}^{p} \xi_i$ and is oriented by the complex structure and the orientation of M, and $\mu\colon T^k \times D(\xi) \to D(\xi)$ is defined by $\mu((\lambda_1, \cdots, \lambda_k), (v_1, \cdots, v_p)) = (\varphi_1(\lambda', \lambda_1 \cdot v_1), \varphi_2(\lambda', \lambda_2 \cdot v_2), \cdots, \varphi_p(\lambda', \lambda_p \cdot v_p))$ for $(\lambda_1, \cdots, \lambda_k) \in T^k$, $\lambda' = (\lambda_{p+1}, \cdots, \lambda_k)$, and $(v_1, \cdots, v_p) \in D(\xi)$, $v_i \in \xi_i$.

$\sigma\rho = 1$ is clear. $\rho\sigma([M, \varphi]) = [D(\nu), \bar{\varphi}]$ where $D(\nu)$ is identified equivariantly with a tubular neighborhood of the stationary point set of $T^k{}_{(1,\cdots,p)}$ in M. From [2; (5.2)] we have $[D(\nu), \bar{\varphi}] = [M, \varphi]$. q.e.d.

By the same way we have

Lemma 4-5. $\Omega_n(T^k; \mathfrak{F}_{p+1}, \mathfrak{F}_p) \cong \bigoplus_{n=m+2\sum n_i} \hat{\Omega}_m(T^{k-p}; n_1, \cdots, n_p)$.

Theorem 4-6. *There is an exact sequence*

$$(*) \qquad 0 \to \Omega_m(T^k; n_1, \cdots, n_p) \xrightarrow{F} \bigoplus_{m=r+2n_{p+1}} \Omega_r(T^{k-1}; n_1, \cdots, n_p, n_{p+1}) \xrightarrow{S} \hat{\Omega}_{m-1}(T^k; n_1, \cdots, n_p) \to 0 .$$

Proof. First we define the homomorphism F. Given $x=[(M, \varphi), (\xi_1, \varphi_1), \cdots, (\xi_p, \varphi_p)] \in \Omega_m(T^k; n_1, \cdots, n_p)$, let N be a connected component of the stationary point set of T_1^k in M. Let ν be the normal bundle of N in M. N may be oriented by the usual way. Then we define F by sending x to $\oplus [(N, \varphi'), (\xi_1', \varphi_1'), \cdots, (\xi_p', \varphi_p'), (\nu, \bar{\varphi}')]$, where, considering $T^{k-1}=T^k_{(2,\cdots,k)}$, $\varphi'=\varphi \mid T^{k-1}\times N$, $\xi_i'=\xi_i \mid N$, $\varphi_i'=\varphi_i \mid T^{k-1}\times \xi_i'$ and $\bar{\varphi}'=\bar{\varphi} \mid T^{k-1}\times \nu$ with $\bar{\varphi}$ being the T^k-action by bundle maps covering φ, and where the sum is taken over all connected components of the stationary point set of T_1^k.

Next we define the homomorphism S. Given $y=[(M, \varphi), (\xi_1, \varphi_1), \cdots, (\xi_{p+1}, \varphi_{p+1})] \in \Omega_r(T^{k-1}; n_1, \cdots, n_{p+1})$ $(m=r+2n_{p+1})$, let π: $S(\xi_{p+1}) \to M$ be the sphere bundle of ξ_{p+1}, and let $S(\xi_{p+1})$ be oriented as the boundary of $D(\xi_{p+1})$, $D(\xi_{p+1})$ oriented by the complex structure and the given orientation of M. Let $\pi^*\xi_i$ be the induced bundle on $S(\xi_{p+1})$ from ξ_i by π for $i=1, \cdots, p$. Define $\mu: T^k \times S(\xi_{p+1}) \to S(\xi_{p+1})$ by $\mu((\lambda_1, \cdots, \lambda_k), v) = \varphi_{p+1}((\lambda_2, \cdots, \lambda_k), \lambda_1 \cdot v)$ for $(\lambda_1, \cdots, \lambda_k) \in T^k$ and $v \in S(\xi_{p+1})$, and μ_i: $T^k \times \pi^*\xi_i \to \pi^*\xi_i$ by $\mu_i((\lambda_1, \cdots, \lambda_k), (u, v)) = (\varphi_i((\lambda_2, \cdots, \lambda_k), u), \mu((\lambda_1, \cdots, \lambda_k), v))$ for $u \in \xi_i$ and $v \in S(\xi_{p+1})$. Then we define S by sending y to $[(S(\xi_{p+1}), \mu), (\pi^*\xi_1, \mu_1), \cdots, (\pi^*\xi_p, \mu_p)]$.

We take the direct sum of the sequences $(*)$ over all $(m, n_1, \cdots, n_p)$ with $n=m+2\sum_{i=1}^p n_i$, after which this becomes precisely the sequence

$$0 \to \Omega_n(T^{k+p}; \mathfrak{F}_{k+p+1}, \mathfrak{F}_p) \xrightarrow{j} \Omega_n(T^{k+p}; \mathfrak{F}_{k+p+1}, \mathfrak{F}_{p+1}) \xrightarrow{\partial} \Omega_{n-1}(T^{k+p}; \mathfrak{F}_{p+1}, \mathfrak{F}_p) \to 0$$

of Theorem 3-1, using the identifications of Lemmas 4-4 and 4-5. Thus the exactness of the sequence $(*)$ follows. q.e.d.

Let t: $\oplus \Omega_r(T^{k-1}; n_1, \cdots, n_p, n_{p+1}) \to \oplus' \Omega_r(T^{k-1}; n_1, \cdots, n_p, n_{p+1})$ be the projection where the sum $\oplus$ is taken over all (r, n_{p+1}) with $m=r+2n_{p+1}$, and $\oplus'$ is taken over all (r, n_{p+1}) with $m=r+2n_{p+1}$ and $n_{p+1} \neq 1$.

Then we have

Corollary 4-7. *By the composition tF,*

$$\Omega_m(T^k; n_1, \cdots, n_p) \cong \oplus \Omega_r(T^{k-1}; n_1, \cdots, n_p, n_{p+1})$$

where the sum is taken over all (r, n_{p+1}) *with* $m=r+2n_{p+1}$ *and* $n_{p+1} \neq 1$.

Proof. To prove this, it sufficess to show that the homomorphism S maps the summand $\Omega_{m-2}(T^{k-1}; n_1, \cdots, n_p, 1)$ isomorphically onto $\hat{\Omega}_{m-1}(T^k; n_1, \cdots, n_p)$. We construct the inverse R: $\hat{\Omega}_{m-1}(T^k; n_1, \cdots, n_p) \to \Omega_{m-2}(T^{k-1}; n_1, \cdots, n_p, 1)$ for S on this summand as follows.

Given $z=[(M, \varphi), (\xi_1, \varphi_1), \cdots, (\xi_p, \varphi_p)] \in \hat{\Omega}_{m-1}(T^k; n_1, \cdots, n_p)$, T_1^k freely acts on M, so we obtain the n_i-dimensional complex vector bundles $\xi_i' \to M'$ by deviding out $\xi_i \to M$ by the actions of T_1^k for $i=1, \cdots, p$. Considering $T^{k-1}=T^k_{(2,\cdots,p)}$, φ and φ_i induce T^{k-1}-actions φ': $T^{k-1} \times M' \to M'$ and φ_i': $T^{k-1} \times \xi_i' \to \xi_i'$ with φ_i' being actions by bundle maps covering φ'. Let $L \to M'$ be the complex line bundle associated to the principal S^1-bundle $M \to M'$. We may give a T^{k-1}-action ψ on L by bundle maps which covers φ' and restricts to $\varphi | T^{k-1} \times M$ on $S(L)=M$. Then we define R by sending z to $[(M', \varphi'), (\xi_1', \varphi_1'), \cdots, (\xi_p', \varphi_p'), (L, \psi)]$. It is easily checked that R is the required inverse for S on the summand. q.e.d.

Proof of Theorem 4-1. $\Omega_m(T^k; \mathfrak{F}_{k+1}, \mathfrak{F}_0)$ is identified with $\Omega_m(T^k; 0)$ the group of cobordism classes of $(T^k, 1)$-manifold-bundles $((M, \varphi), (\xi, \psi))$ with $\dim M=m$ and $\dim_C \xi=0$ (i.e. $(M, \varphi)=(\xi, \psi)$). By repetition of Corollary 4-7, $\Omega_m(T^k; 0)$ is isomorphic to $\oplus \Omega_r(T^0; n_1, \cdots, n_k)$ where the sum is taken over all $(r, n_1, \cdots, n_k)$ with $m=r+2\sum_{i=1}^k n_i$ and $n_i \neq 1$ for $i=1, \cdots, k$. Corresponding bundles to their classifying maps, $\Omega_r(T^0; n_1, \cdots, n_k)$ is isomorphic to $\Omega_r(BU(n_1) \times \cdots \times BU(n_k))$. Theorem 4-1 thus follows. q.e.d.

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