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A GEOMETRIC MEANING OF THE MULTIPLICITY OF INTEGRABLE DISCRETE CLASSES IN $L^2(\Gamma\backslash G)$

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Introduction

Let Γ be a discrete subgroup of a group G of motions of a noncompact symmetric space M of inner type such that the quotient $\Gamma\backslash M$ is compact. Let K be the isotropy group at a point of M. Let ω be a discrete class of G such that K-finite matrix coefficients of ω belong to $L^1(G)$. Depending on some parameter corresponding to ω we can associate a homogeneous vector bundle over M. The Dirac operator D which is a first order elliptic G-invariant differential operator acts on the space of C^∞ sections of the above vector bundle. We then prove (Theorem 4, §3) that the multiplicity $N_\omega(\Gamma)$ of ω in the (right) regular representation of G on $L^2(\Gamma\backslash G)$ is the dimension of the space $H(\Gamma; *)$ of sections which are annihilated by D (Dirac spinors) and which are Γ-invariant and obtain a formula for the same (Corollary to Theorem 4, §3).

We remark that algebraic formulas for $N_\omega(\Gamma)$ are already available in several cases (Langlands [6]). Also when the parameter corresponding to ω satisfies some further conditions, Schmid [9] obtained geometric meaning to the multiplicity $N_\omega(\Gamma)$ (similar to Theorem 4), working with G/T rather than the symmetric space G/K where T is a Cartan subgroup of G contained in K.

Our method of proof is as follows: The space $H(\Gamma; *)$ is the direct sum of two subspaces $H^+(\Gamma; *)$ and $H^-(\Gamma; *)$. First we prove that one of these two spaces vanishes (Theorem 2, §1). Then using the Lefschetz Theorem of Atiyah and Singer [1] we obtain a formula for the difference $\dim H^+(\Gamma; *) - \dim H^-(\Gamma; *)$ (Theorem 3, §2). Let us divide our problem into two parts; namely,

(1) to prove $\dim H(\Gamma; *) = N_\omega(\Gamma)$ and

(2) to compute explicitly the above number.

When Γ has no elliptic elements other than the identity, we prove (1) by directly showing** that the expression for $\dim H^+(\Gamma; *) - \dim H^-(\Gamma; *)$ given by Theorem

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3 is, up to sign, just the algebraic formula for $N_\omega(\Gamma)$ obtained from the results in [3(b)] and [6(b)]. But the proof of (1) for general Γ can be reduced to the above case. Thus (1) is proved for all Γ. Then Theorem 3 yields part (2). This will be done in §3.

Finally in §4, we shall make several remarks when $M=G/K$ is a hermitian symmetric space. The space of Dirac spinors is then interpreted into the terminology of sheaf cohomology; hence our results appear to be related with the earlier works by several authors [4], [5], [6(a)], [7] and [9]. In view of the result by Trombi and Varadarajan [10], one can compare those results (see final Remarks).

Throughout the paper $\boldsymbol{Z}$, $\boldsymbol{R}$ and $\boldsymbol{C}$ will as usual denote the ring of rational integers, the field of real numbers and that of complex numbers respectively. For a finite set S, $|S|$ will denote the number of elements in S. For a C^∞ vector bundle E, $C^\infty(E)$ will denote the space of C^∞ sections of E.

1. Vanishing theorem

Let G be a connected noncompact semisimple Lie group with a faithful finite dimensional representation. We further make the convenient (but otherwise unnecessary) assumption that the complexification G^C of G is simply connected. Let $\mathfrak{g}$ be the Lie algebra of left invariant vector fields on G and $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let K be the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Let $T\subset K$ be a compact Cartan subgroup of G and $\mathfrak{t}\subset\mathfrak{k}$ the corresponding Lie algebra. Let $\mathfrak{g}^C$ denote the complexification of $\mathfrak{g}$. For any subspace $\mathfrak{m}$ of $\mathfrak{g}$ we dentoe by $\mathfrak{m}^C$ the complex subspace of $\mathfrak{g}^C$ generated by $\mathfrak{m}$. Let $\sum$ be the set of roots of $(\mathfrak{g}^C, \mathfrak{t}^C)$. Compact and noncompact roots have their usual meaning. P denotes the set of positive roots of $\sum$ with respect to some fixed order, P_k the set of positive compact roots and P_n the set of positive noncompact roots. Let $\rho=\frac{1}{2}\sum_{\alpha\in P}\alpha$, $\rho_k=\frac{1}{2}\sum_{\alpha\in P_k}\alpha$ and $\rho_n=\frac{1}{2}\sum_{\alpha\in P_n}\alpha$.

Since G was assumed to have a compact Cartan subgroup, G has discrete series, denoted by $\mathcal{E}_2(G)$. According to Harish-Chandra [3(b)], $\mathcal{E}_2(G)$ is parametrized by the set $\mathcal{F}_0'$ descirbed below:

Let

$$\mathcal{F}=\left\{\lambda\in\operatorname{Hom}(\mathfrak{t}^C, \boldsymbol{C});\frac{2\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}\in\boldsymbol{Z}, \text{ for } \alpha\in P\right\}$$

$$\mathcal{F}'=\{\lambda\in\mathcal{F};\langle\lambda+\rho, \alpha\rangle\neq 0 \text{ for } \alpha\in P\},$$

where $\langle\ ,\ \rangle$ denotes the usual bilinear form of $\mathcal{F}$ induced from the Killing form. Then

$$\mathcal{F}_0'=\{\lambda\in\mathcal{F}';\langle\lambda+\rho, \alpha\rangle>0 \text{ for } \alpha\in P_k\}$$

If $\lambda\in\mathcal{F}_0'$, $\lambda+\rho_n$ is the highest weight (with respect to P_k) of a representation $\tau_{\lambda+\rho_n}$ of $\mathfrak{k}^C$ on a space $V_{\lambda+\rho_n}$.

Let W be the Weyl group for $(\mathfrak{g}^C, \mathfrak{t}^C)$, W_G the subgroup of W generated by the reflections given by compact roots and set $W^1=\{\sigma\in W;\ \sigma P\supset P_k\}$. The map $W_G\times W^1\to W$ given by $(s, \sigma)\mapsto s\sigma$ is then a bijection. Put

$$D=\{\lambda\in\mathcal{F};\ \langle\lambda, \alpha\rangle\geq 0 \text{ for } \alpha\in P\}\,.$$

As in [8(a), Remark 8.1], we then have a bijection

$$D\times W^1\to\mathcal{F}_0'$$

given by $(\lambda, \sigma)\mapsto\lambda^{(\sigma)}$, where $\lambda^{(\sigma)}=\sigma(\lambda+\rho)-\rho$.

Let $SO(\mathfrak{p})$ denote the rotation group of $\mathfrak{p}$ with respect to the metric defined by the restriction to $\mathfrak{p}$ of the Killing form. Let $\mathrm{Spin}(\mathfrak{p})$ denote the unique connected two-fold covering group of $SO(\mathfrak{p})$. $\mathrm{Spin}(\mathfrak{p})$ is a subgroup of the group of invertible elements in the Clifford algebra $\mathrm{Cliff}(\mathfrak{p})$. One has for $x\in\mathrm{Spin}(\mathfrak{p})$ and $e\in\mathfrak{p}$, $xex^{-1}\in\mathfrak{p}$. The algebra $\mathrm{Cliff}(\mathfrak{p})$ has a unique simple module L, through a representation $\mathcal{E}$: $\mathrm{Cliff}(\mathfrak{p})\to\mathrm{End}(L)$. This gives rise to a map (also denoted by $\mathcal{E}$)

$$\mathcal{E}:\ \mathfrak{p}\otimes L\to L$$

given by $\mathcal{E}(x\otimes l)=\mathcal{E}(x)l$. Also, the restriction of $\mathcal{E}$ to $\mathrm{Spin}(\mathfrak{p})$ is "the spin representation" denoted by σ. It decomposes into the direct sum of two "half spin representations"

$$\sigma^{\pm}:\ \mathrm{Spin}(\mathfrak{p})\to\mathrm{Aut}(L^{\pm})\,.$$

Let $\mathfrak{so}(\mathfrak{p})$ denote the Lie algebra of $SO(\mathfrak{p})$ and let $\sigma^{\pm}$: $\mathfrak{so}(\mathfrak{p})\to\mathrm{End}(L^{\pm})$ denote also the representations of $\mathfrak{so}(\mathfrak{p})$ which are differentials of $\sigma^{\pm}$: $\mathrm{Spin}(\mathfrak{p})\to\mathrm{Aut}(L^{\pm})$. Since $[\mathfrak{k}, \mathfrak{p}]\subset\mathfrak{p}$, we have an adjoint homomorphism α: $\mathfrak{k}\to\mathfrak{so}(\mathfrak{p})$ (the corresponding group homomorphism will also be denoted by α). Let $\chi^{\pm}=\sigma^{\pm}\circ\alpha$. Denoting by j one of the sign $\pm$, we notice that the set of weights of χ^j is either $\{\rho_n-\langle Q\rangle;\ Q\subset P_n,\ |Q|: \text{even}\}$ or $\{\rho_n-\langle Q\rangle;\ Q\subset P_n,\ |Q|: \text{odd}\}$ where $\langle Q\rangle=\sum_{\alpha\in Q}\alpha$([8(a), Remark 2.1]). Here we may assume that $\chi^{\pm}$ are defined such that the set of weights of χ^j is $\{\rho_n-\langle Q\rangle;\ Q\subset P_n, \text{ the sign of } (-1)^{|Q|}=j\}$ for $j=\pm$.

REMARK. The representations $\chi^{\pm}$ of $\mathfrak{k}$ may not give rise to representations of K. Similarly, for $\lambda\in\mathcal{F}_0'$ the representation $\tau_{\lambda+\rho_n}$ of $\mathfrak{k}$ may not give rise to a representation of K. However, it is easy to see that the representation $\chi^{\pm}\otimes\tau_{\lambda+\rho_n}$ of $\mathfrak{k}$ give rise to representations of K.

We now fix a $\lambda\in\mathcal{F}_0'$ and write τ for $\tau_{\lambda+\rho_n}$. We now consider the Dirac operator denoted by D on the symmetric space $M=G/K$. Since we have a principal K-bundle $G\to M$, the representations $\chi^{\pm}\otimes\tau$ of K induce the associated vector bundles $E_{L^{\pm}\otimes V_{\lambda+\rho_n}}$ over M. The spaces $C^\infty(E_{L^{\pm}\otimes V_{\lambda+\rho_n}})$ of C^∞ sections of $E_{L^{\pm}\otimes V_{\lambda+\rho_n}}$ can be identified, in the usual manner, with certain subspaces

$$(L^{\pm}\otimes C^\infty(G)\otimes V_{\lambda+\rho_n})^0$$

of the spaces

$$L^{\pm}\otimes C^\infty(G)\otimes V_{\lambda+\rho_n}$$

of $L^{\pm}\otimes V_{\lambda+\rho_n}$-valued C^∞ functions of G: Denoting for $x\in\mathfrak{g}$ and $f\in C^\infty(G)$ by $\nu(x)f\in C^\infty(G)$, the derivation of f with respect to the left invariant vector field x,

$$\begin{aligned}(L^{\pm}\otimes C^\infty(G)\otimes V_{\lambda+\rho_n})^0 = \{&u\in L^{\pm}\otimes C^\infty(G)\otimes V_{\lambda+\rho_n};\\ &(\chi^{\pm}(y)\otimes 1\otimes 1+1\otimes\nu(y)\otimes 1+1\otimes 1\otimes\tau(y))u = 0,\ \forall y\in\mathfrak{k}\}\,.\end{aligned}$$

We denote by η the identification map $C^\infty(E_{L^{\pm}\otimes V_{\lambda+\rho_n}})\rightarrow(L^{\pm}\otimes C^\infty(G)\otimes V_{\lambda+\rho_n})^0$. Via this map the Dirac operator D has the following formula: for $s\in C^\infty(E_{L\otimes V_{\lambda+\rho_n}})$

$$\eta(Ds) = (\sum_{i=1}^{2n}\varepsilon(x_i)\otimes\nu(x_i)\otimes 1)(\eta s)\,,$$

where $\{x_i\}_{i=1}^{2n}$ is an orthonormal basis of $\mathfrak{p}$ and $2n=\dim_R\mathfrak{p}$. The symbol of the Dirac operator is

$$\varepsilon\otimes 1\colon \mathfrak{p}\otimes L^{\pm}\otimes V_{\lambda+\rho_n}\rightarrow L^{\pm}\otimes V_{\lambda+\rho_n}\,.$$

Let $\square=D^2$. Then one can prove [8(a), Proposition 3.1] that

$$\square = \langle\lambda+2\rho,\lambda\rangle-\hat{\pi}(\Omega) \tag{1.1}$$

where $\hat{\pi}(\Omega)$ denotes the action of the Casimir operator $\Omega\in U(\mathfrak{g}^C)$ (the universal enveloping algebra) on $C^\infty(E_{L\otimes V_{\lambda+\rho_n}})$.

We remark [8(a), Lemma 4.1] that there exists an inner product (,) in L such that for $x\in\mathfrak{p}$ and $l, l'\in L$

$$(\varepsilon(x)l, l')+(l, \varepsilon(x)l') = 0\,.$$

Then one has also for any $y\in\mathfrak{k}$,

$$(\chi(y)l, l')+(l, \chi(y)l') = 0\,.$$

We choose an inner product (,) in $V_{\lambda+\rho_n}$ such that for $y\in\mathfrak{k}$ and $v, v'\in V_{\lambda+\rho_n}$,

$$(\tau(y)v, v')+(v, \tau(y)v') = 0\,.$$

Let Γ be a discrete subgroup of G. We denote by $C^\infty(\Gamma; E_{L\otimes V_{\lambda+\rho_n}})$ the space of Γ-invariant sections in $C^\infty(E_{L\otimes V_{\lambda+\rho_n}})$. It is easy to see that D maps $C^\infty(\Gamma; E_{L\otimes V_{\lambda+\rho_n}})$ into itself. Under the isomorphism one has

$$\eta(C^\infty(\Gamma; E_{L\otimes V_{\lambda+\rho_n}})) = (L\otimes C^\infty(\Gamma\backslash G)\otimes V_{\lambda+\rho_n})^0\,.$$

For two elements $s, s' \in C^\infty(\Gamma; E_{L\otimes V\lambda+\rho_n})$, where

$$\eta(s) = \sum_i l_i \otimes f_i \otimes v_i \text{ and}$$
$$\eta(s') = \sum_j l_j' \otimes f_j' \otimes v_j'$$

we define

$$(s, s')_\Gamma = \sum_{i,j} (l_i, l_j')(v_i, v_j') \int_{\Gamma\backslash G} f_i \bar{f}_j' dx$$

with respect to a G-invariant measure on $\Gamma\backslash G$.

When $\Gamma = \{1\}$, $(\ ,\ ) = (\ ,\ )_{\{1\}}$ is the usual inner product defined in [8(a)] and one can prove the following (see remark for [8(a), Lemma 4.3]):

If $s \in C^\infty(E_{L\otimes V\lambda+\rho_n})$ and if both $(s, s) < \infty$ and $(\Box s, \Box s) < \infty$, then

$$(1.2)\qquad \begin{aligned} &(Ds, Ds) < \infty \text{ and one has} \\ &(\Box s, s) = (s, \Box s) = (Ds, Ds). \end{aligned}$$

Similarly, if we further assume that Γ is finitely generated, one can prove that if $(s, s)_\Gamma < \infty$ and if $(\Box s, \Box s)_\Gamma < \infty$ for $s \in C^\infty(\Gamma; E_{L\otimes V\lambda+\rho_n})$, then

$$(1.3)\qquad \begin{aligned} &(Ds, Ds)_\Gamma < \infty \text{ and} \\ &(\Box s, s)_\Gamma = (s, \Box s)_\Gamma = (Ds, Ds)_\Gamma. \end{aligned}$$

In fact, by Borel's theorem [2], one can then take a normal subgroup Γ' such that $[\Gamma : \Gamma'] < \infty$ where the quotient $\Gamma'\backslash M$ is a manifold. We may assume that $(s, s')_\Gamma = (s, s')_{\Gamma'}$ for $s, s' \in C^\infty(\Gamma; E_{L\otimes V\lambda+\rho_n})$ which is naturally imbedded in $C^\infty(\Gamma'; E_{L\otimes V\lambda+\rho_n})$. Since our riemannian metric on $\Gamma'\backslash M$ is complete, (1.3) also follows from Andreotti-Vesentini's observation.

We now define

$$H_2^\pm(E_{V\lambda+\rho_n}) = \{s \in C^\infty(E_{L\pm\otimes V\lambda+\rho_n});\ Ds = 0 \text{ and } (s, s) < \infty\}.$$

Similarly we define

$$H_2^\pm(\Gamma; E_{V\lambda+\rho_n}) = \{s \in C^\infty(\Gamma; E_{L\pm\otimes V\lambda+\rho_n}\};\ Ds = 0 \text{ and } (s, s)_\Gamma < \infty\}.$$

For $\sigma \in W^1$, we define

$$j(\sigma) = \begin{cases} + & \text{if } |\sigma(-P) \cap P| \text{ is even} \\ - & \text{if } |\sigma(-P) \cap P| \text{ is odd.} \end{cases}$$

For an element $\lambda \in \mathcal{F}_0'$, we write $j(\lambda) = j(\sigma)$ when $\lambda = \mu^{(\sigma)} = \sigma(\mu + \rho) - \rho$ with $\sigma \in W^1$, $\mu \in D$. Concerning $H_2^\pm(E_{V\lambda+\rho_n})$ one has the vanishing theorem [8(a), Theorem 2] which depends on (1.2) and [8(a), Lemma 8.1] for its proof. In a similar way using (1.3) and [8(a), Lemma 8.1] we now have the following

Theorem 1. *Let $\lambda \in D$ and $\sigma \in W^1$ so that $\lambda^{(\sigma)} \in \mathcal{F}_0'$. Assume that $\langle \sigma\lambda, \alpha \rangle \neq 0$ for any noncompact root α. If Γ is a finitely generated discrete subgroup of G,* *then*

$$H_2^j(\Gamma; E_{V\lambda^{(\sigma)}+\rho_n}) = 0 \qquad \text{for } j \neq j(\sigma)\,.$$

Now let $\mathcal{E}_1(G)$ be the set of all equivalence classes of irreducible unitary representations of G whose K-finite matrix coefficients are in $L^1(G)$. Then, according to Trombi and Varadarajan, $\mathcal{E}_1(G) \subset \mathcal{E}_2(G)$ ([10, Corollary 3.4]) and, for $\lambda \in \mathcal{F}_0'$, if $\omega(\lambda+\rho) \in \mathcal{E}_1(G)$, then

$$|\langle \lambda+\rho, \alpha\rangle| > |\langle w\rho, \alpha\rangle|$$

for any $\alpha \in P_n$ and $w \in W$, which follows immediately from [10, Theorem 8.2)]. Thus, for $\lambda \in D$ and $\sigma \in W^1$, if $\omega(\lambda^{(\sigma)}+\rho) \in \mathcal{E}_1(G)$, then according to the above necessary conditions, we have in paricular,

$$|\langle \sigma\lambda+\sigma\rho, \alpha\rangle| > |\langle \sigma\rho, \alpha\rangle| \text{ for any } \alpha \in P_n\,,$$

which implies $\langle \sigma\lambda, \alpha\rangle \neq 0$ for any $\alpha \in P_n$. Thus, we have using Theorem 1, the following

Theorem 2. *Let Γ be as in Theorem 1. Let $\lambda \in \mathcal{F}_0'$ and suppose $\omega(\lambda+\rho) \in \mathcal{E}_1(G)$. Then*

$$H_2^j(\Gamma; E_{V\lambda+\rho_n}) = 0 \qquad \text{for } j \neq j(\lambda)\,.$$

In the subsequent sections, we shall be concerned only with the case that $\Gamma = \{1\}$ or $\Gamma \backslash G$ is compact. When $\Gamma \backslash G$ is compact, $H_2^j(\Gamma; *)$ will be denoted by $H^j(\Gamma; *)$.

2. Computation of the Lefschetz number

We henceforth assume that Γ is a discrete subgroup of G such that $\Gamma \backslash G$ is compact. It is well known that Γ then satisfies the condition of Theorems 1, 2. Since the Dirac operator is elliptic, the spaces of Dirac spinors $H^{\pm}(\Gamma; E_{V\lambda+\rho_n})$ are then finite dimensional. Setting

$$\chi(\Gamma; \lambda) = \dim H^+(\Gamma; E_{V\lambda+\rho_n}) - \dim H^-(\Gamma; E_{V\lambda+\rho_n})$$

for $\lambda \in \mathcal{F}_0'$, we shall compute $\chi(\Gamma; \lambda)$ in this section. The result, as it should be, is compatible with that of Schmid [9, Theorem 3]. In order to state the result, several definitions and notations must be prepared, most of which are similar to those in [9].

For an element $\gamma \in T$, we denote by $\sum_\gamma$ the set of roots $\alpha \in \sum$ such that $e^{\alpha}(\gamma) = 1$, and define $P_\gamma = P \cap \sum_\gamma$ and $\rho_\gamma = \frac{1}{2}\sum_{\alpha \in P_\gamma} \alpha$. Let G_γ be the centralizer of γ in G, and G_γ^0 the identity component of G_γ. Let W_γ be the Weyl group for (G_γ^0, T), i.e., the quotient of the normalizer of T in G_γ^0 by T. Then, W_γ can be regarded as a subgroup of W_G. Let $\mathfrak{g}_\gamma$ be the Lie algebra of G_γ^0, i.e. the Lie subalgebra of $\mathfrak{g}$ consisting of $\mathrm{Ad}(\gamma)$-fixed elements in $\mathfrak{g}$, so that P_γ is a positive root system for $(\mathfrak{g}_\gamma^C, \mathfrak{t}^C)$ where $\mathfrak{g}_\gamma^C$ is the complexification of $\mathfrak{g}_\gamma$. Let W_γ^C be the Weyl group for $(\mathfrak{g}_\gamma^C, \mathfrak{t}^C)$. Then W_γ can be regarded as a subgroup of W_γ^C. Put $K_\gamma = G_\gamma \cap K$ and $n_\gamma = \frac{1}{2} \dim_R G_\gamma / K_\gamma$.

For $\lambda \in \mathcal{F}_0'$, we define the function Ψ_λ' on T by

$$\Psi_\lambda'(\gamma) = \frac{\sum_{w \in W_\gamma \backslash W_G} \mathcal{E}(w) e^{w(\lambda+\rho)-\rho}(\gamma) \prod_{\alpha \in P_\gamma} \langle w(\lambda+\rho), \alpha \rangle}{\prod_{\alpha \in P - P_\gamma} (1 - e^{-\alpha}(\gamma))}$$

for $\gamma \in T$, where $\mathcal{E}(w) = (-1)^{|wP \cap (-P)|}$. As in [9], Ψ_λ' is then well-defined and invariant under the action of W_G and this function can be extended to an $\mathrm{Ad}(G)$ invariant function on the set of all elliptic elements of G. Thus, if we put

$$\Psi_\lambda(\gamma) = j(\lambda)(-1)^{n+n_\gamma}[G_\gamma : G_\gamma^0]^{-1} |W_\gamma^C|^{-1} \prod_{\alpha \in P_\gamma} \langle \rho_\gamma, \alpha \rangle^{-1} \Psi_\lambda'(\gamma)$$

where $n = \frac{1}{2} \dim_R G/K = |P_n|$, Ψ_λ is also defined for any elliptic elemten γ. (If γ is regular, i.e. $P_\gamma = \phi$, put $\prod_{\alpha \in P_\gamma} \langle \rho_\gamma, \alpha \rangle = \prod_{\alpha \in P_\gamma} \langle w(\lambda+\rho), \alpha \rangle = 1$.)

We normalize the Haar measure on G_γ as follows. It is well known that for any discrete subgroup Γ_0 of G_γ such the $\Gamma_0 \backslash G_\gamma / T$ is a compact manifold, the Euler number $e(\Gamma_0 \backslash G_\gamma / T)$ has the same sign as $(-1)^{n_\gamma}$ and that there exists a unique Harr measure such that

$$v(\Gamma_0 \backslash G_\gamma) = (-1)^{n_\gamma} e(\Gamma_0 \backslash G_\gamma / T)$$

for any such discrete subgroup Γ_0, where $v(\Gamma_0 \backslash G_\gamma)$ is the volume of $\Gamma_0 \backslash G_\gamma$ with respect to that measure.

The purpose of this section is to prove

Theorem 3. *Put* $\Gamma_\gamma = G_\gamma \cap \Gamma$ *and normalize the Haar measure on* G_γ *as above. Then*

$$\chi(\Gamma; \lambda) = j(\lambda) \sum_\gamma v(\Gamma_\gamma \backslash G_\gamma) \Psi_\lambda(\gamma)$$

where γ *runs over the (finite) set of* Γ*-conjugacy classes of elliptic elements in* Γ.

REMARK. For reference, we remark on the difference of our normalization of the Haar measure on G_γ from that in [9]. Dentoe by $v'(\Gamma_0 \backslash G_\gamma)$ the volume of $\Gamma_0 \backslash G_\gamma$ under the normalization in [9]. By means of computation of Gauss-Bonnet type, Schmid gets

$$e(\Gamma_0 \backslash G_\gamma / T) = (2\pi)^{-m_\gamma} (-1)^{n_\gamma} |W_\gamma^C| \prod_{\alpha \in P_\gamma} \langle \rho_\gamma, \alpha \rangle v'(\Gamma_0 \backslash G_\gamma)$$

for Γ_0 such that $\Gamma_0 \backslash G_\gamma / T$ is a compact manifold ([9, (5.12) and (5.14)]), where m_γ

$=\frac{1}{2}\dim_R G_\gamma/T=|P_\gamma|$. Hence, if we define as in [9]

$$\Phi_\lambda(\gamma)=j(\lambda)(-1)^{n+n_\gamma}(2\pi)^{-m_\gamma}[G_\gamma\colon G_\gamma^0]^{-1}\Psi_\lambda'(\gamma),$$

then Theorem 3 will imply

$$\chi(\Gamma;\lambda)=j(\lambda)\sum v'(\Gamma_\gamma\backslash G_\gamma)\Phi_\lambda(\gamma).$$

By Borel [2], we can choose a normal subgroup Γ' of Γ such that $[\Gamma\colon\Gamma']<\infty$ and Γ' contains no elliptic elements other than the identity. Put $F=\Gamma/\Gamma'$. The finite group F then acts on the principal K-bundle $\Gamma'\backslash G\to X$ where $X=\Gamma'\backslash G/K$ is a compact manifold, so on the spinor bundle $E_{L\otimes V\lambda+\rho_n}$ (regarded as that over X). Since this action commutes with the Dirac operator, any element $f\in F$ acts on $H^\pm(\Gamma';E_{V\lambda+\rho_n})$. Denote by $f|H^\pm(\Gamma';E_{V\lambda+\rho_n})$ these induced actions. It is then easy to see that

$$\chi(\Gamma;\lambda)=|F|^{-1}\textstyle\sum_{f\in F}L(f;\lambda) \tag{2.1}$$

where

$$L(f;\lambda)=\text{trace}\{f|H^+(\Gamma';E_{V\lambda+\rho_n})\}-\text{trace}\{f|H^-(\Gamma';E_{V\lambda+\rho_n})\}.$$

In order to compute the Lefschetz number $L(f;\lambda)$, we must look into the fixed point set X^f of f in X. The following lemmas are similar to those of Hirzebruch's and Schmid's ([4], [9]).

Lemma 1. *Let M^γ be the fixed point set of $\gamma\in G$ in $M=G/K$. Then $M^\gamma\neq\phi$ if and only if γ is elliptic. In this case, M^γ is itself a connected symmetric space on which G_γ and G_γ^0 act transitively. Assume moreover that $\gamma\in K$. Then M^γ is isomorphic to $G_\gamma/K_\gamma\cong G_\gamma^0/K_\gamma^0$ and the imbedding $M^\gamma\hookrightarrow M$ is given by $gK_\gamma\mapsto gK$ for $g\in G_\gamma$, where K_γ^0 is the identity component of K_γ.*

Proof. Assume that $\gamma\in K$ so that $\{K\}\in M^\gamma$. We show that G_γ^0 acts transitively on M^γ. If $gK\in M^\gamma$, then $g^{-1}\gamma g\in K$. Let $g=(\exp y)k$ be the Cartan decomposition of g as an element in G, where $k\in K$ and $y\in\mathfrak{p}$. Since $k^{-1}(\exp y)^{-1}\gamma(\exp y)k=k'\in K$, we have $(\exp \mathrm{Ad}\,\gamma(y))\gamma k=(\exp y)kk'$. By the uniqueness of the Cartan decomposition, $y=\mathrm{Ad}\,\gamma(y)$. If we denote by $\mathfrak{p}^\gamma$ the subspace of $\mathfrak{p}$ consisting of Ad γ-fixed elements in $\mathfrak{p}$, we thus have $y\in\mathfrak{p}^\gamma$, which implies $\exp y\in G_\gamma^0$. Hence $gK=(\exp y)K\in G_\gamma^0K$, which means that $M^\gamma\cong G_\gamma^0/K_\gamma^0=G_\gamma/K_\gamma$. When $\gamma\notin K$, take $g_0\in G$ such that $g_0^{-1}\gamma g_0\in K$ and put $\gamma'=g_0^{-1}\gamma g_0$. Then $M^\gamma=g_0M^{\gamma'}$ and Lemma easily follows from the special case. q.e.d.

Lemma 2. *Denote by R_f the set Γ'-conjugacy classes in f (as a coset) which consists of elliptic elements. This is a finite set. For $\gamma\in R_f$, put $X^\gamma=\Gamma_\gamma'\backslash M^\gamma$, where $\Gamma_\gamma'=\Gamma'\cap G_\gamma$. Then X^f can be regarded as the disjoint union $\cup_{\gamma\in R_f}X^\gamma$.*

The proof is similar to that of [9, Lemma 11].

Let $\mathcal{E}_\lambda$ be the elliptic complex consisting of the Dirac operator D: $C^\infty(\Gamma'; E_{L^+\otimes V\lambda+\rho_n}) \to C^\infty(\Gamma'; E_{L^-\otimes V\lambda+\rho_n})$ over $X=\Gamma'\backslash M$. Let Φ be the cyclic subgroup generated by f in F. The symbol class of $\mathcal{E}_\lambda$ is then associated to the K-structure of X in the sense of [1, §2] and given as follows. As in §1, the Clifford multiplication ε gives a K-module homomorphism

$$(2.2) \qquad \mathfrak{p} \to \mathrm{Hom}\,(L^+\otimes V_{\lambda+\rho_n}, L^-\otimes V_{\lambda+\rho_n}),$$

which induces a Φ-map

$$(2.3) \qquad (\Gamma'\backslash G\times\mathfrak{p})\times_K(L^+\otimes V_{\lambda+\rho_n})\to(\Gamma'\backslash G\times\mathfrak{p})\times_K(L^-\otimes V_{\lambda+\rho_n})$$

where $\Gamma'\backslash G\times\mathfrak{p}$ is considered as a principal K-bundle over the tangent bundle TX which is isomorphic to the associated vector bundle $\Gamma'\backslash G\times_K\mathfrak{p}$ and where $(\Gamma'\backslash G\times\mathfrak{p})\times_K(L^\pm\otimes V_{\lambda+\rho_n})$ are the vector bundles over TX associated to the K-modules $L^\pm\otimes V_{\lambda+\rho_n}$. Denoting by π the projection $TX\to X$, it is clear that

$$\pi^{-1}(E_{L^\pm\otimes V\lambda+\rho_n}) = (\Gamma'\backslash G\times\mathfrak{p})\times_K(L^\pm\otimes V_{\lambda+\rho_n}),$$

where $E_{L^\pm\otimes V\lambda+\rho_n}=\Gamma'\backslash G\times_K(L^\pm\otimes V_{\lambda+\rho_n})$. Since the map (2.3) is an isomorphism outside the zero-scetion of TX, it gives an element $\sigma(\mathcal{E}_\lambda)\in K_\Phi(TX)$ called the symbol class of $\mathcal{E}_\lambda$, where K_Φ denotes the Φ-equivariant K-theory. By Lemma 2, the fixed point set X^f of f in X splits as $X^f=\bigcup_{\gamma\in R_f} X^\gamma$. Let i_γ: $X^\gamma\hookrightarrow X$ be the imbedding for each $\gamma\in R_f$, and i_γ^*: $K_\Phi(TX)\to K_\Phi(TX^\gamma)$ the induced homomorphism. Since Φ acts trivially on X^γ and so on its tangent bundle TX^γ, one can define the cohomology class

$$\mathrm{ch}\; i_\gamma^*\sigma(\mathcal{E}_\lambda)(f)\in H^*(TX^\gamma; \boldsymbol{C})$$

as in [1, §3].

Denoting by N^γ the normal bundle of X^γ in X, one has an eigenspace decomposition by the f-action

$$(2.4) \qquad N^\gamma = N^\gamma(-1)\oplus\sum_{0<\theta<\pi} N^\gamma(\theta)$$

as in [1, (3.3)]. The characteristic classes $\mathcal{R}(N^\gamma(-1))$, $\mathcal{S}^\theta(N^\gamma(\theta))\in H^*(X^\gamma; \boldsymbol{C})$ are then defined there ([1, (3.7)]). Denote by $\mathcal{J}(X^\gamma)\in H^*(X^\gamma; \boldsymbol{C})$ the index class of X^γ ([1, §2]). We notice that $H^*(TX^\gamma; \boldsymbol{C})$ has a natural $H^*(X^\gamma; \boldsymbol{C})$-module strucutre. Lefschetz Theorem [1, (3.9)] then says that

$$(2.5) \qquad L(f; \lambda) = \sum_{\gamma\in R_f} L(\gamma; \lambda)$$

where

$$(2.6) \qquad L(\gamma; \lambda) = \left\{\frac{\mathrm{ch}\; i_\gamma^*\sigma(\mathcal{E}_\lambda)(f)\,\mathcal{R}(N^\gamma(-1))\prod_{0<\theta<\pi}\mathcal{S}^\theta(N^\gamma(\theta))\,\mathcal{J}(X^\gamma)}{\det(1-f\,|\,N^\gamma)}\right\}[TX^\gamma].$$

Note that $\dim_R X^\gamma = 2n_\gamma$ in our case and the fundamental class $[TX^\gamma]$ is given by the orientation induced from the almost complex structure on TX^γ ([1, Proposition (2.11)]).

By Lemmas 1, 2 we can choose a suitable maximal compact subgroup K_γ of G_γ such that $X^\gamma \cong \Gamma_\gamma' \backslash G_\gamma / K_\gamma$ for each $\gamma \in R_f$ (when $\gamma \in K$, we can choose $K_\gamma = K \cap G_\gamma$ as in Lemma 1). Let K_γ^0 be the identity component of K_γ and put $\tilde{X}^\gamma = \Gamma_\gamma' \backslash G_\gamma / K_\gamma^0$. Then $\tilde{X}^\gamma$ is orientable and the covering $p_\gamma \colon \tilde{X}^\gamma \to X^\gamma$ is of order $[K_\gamma \colon K_\gamma^0]$. Denoting the covering map $T\tilde{X}^\gamma \to TX^\gamma$ also by the same letter p_γ, the formula (2.6) can be then rewritten as

$$(2.7)\quad L(\gamma;\lambda) = [K_\gamma : K_\gamma^0]^{-1} \left\{ \frac{p_\gamma^*(\mathrm{ch}\, i_\gamma^* \sigma(\mathcal{E}_\lambda)(f)) p_\gamma^*(\mathcal{R}(N^\gamma(-1)) \prod_{0<\theta<\pi} \mathcal{S}^\theta(N^\gamma(\theta)) \mathcal{J}(X^\gamma))}{\det(1-f \mid N^\gamma)} \right\} [T\tilde{X}^\gamma] .$$

Since $\tilde{X}^\gamma$ is orientable, evaluation of cohomology classes of $T\tilde{X}^\gamma$ can be replaced by one on $\tilde{X}^\gamma$ by means of the Thom isomorphism. Furthermore, since the symbol class $\sigma(\mathcal{E}_\lambda)$ is associated to the K-structure under the map (2.2), the formula (2.7) can be rewritten in a more useful form by a similar argument to the one in [1, §2] (see also [1, §3, Remark to Theorem (3.9)]). For the sake of convenience, we introduce the following characteristic class. Let U be a compact Lie group and γ an element in the center of U. Assume that there is given a U-module V. For any principal U-bundle P over any compact manifold X, the cyclic group $\{\gamma\}$ generated by γ acts on the associated vector bundle $E_V = P \times_U V$ over X by

$$\gamma(p, v) = (p\gamma, v) \text{ for } (p, v) \in P \times_U V$$

(the action of $\{\gamma\}$ on X is trivial). Hence E_V determines an element in $K_{\{\gamma\}}(X)$ and a cohomology class $\mathrm{ch}\, E_V(\gamma) \in H^*(X; \boldsymbol{C})$ can be defined. Since the map

$$P \mapsto \mathrm{ch}\, E_V(\gamma) \in H^*(X; \boldsymbol{C})$$

is functorial, this defines an element of the ring of characteristic classes $H_U^*(\boldsymbol{C})$ for the group U. We denote this element by $\mathrm{ch}\, V(\gamma) \in H_U^*(\boldsymbol{C})$, so that

$$\mathrm{ch}\, V(\gamma)\,(P) = \mathrm{ch}\, E_V(\gamma)$$

for any principal U-bundle P. We note that $\mathrm{ch}\, V(\gamma)$ is given explicitly as follows. Let T be a maximal torus of U containing γ and W the Weyl gorup for (U, T). Using the identification of $H_U^*(\boldsymbol{C})$ with $H_T^*(\boldsymbol{C})^W$ consisting of W-fixed elements in $H_T^*(\boldsymbol{C})$, $\mathrm{ch}\, V \in H_U^*(\boldsymbol{C})$ is given by $\sum_\mu e^\mu \in H_T^*(\boldsymbol{C})^W$ where μ runs over the set of weights of V. Since γ is in the center of U, $\sum_\mu e^\mu(\gamma) e^\mu$ defines an element of $H_T^*(\boldsymbol{C})^W$ hence of $H_U^*(\boldsymbol{C})$. In case $U = T$, it is easy to see that $e^\mu(\gamma) e^\mu = \mathrm{ch}\, V_\mu(\gamma)$ for any 1-dimensional T-module V_μ with character e^μ. The functoriality of characteristic classes now implies that

$$\text{ch } V(\gamma) = \textstyle\sum_{\mu} e^{\mu}(\gamma) e^{\mu} . \tag{2.8}$$

We now put $P^{(\gamma)} = \Gamma_\gamma' \backslash G_\gamma$ considered as a principal K_γ^0-bundle over $\tilde{X}^\gamma = \Gamma_\gamma' \backslash G_\gamma / K_\gamma^0$. Since the symbol class $\sigma(\mathcal{E}_\lambda)$ is given by the map (2.2), its restriction $i_\gamma^* \sigma(\mathcal{E}_\lambda)$ is in a similar way given by the composite map

$$\mathfrak{p}^\gamma \hookrightarrow \mathfrak{p} \to \text{Hom}(L^+ \otimes V_{\lambda+\rho_n}, L^- \otimes V_{\lambda+\rho_n}) , \tag{2.9}$$

where $\mathfrak{p}^\gamma$ is the subspace of Ad γ-fixed elements in $\mathfrak{p}$. Again the pull back $p_\gamma^* i_\gamma^* \sigma(\mathcal{E}_\lambda)$ by the map p_γ: $T\tilde{X}^\gamma \to TX^\gamma$ is also given by the K_γ^0-module homomorphism (2.9) in a similar way. Since the f-actions on our vector bundles equal the γ-actions as in the above paragraph when restricted to X^γ or TX^γ, we have $p_\gamma^*(\text{ch } i_\gamma^* \sigma(\mathcal{E}_\lambda)(f)) = \text{ch } p_\gamma^* i_\gamma^* \sigma(\mathcal{E}_\lambda)(\gamma)$. We now notice the following two points:

(1) the class $p_\gamma^* i_\gamma^* \sigma(\mathcal{E}_\lambda) \in K_{(\gamma)}(T\tilde{X}^\gamma)$ is associated to the K_γ^0-structure on $\tilde{X}^\gamma$ by (2.9) where $T\tilde{X}^\gamma = P^{(\gamma)} \times_{K_\gamma^0} \mathfrak{p}^\gamma$,

(2) a maximal torus of K_γ^0 has no fixed non-zero vector in $\mathfrak{p}^\gamma$. By (2), if we denote by $e_\gamma \in H_{K_\gamma^0}^*(\boldsymbol{Z})$ the restriction of the Euler class of $H_{SO(\mathfrak{p}^\gamma)}^*(\boldsymbol{Z})$ via the representation $K_\gamma^0 \to SO(\mathfrak{p}^\gamma)$, then $e_\gamma \neq 0$. Here we assume that the orientation of $\tilde{X}^\gamma$ has been fixed such that $e_\gamma(P^{(\gamma)})[\tilde{X}^\gamma]$ is equal to the Euler number $e(\tilde{X}^\gamma)$ of $\tilde{X}^\gamma$. In view of (1) and (2), using an argument similar to the one in [1, §2], one can see that the characteristic class $\text{ch}\, L^+ \otimes V_{\lambda+\rho_n}(\gamma) - \text{ch}\, L^- \otimes V_{\lambda+\rho_n}(\gamma) \in H_{K_\gamma^0}^*(\boldsymbol{C})$, defined as above, can be divided by e_γ, and in the formula (2.7) $\text{ch } p_\gamma^* i_\gamma^* \sigma(\mathcal{E}_\lambda)(\gamma)$ can be replaced by

$$\left(\frac{\text{ch}\, L^+ \otimes V_{\lambda+\rho_n}(\gamma) - \text{ch}\, L^- \otimes V_{\lambda+\rho_n}(\gamma)}{e_\gamma}\right)(P^{(\gamma)}) \in H^*(\tilde{X}^\gamma; \boldsymbol{C})$$

in the case of evaluation on $\tilde{X}^\gamma$; i.e., we have

$$\begin{aligned} L(\gamma;\lambda) = {} & (-1)^{n_\gamma}[K_\gamma : K_\gamma^0]^{-1} \times \\ & \times \left\{ \left(\frac{\text{ch}\, L^+ \otimes V_{\lambda+\rho_n}(\gamma) - \text{ch}\, L^- \otimes V_{\lambda+\rho_n}(\gamma)}{e_\gamma}\right)(P^{(\gamma)}) \right. \\ & \left. \frac{p_\gamma^*(\mathcal{R}(N^\gamma(-1)) \prod_{0<\theta<\pi} \mathcal{S}^\theta(N^\gamma(\theta)) \mathcal{J}(X^\gamma))}{\det(1 - f \mid N^\gamma)} \right\} [\tilde{X}^\gamma] . \end{aligned} \tag{2.10}$$

Using this formula, we shall prove the following lemma which is crucial for the proof of Theorem 3.

Lemma 3. $L(\gamma;\lambda) = j(\lambda) v(\Gamma_\gamma' \backslash G_\gamma) \Psi_\lambda(\gamma)$.

If we take for granted Lemma 3, Theorem 3 easily follows from Hirzebruch-Schmid's observation, i.e.

$$\chi(\Gamma;\lambda) = |F|^{-1}\textstyle\sum_{f\in F} L(f;\lambda) \tag{2.1}$$
$$= |F|^{-1} \sum_{\gamma\in R_f,\, f\in F} L(\gamma;\lambda) \tag{2.5}$$
$$= j(\lambda)[\Gamma:\Gamma']^{-1} \sum_{\substack{\gamma:\ \Gamma'\text{-conjugacy classes}\\ \text{of elliptic elements in } \Gamma}} v(\Gamma'_\gamma\backslash G_\gamma)\Psi_\lambda(\gamma) \quad \text{(Lemma 3)}$$
$$= j(\lambda) \sum_{\substack{\gamma:\ \Gamma\text{-conjugacy classes}\\ \text{of elliptic elements in } \Gamma}} v(\Gamma_\gamma\backslash G_\gamma)\Psi_\lambda(\gamma)$$

where the last equality follows form [9, (5.16), (5.17)].

We shall first prove Lemma 3 under the assumption that $\gamma\in T$. Then T is a maximal torus of K^0_γ. Since W_γ is regarded as the Weyl group for (K^0_γ, T), we have the identification $H^*_{K^0_\gamma}(\boldsymbol{C})\cong H^*_T(\boldsymbol{C})^{W_\gamma}$. By Weyl's character formula and the definition of $L^\pm$, we have

$$\operatorname{ch} L^+\otimes V_{\lambda+\rho_n} - \operatorname{ch} L^-\otimes V_{\lambda+\rho_n} = \frac{\sum_{s\in W_G} \varepsilon(s) e^{s(\lambda+\rho)}}{\Delta_k}\,\Delta_n$$

where $\Delta_k = \prod_{\alpha\in P_k}(e^{\alpha/2}-e^{-\alpha/2})$ and $\Delta_n=\prod_{\alpha\in P_n}(e^{\alpha/2}-e^{-\alpha/2})$. By (2.8), we then have

$$\begin{aligned}(2.11)\qquad &\operatorname{ch} L^+\otimes V_{\lambda+\rho_n}(\gamma) - \operatorname{ch} L^-\otimes V_{\lambda+\rho_n}(\gamma)\\ &= \frac{\sum_{s\in W_G} \varepsilon(s) e^{s(\lambda+\rho)}(\gamma) e^{s(\lambda+\rho)}}{\prod_{\alpha\in P_k}(e^{\alpha/2}(\gamma)e^{\alpha/2}-e^{-\alpha/2}(\gamma)e^{-\alpha/2})} \prod_{\alpha\in P_n}(e^{\alpha/2}(\gamma)e^{\alpha/2}-e^{-\alpha/2}(\gamma)e^{-\alpha/2})\,.\end{aligned}$$

Note that even if $e^{\pm\alpha/2}(\gamma)$ or $e^{\pm\alpha/2}$ have no meaning, the right hand side of (2.11) can still be interpreted as

$$\begin{aligned}&\Big(\sum_{s\in W_G}\varepsilon(s)e^{s(\lambda+\rho)}(\gamma)e^{s(\lambda+\rho)}\Big)e^{\rho_n-\rho_k}(\gamma)e^{\rho_n-\rho_k}\\ &\quad\times\prod_{\alpha\in P_n}(1-e^{-\alpha}(\gamma)e^{-\alpha})\prod_{\alpha\in P_k}(1-e^{-\alpha}(\gamma)e^{-\alpha})^{-1},\end{aligned}$$

which is meaningful.

Put $P^\gamma_n = P_n\cap P_\gamma$. The set of weights of the K^0_γ-module $\mathfrak{p}^\gamma\otimes\boldsymbol{C}$ is then $\sum^\gamma_n = P^\gamma_n\cup(-P^\gamma_n)$. Concerning the Euler class $e_\gamma\in H^*_{K^0_\gamma}(\boldsymbol{Z})$, it is easily seen that $e_\gamma = \varepsilon_\gamma\prod_{\alpha\in P^\gamma_n}\alpha\in H^*_{K^0_\gamma}(\boldsymbol{Z})\cong H^*_T(\boldsymbol{Z})^{W_\gamma}$ where $\varepsilon^\gamma=\pm1$ depending on the orientation of $\tilde{X}^\gamma$ (or $\mathfrak{p}^\gamma$). We may asume that the orientation of $\tilde{X}^\gamma$ is fixed so that the Euler class is

$$e_\gamma = \prod_{\alpha\in P^\gamma_n}\alpha\,. \tag{2.12}$$

Concerning the index class, we have by definition

$$(2.13)\qquad \mathcal{J}(\tilde{X}^\gamma)=\Big(\prod_{\alpha\in P_n^\gamma}\frac{-\alpha}{1-e^{\alpha}}\prod_{\alpha\in P_n^\gamma}\frac{\alpha}{1-e^{-\alpha}}\Big)(P^{(\gamma)})$$
$$=\Big(\prod_{\alpha\in\Sigma_n^\gamma}\frac{\alpha}{1-e^{\alpha}}\Big)(P^{(\gamma)})\,.$$

Let $\check{\mathfrak{p}}^\gamma$ be the orthogonal complement of $\mathfrak{p}^\gamma$ in $\mathfrak{p}$. The normal bundle N^γ of X^γ in X can then be regarded as the associated vector bundle $P^{(\gamma)}\times_{K_\gamma}\check{\mathfrak{p}}^\gamma$ where $P^{(\gamma)}$ is considered as a principal K_γ-bundle over X^γ. We have $\check{\mathfrak{p}}^\gamma\otimes \boldsymbol{C}=\sum_{\alpha\in P_n-P_n^\gamma}\mathfrak{g}^{\pm\alpha}$ as T-modules where $\mathfrak{g}^\alpha$ denotes the eigenspace in $\mathfrak{g}^C$ of a root $\alpha\in\sum$. If we put $\mathfrak{q}(\alpha)=\check{\mathfrak{p}}^\gamma\cap(\mathfrak{g}^\alpha\oplus\mathfrak{g}^{-\alpha})$ for a root α, γ acts on $\mathfrak{q}(\alpha)$ as

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

where $0<\theta<\pi$ is given by $e^{i\theta}=e^{\pm\alpha}(\gamma)$. Hence in the decomposition (2.4) of N^γ, we have

$$N^\gamma(\theta)=P^{(\gamma)}\times_{K_\gamma}\Big(\sum_{\alpha\in P_n-P_n^\gamma,\ e^{\pm\alpha}(\gamma)=e^{i\theta}}\mathfrak{q}(\alpha)\Big)\qquad\text{for } 0<\theta<\pi$$

and

$$N^\gamma(-1)=P^{(\gamma)}\times_{K_\gamma}\Big(\sum_{\alpha\in P_n-P_n^\gamma,\ e^{\alpha}(\gamma)=-1}\mathfrak{q}(\alpha)\Big)\,.$$

By definition, we then have

$$\mathcal{S}^\theta(N^\gamma(\theta))=\Big\{\prod_{\substack{\alpha\in P_n-P_n^\gamma\\ e^{i\theta}=e^{\pm\alpha}(\gamma)}}\Big(\frac{1-e^{\alpha}(\gamma)e^{\alpha}}{1-e^{\alpha}(\gamma)}\Big)\Big(\frac{1-e^{-\alpha}(\gamma)e^{-\alpha}}{1-e^{-\alpha}(\gamma)}\Big)\Big\}^{-1}(P^{(\gamma)})$$

and

$$\mathcal{R}(N^\gamma(-1))=\Big\{\prod_{\substack{\alpha\in P_n-P_n^\gamma\\ e^{\alpha}(\gamma)=-1}}\Big(\frac{1+e^{\alpha}}{2}\Big)\Big(\frac{1+e^{-\alpha}}{2}\Big)^{-1}(P^{(\gamma)})$$

where $P^{(\gamma)}$ is considered as a K_γ-bundle over X^γ. Hence

$$(2.14)\qquad p_\gamma^*\Big(\mathcal{R}(N^\gamma(-1))\prod_{0<\theta<\pi}\mathcal{S}^\theta(N^\gamma(\theta))\Big)$$
$$=\Big\{\prod_{\alpha\in P_n-P_n^\gamma}\Big(\frac{1-e^{\alpha}(\gamma)e^{\alpha}}{1-e^{\alpha}(\gamma)}\Big)\Big(\frac{1-e^{-\alpha}(\gamma)e^{-\alpha}}{1-e^{-\alpha}(\gamma)}\Big)\Big\}^{-1}(P^{(\gamma)})$$

as elements in $H^*(\tilde{X}^\gamma;\boldsymbol{C})$ where $P^{(\gamma)}$ is considered as a K_γ^0-bundle over $\tilde{X}^\gamma$. By the above argument it is also easy to see that

$$(2.15)\qquad \det(1-f\,|\,N^\gamma)=\prod_{\alpha\in P_n-P_n^\gamma}(1-e^{\alpha}(\gamma))(1-e^{-\alpha}(\gamma))\,.$$

Noting that $n_\gamma=|P_n^\gamma|$, $e^{\alpha}(\gamma)=1$ for $\alpha\in P_n^\gamma$ and $p_\gamma^*\mathcal{J}(X^\gamma)=\mathcal{J}(\tilde{X}^\gamma)$, we thus have

$$(2.16)\qquad L(\gamma;\lambda)=(-1)^n[K_\gamma\colon K_\gamma^0]^{-1}$$
$$\left\{\left(\frac{\sum_{s\in W_G}\mathcal{E}(s)e^{s(\lambda+\rho)}(\gamma)e^{s(\lambda+\rho)}}{\prod_{\alpha\in P}(e^{\alpha/2}(\gamma)e^{\alpha/2}-e^{-\alpha/2}(\gamma)e^{-\alpha/2})}\prod_{\alpha\in P_n^\gamma}\alpha\right)(P^{(\gamma)})\right\}[\tilde{X}^\gamma],$$

by substituting (2.1)~(2.15) into (2.10).

Consider the compact manifold $Y^\gamma=\Gamma_\gamma'\backslash G_\gamma/T$ and the fibre bundle $h\colon Y^\gamma\to\tilde{X}^\gamma$ with the standard fibre $S^\gamma=K_\gamma^0/T$. Then $P^{(\gamma)}$ can also be regarded as a principal T-bundle over Y^γ which will be denoted by $P^{(\gamma)}/Y^\gamma$. Let $T\tilde{X}^\gamma$, TY^γ be the tangent bundles of $\tilde{X}^\gamma$, Y^γ respectively. Let $\widetilde{TS}^\gamma$ be the vector bundle over Y^γ tangential along fibres. Then

$$(2.17)\qquad TY^\gamma=h^{-1}(T\tilde{X}^\gamma)\oplus\widetilde{TS}^\gamma.$$

We can fix the orientation of the vector bundle $\widetilde{TS}^\gamma$ so that its Euler class $e_{\widetilde{TS}^\gamma}\in H^*(Y^\gamma;\boldsymbol{Z})$ is given by

$$(2.18)\qquad e_{\widetilde{TS}^\gamma}=(\prod_{\alpha\in P_k}\alpha)(P^{(\gamma)}/Y^\gamma)$$

where $\prod_{\alpha\in P_k^\gamma}\alpha\in H_T^*(\boldsymbol{Z})$. Fix the orientation of TY^γ induced from those of $T\tilde{X}^\gamma$ and $\widetilde{TS}^\gamma$. Denoting by e_{Y^γ}, $e_{\tilde{X}^\gamma}$ the Euler classes of TY^γ, $T\tilde{X}^\gamma$ respectively, we have

$$(2.19)\qquad e_{Y^\gamma}=(h^*e_{\tilde{X}^\gamma})e_{\widetilde{TS}^\gamma}\text{ in }H^*(Y^\gamma;\boldsymbol{Z})$$

by the multiplicative property of the Euler classes for (2.17). For a top-degree cohomology class $u\in H^{2n_\gamma}(\tilde{X}^\gamma;\boldsymbol{C})$, we have $u=e(\tilde{X}^\gamma)^{-1}u[\tilde{X}^\gamma]e_{\tilde{X}^\gamma}$ since the Euler number $e(\tilde{X}^\gamma)=e_{\tilde{X}^\gamma}[\tilde{X}^\gamma]$ is non-zero. Hence we have

$$\{(h^*u)e_{\widetilde{TS}^\gamma}\}[Y^\gamma]=e(\tilde{X}^\gamma)^{-1}u[\tilde{X}^\gamma]e_{Y^\gamma}[Y^\gamma]$$

by (2.19). On the other hand, since $e_{Y^\gamma}[Y^\gamma]=e(Y^\gamma)=e(\tilde{X}^\gamma)e(S^\gamma)$ and $e(S^\gamma)=|W_\gamma|$ where $e(Y^\gamma)$, $e(S^\gamma)$ denote the Euler numbers of Y^γ, S^γ respectively, we have

$$(2.20)\qquad u[\tilde{X}^\gamma]=|W_\gamma|^{-1}\{(h^*u)e_{\widetilde{TS}^\gamma}\}[Y^\gamma]$$

for any $u\in H^*(\tilde{X}^\gamma;\boldsymbol{C})$.

For $c\in H_{K_\gamma^0}^*(\boldsymbol{C})$, $\bar{c}\in H_T^*(\boldsymbol{C})$ denotes the image of c by the canonical injection $H_{K_\gamma^0}^*(\boldsymbol{C})\hookrightarrow H_T^*(\boldsymbol{C})$. Then $h^*(c(P^{(\gamma)}))=\bar{c}(P^{(\gamma)}/Y^\gamma)$. Thus,

$$(2.21)\qquad e_{Y^\gamma}=(\prod_{\alpha\in P\gamma}\alpha)(P^{(\gamma)}/Y^\gamma)$$

by (2.18) and (2.19). Furthermore, applying (2. 20) for (2.16), we have

$$(2.22)\qquad L(\gamma;\lambda)=(-1)^n[K_\gamma: K_\gamma^0]^{-1}|W_\gamma|^{-1}$$
$$\left\{\left(\frac{\sum_{s\in W_G}\varepsilon(s)e^{s(\lambda+\rho)}(\gamma)e^{s(\lambda+\rho)}}{\prod_{\alpha\in P}(e^{\alpha/2}(\gamma)e^{\alpha/2}-e^{-\alpha/2}(\gamma)e^{-\alpha/2})}\prod_{\alpha\in P_\gamma}\alpha\right)(P^{(\gamma)}/Y^\gamma)\right\}[Y^\gamma]$$

Putting for $\mu\in\mathcal{F}$

$$A_\mu(\gamma)=\left\{\left(\frac{e^\mu(\gamma)e^\mu\prod_{\alpha\in P_\gamma}\alpha}{\prod_{\alpha\in P}(e^{\alpha/2}(\gamma)e^{\alpha/2}-e^{-\alpha/2}(\gamma)e^{-\alpha/2})}\right)(P^{(\gamma)}/Y^\gamma)\right\}[Y^\gamma],$$

by arguments similar to the ones in [9, pp. 37, 38], one can see that

$$A_\mu(\gamma)=|W_\gamma^C|^{-1}e^{\mu-\rho_\gamma}(\gamma)\prod_{\alpha\in P_\gamma}(\langle\mu,\alpha\rangle\langle\rho_\gamma,\alpha\rangle^{-1})$$
$$\times\prod_{\alpha\in P-P_\gamma}(e^{\alpha/2}(\gamma)-e^{-\alpha/2}(\gamma))^{-1}\{(\prod_{\alpha\in P_\gamma}\alpha)(P^{(\gamma)}/Y^\gamma)\}[Y^\gamma].$$

Hence, by (2.21) and (2.22), we have

$$L(\gamma;\lambda)=(-1)^n[K_\gamma: K_\gamma^0]^{-1}|W_\gamma|^{-1}\sum_{s\in W_G}\varepsilon(s)A_{s(\lambda+\rho)}(\gamma)$$
$$=(-1)^n[K_\gamma: K_\gamma^0]^{-1}|W_\gamma|^{-1}|W_\gamma^C|^{-1}\prod_{\alpha\in P_\gamma}\langle\rho_\gamma,\alpha\rangle^{-1}$$
$$\times\{\sum_{s\in W_G}\varepsilon(s)e^{s(\lambda+\rho)-\rho_\gamma}(\gamma)\prod_{\alpha\in P_\gamma}\langle s(\lambda+\rho),\alpha\rangle\prod_{\alpha\in P-P_\gamma}(e^{\alpha/2}(\gamma)-e^{-\alpha/2}(\gamma))^{-1}\}e(Y^\gamma)$$
$$=(-1)^n[K_\gamma: K_\gamma^0]^{-1}|W_\gamma^C|^{-1}\prod_{\alpha\in P_\gamma}\langle\rho_\gamma,\alpha\rangle^{-1}$$
$$\times|W_\gamma|^{-1}\{\sum_{\substack{w\in W_\gamma\backslash W_G\\ s\in W_\gamma}}\varepsilon(w)e^{sw(\lambda+\rho)-\rho}(\gamma)(\varepsilon(s)\prod_{\alpha\in P_\gamma}\langle sw(\lambda+\rho),\alpha\rangle)\prod_{\alpha\in P-P_\gamma}(1-e^{-\alpha}(\gamma)^{-1}\}e(Y^\gamma),$$

which by the definition of Ψ_λ' equals

$$(-1)^n[G_\gamma: G_\gamma^0]^{-1}|W_\gamma^C|^{-1}\prod_{\alpha\in P_\gamma}\langle\rho_\gamma,\alpha\rangle^{-1}\Psi_\lambda'(\gamma)e(Y^\gamma),$$

since

$$e^{sw(\lambda+\rho)-\rho}(\gamma)=e^{w(\lambda+\rho)-\rho}(\gamma),$$

$$\prod_{\alpha\in P_\gamma}\langle w(\lambda+\rho),\alpha\rangle=\varepsilon(s)\prod_{\alpha\in P_\gamma}\langle sw(\lambda+\rho),\alpha\rangle$$

for $s\in W_\gamma$ and

$$[K_\gamma: K_\gamma^0]=[G_\gamma: G_\gamma^0].$$

Since we have normalized the Haar measure on G_γ so that $v(\Gamma_\gamma'\backslash G_\gamma)=(-1)^{n_\gamma}e(Y^\gamma)$, we thus have by the definition of Ψ_γ

$$L(\gamma;\lambda)=j(\lambda)v(\Gamma_\gamma'\backslash G_\gamma)\Psi_\lambda(\gamma),$$

which proves Lemma 3 under the assumption that $\gamma\in T$.

For an elliptic element $\gamma\in\Gamma$ in general, by such a routine method as translating γ into T by an inner automorphism of G, applying the above special case for it and then going back to the definition of Ψ_λ and $v(\Gamma_\gamma'\backslash G_\gamma)$, one can see that the above formula is still valid for the general case. We have thus completed the proof of Lemma 3, hence Theorem 3.

3. The multiplicity $N_\omega(\Gamma)$ for $\omega\in\mathcal{E}_1(G)$ and dim $H^\pm(\Gamma; E_{V_{\lambda+\rho_n}})$

Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Denote by $\hat{G}$ the set of all equivalence classes of irreducible unitary representations of G. It is well known that the multiplicity $N_\omega(\Gamma)$ of $\omega\in\hat{G}$ in the right regular representation $L^2(\Gamma\backslash G)$ is finite. We shall first recall several known facts which will be needed later. For $\omega\in\mathcal{E}_1(G)$, the formula for $N_\omega(\Gamma)$ is more or less known by Langlands' implication [6(b)] combined with Harish-Chandra's results [3(b)]. Under the assumption that Γ contains no elliptic elements other than the identity, such formula is very simple; namely,

$$N_\omega(\Gamma)=v(\Gamma\backslash G)\,d_\omega \text{ for } \omega\in\mathcal{E}_1(G) \tag{3.1}$$

where d_ω is the formal degree of ω (for the proof, we recall a "conjecture of Selberg" [3(b), Theorem 11] was crucial). We note that $v(\Gamma\backslash G)$ is the volume of $\Gamma\backslash G$ under a Haar measure on G and d_ω is uniquely determined depending on it. The method to compute d_ω is then given by Harish-Chandra. By [3(b), Theorem 16], we know

$$d_{\omega(\lambda+\rho)}=c_G^{-1}\,|W_G|\,|\prod_{\alpha\in P}\langle\lambda+\rho,\alpha\rangle| \tag{3.2}$$

for $\lambda\in\mathcal{F}_0'$, where c_G is a certain positive constant depending proportionally on the Haar measure on G ([3(c), Lemma 5]). For c_G, consider the following Haar measure on G. Let dx be the Euclidean measure on $\mathfrak{g}$ given by the Euclidean norm $||x||^2=-\mathrm{tr}\,(\mathrm{ad}\,x\;\mathrm{ad}\,\theta(x))$ for $x\in\mathfrak{g}$ where θ is the Cartan involution. Put

$$\xi(x)=\left|\det\left(\frac{e^{\mathrm{ad}\,x/2}-e^{-\mathrm{ad}\,x/2}}{\mathrm{ad}\,x}\right)\right|^{1/2} \text{ for } x\in\mathfrak{g}\,.$$

For $g=\exp x\in G$, $dg=\xi(x)^2\,dx$ then defines a measure on a neighborhood of the identity of G through the exponential mapping, which extends to the Haar measure on G. On the Cartan subgroup T, we consider the Haar measure given by the Euclidean measure on $\mathfrak{t}$.

Lemma 4 (Harish-Chandra)*. *Under the above normalization of the Haar measure,*

$$c_G=|W_G|(2\pi)^m v(T)$$

* See [3(c), Lemma 6], which gives an explicit formula for c_G under another normalization of the Haar measure. His unpublished proof of this lemma is based on the above Lemma 4.

where $m=\frac{1}{2}\dim_R G/T$ *and* $v(T)$ *is the total volume of* T *unedr the above measure on* T.

It is easy to see that the Haar measure $v(T)^{-1}dg$ on G coincides with that normalized in [9], where dg is normalized in Lemma 4. On the other hand, our normalization in §2 equals

$$(2\pi)^{-m}|W|\prod_{\alpha\in P}\langle\rho,\alpha\rangle$$

times that in [9] (see Remark to Theorem 3). Hence, by (3.1), (3.2) and Lemma 4, we have

$$N_{\omega(\lambda+\rho)}(\Gamma)=v(\Gamma\backslash G)|W|^{-1}\prod_{\alpha\in P}\langle\rho,\alpha\rangle^{-1}|\prod_{\alpha\in P}\langle\lambda+\rho,\alpha\rangle| \tag{3.3}$$

for $\lambda\in\mathcal{F}_0'$ such that $\omega(\lambda+\rho)\in\mathcal{E}_1(G)$, under our normalizaation of the Haar measure on G in §2.

On the other hand, in the above special case, we have by Theorems 2, 3

$$\dim H^{j(\lambda)}(\Gamma; Ev_{\lambda+\rho_n})=j(\lambda)v(\Gamma\backslash G)|W|^{-1}\prod_{\alpha\in P}\langle\rho,\alpha\rangle^{-1}\prod_{\alpha\in P}\langle\lambda+\rho,\alpha\rangle. \tag{3.4}$$

Hence, noting that $j(\lambda)\prod_{\alpha\in P}\langle\lambda+\rho,\alpha\rangle=|\prod_{\alpha\in P}\langle\lambda+\rho,\alpha\rangle|$, we have by (3.3) and (3.4)

$$\dim H^{j(\lambda)}(\Gamma; Ev_{\lambda+\rho_n})=N_{\omega(\lambda+\rho)}(\Gamma) \tag{3.5}$$

for $\lambda\in\mathcal{F}_0'$ such that $\omega(\lambda+\rho)\in\mathcal{E}_1(G)$, when Γ contains no elliptic elements other than the identity.

We shall now prove the above formula without the above assumption for Γ. For $\omega\in\hat{G}$, π_ω will denote an irreducible representation of G belonging to ω and also its infinitesimal character. Let $\chi^\pm\otimes\tau_{\lambda+\rho_n}$ be the representations of K on $L^\pm\otimes V_{\gamma+\rho_n}$ as in §1. Then, similar to [7, Theorem 3], we have by (1.1)

$$\dim H^\pm(\Gamma; Ev_{\lambda+\rho_n})=\sum_{\pi_\omega(\Omega)=\langle\lambda+2\rho,\lambda\rangle}N_\omega(\Gamma)(\pi_\omega|K;\chi^\pm\otimes\tau_{\lambda+\rho_n}) \tag{3.6}$$

where $(\pi_\omega|K;\chi^\pm\otimes\tau_{\lambda+\rho_n})$ denotes the intertwining number between the restricion $\pi_\omega|K$ of π_ω and $\chi^\pm\otimes\tau_{\lambda+o_n}$ (note that $N_\omega(\Gamma)=N_{\omega^*}(\Gamma)$ where ω^* is the contragredient class of ω). We now take a normal subgroup Γ' of Γ as in §2. Since $L^2(\Gamma\backslash G)$ can be naturally imbedded in $L^2(\Gamma'\backslash G)$,

$$N_\omega(\Gamma)\leq N_\omega(\Gamma') \text{ for all } \omega\in\hat{G}. \tag{3.7}$$

Lemma 5. *Let* ω_0 *be an element in* $\hat{G}$ *such that*

$$\pi_{\omega_0}(\Omega)=\langle\lambda+2\rho,\lambda\rangle \text{ and } (\pi_{\omega_0}|K;\chi^j\otimes\tau_{\lambda+\rho_n})>0.$$

Suppose that $\dim H^j(\Gamma'; Ev_{\lambda+\rho_n})=N_{\omega_0}(\Gamma')$ *for the above* Γ'. *Then*

$\dim H^j(\Gamma; E_{V\lambda+\rho_n}) = N_{\omega_0}(\Gamma)$.

Proof. If $N_{\omega_0}(\Gamma')=0$, then we have $\dim H^j(\Gamma; E_{V\lambda+\rho_n})=N_{\omega_0}(\Gamma)=0$ by (3.6) and (3.7). Assume that $N_{\omega_0}(\Gamma')>0$. By (3.6), we have

$$\dim H^j(\Gamma'; E_{V\lambda+\rho_n}) = N_{\omega_0}(\Gamma')(\pi_{\omega_0}|K; \chi^j\otimes\tau_{\lambda+\rho_n}) + \sum_{\substack{\pi_\omega(\Omega)=\langle\lambda+2\rho,\lambda\rangle \\ \omega\neq\omega_0}} N_\omega(\Gamma')(\pi_\omega|K; \chi^j\otimes\tau_{\lambda+\rho_n}).$$

In the right hand side, the second term is non-negative while $\dim H^j(\Gamma'; E_{V\lambda+\rho_n}) = N_{\omega_0}(\Gamma') \leq N_{\omega_0}(\Gamma')(\pi_{\omega_0}|K; \chi^j\otimes\tau_{\lambda+\rho_n})$ by assumption, which implies $(\pi_{\omega_0}|K; \chi^j \otimes\tau_{\lambda+\rho_n})=1$. Therefore the second term vanishes, which implies

$$\sum_{\substack{\pi_\omega(\Omega)=\langle\lambda+2\rho,\lambda\rangle \\ \omega\neq\omega_0}} N_\omega(\Gamma)(K; \pi_\omega|\chi^j\otimes\tau_{\lambda+\rho_n}) = 0$$

in view of (3.7). Hence by (3.6)

$$\dim H^j(\Gamma; E_{V\lambda+\rho_n}) = N_{\omega_0}(\Gamma). \qquad \text{q.e.d.}$$

Let $\lambda\in\mathcal{F}_0'$ such that $\omega(\lambda+\rho)\in\mathcal{E}_1(G)$. By Theorem 2 (setting $\Gamma=\{1\}$) and [8(a), Theorem 1], we know

$$(\pi_{\omega(\lambda+\rho)}|K; \chi^{j(\lambda)}\otimes\tau_{\lambda+\rho_n}) = 1.$$

Since Γ' contains no elliptic elements other than the identity, we have by (3.5)

$$N_{\omega(\lambda+\rho)}(\Gamma') = \dim H^{j(\lambda)}(\Gamma'; E_{V\lambda+\rho_n}).$$

Hence, noting that $\pi_{\omega(\lambda+\rho)}(\Omega)=\langle\lambda+2\rho, \lambda\rangle$, we have by Lemma 5,

Theorem 4. *Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Then*

$$N_{\omega(\lambda+\rho)}(\Gamma) = \dim H^{j(\lambda)}(\Gamma; E_{V\lambda+\rho_n})$$

for $\lambda\in\mathcal{F}_0'$ such that $\omega(\lambda+\rho)\in\mathcal{E}_1(G)$.

Corollary (Langlands formula). *For $\lambda\in\mathcal{F}_0'$ such that $\omega(\lambda+\rho)\in\mathcal{E}_1(G)$,*

$$N_{\omega(\lambda+\rho)}(\Gamma) = \sum v(\Gamma_\gamma\backslash G_\gamma)\Psi_\lambda(\gamma)$$

where the summation is as in Theorem 3.

REMARK. As remarked at the beginning of this section, the formula in Corollary will follow directly from the observation in [6(b)] combined with several results in [3(b)]. One will then be able to get Theorem 4 comparing each numbers. We notice that we have made use of such a formula of Selberg's type only when Γ contains no elliptic elements other than the identity.

We might have the equality (3.5) without appealing to Lemma 4 (i.e. the explicit computation of the constant c_G) if we know that (3.5) holds for at least one $\lambda\in\mathcal{F}_0'$. Such a λ can be found, in the case of holomorphic discrete classes by earlier works (see §4) and in the general case by methods of Schmid [9].

4. Some remarks in the case of Hermitian symmetric spaces

In this section, we shall make some remarks for the hermitian symmetric case. Assume that the symmetric space $M=G/K$ is hermitian, i.e., M has a G-invariant complex structure. Regarding $\mathfrak{p}$ as the tangent space at the origin $\{K\}$ of M, we denote by $\mathfrak{p}_+$ (resp. $\mathfrak{p}_-$) the subspace of $\mathfrak{p}^C$ to be identified with the holomorphic (resp. anti-holomorphic) tangent space. We then know

$$\mathfrak{p}^C=\mathfrak{p}_+\oplus\mathfrak{p}_-,\ [\mathfrak{k},\mathfrak{p}_\pm]\subset\mathfrak{p}_\pm,\ [\mathfrak{p}_\pm,\mathfrak{p}_\pm]=0\,.$$

We fix such a positive root system P that satisfies

$$\mathfrak{p}_\pm=\sum_{\alpha\in P_n}\mathfrak{g}^{\pm\alpha}\,.$$

Let Γ be a discrete subgroup of G with a compact quotient $\Gamma\backslash M$. It is then known that $\Gamma\backslash M$ is a (normal) projective variety. Let τ_λ be an irreducible representation of K with highest weight λ on a space V_λ. One can associate to τ_λ a sheaf $\mathcal{O}_\lambda$ over $\Gamma\backslash M$ in the following way. Denote by E_{V_λ} the holomorphic vector bundle over M associated to τ_λ. To each open set U in $\Gamma\backslash M$, there is attached the space of Γ-invariant holomorphic sections of E_{V_λ} over $\widetilde{U}$, where $\widetilde{U}$ is the inverse image in M of U. This gives a presheaf over $\Gamma\backslash M$, whose associated sheaf is denoted by $\mathcal{O}_\lambda$. Denoting as usual by $H^q(\Gamma\backslash M;\mathcal{O}_\lambda)$ the q-the cohomology space with coefficients in $\mathcal{O}_\lambda$, we have the following vanishing

Proposition 1. *For $\lambda\in\mathcal{F}_0'$, take $\mu\in D$, $\sigma\in W^1$ such that $\sigma(\mu+\rho)=\lambda+\rho$. Assume that $\langle\sigma\mu,\alpha\rangle\neq0$ for any $\alpha\in P_n$. Then*

$$H^q(\Gamma\backslash M;\mathcal{O}_\lambda)=0 \text{ for } q\neq q_\lambda\,,$$

where $q_\lambda=|\{\alpha\in P_n;\langle\lambda+\rho,\alpha\rangle>0\}|$.

For the proof, note $H^q(\Gamma\backslash M;\mathcal{O}_\lambda)$ is isomorphic to the space of Γ-invariant q-th harmonic forms on M valued in the vector bundle E_{V_λ} (see [9, Lemma 10]). It then suffices to see the relationship between the Cauchy-Riemann $\bar\partial$ operator and the Dirac operator which was described in [8(b), §2], and to use similar arguments to the ones there.

For the sake of completeness and the subsequent arguments, we shall review it briefly. Consider the $\mathfrak{k}$-module L given by the spin representation of $\mathfrak{so}(\mathfrak{p})$ (§1). In our chioce of the positive root system P, we know that $\langle\rho_n,\alpha\rangle=0$

for $\alpha\in P_k$. Hence we can consider the 1-dimensional representation τ_{ρ_n} of $\mathfrak{k}$ on V_{ρ_n} with highest weight ρ_n. Note that $\tau_{\rho_n}\otimes\tau_\lambda=\tau_{\lambda+\rho_n}$. It is easy to see that $L\otimes V_{\rho_n}$ is isomorphic to $\Lambda\mathfrak{p}_+$ as $\mathfrak{k}$-modules, hence $L\otimes V_{\lambda+\rho_n}=\Lambda\mathfrak{p}_+\otimes V_\lambda$ as $\mathfrak{k}$-modules. Here, since the both $\mathfrak{k}$-modules have K-module structures, they are isomorphisms as K-modules.

Under our assumption, the associated vector bundle $G\times_K\mathfrak{p}_+$ over M can be regarded as the anti-holomorphic cotangent bundle $\bar{T}^*$ of M. The Cauchy-Riemann operator $\bar\partial$ acts on $C^\infty(\Lambda\bar{T}^*\otimes E_{V_\lambda})$, which is regarded as the space of sections of the vector bundle $E_{\Lambda\mathfrak{p}_+\otimes V_\lambda}$ associated to $\Lambda\mathfrak{p}_+\otimes V_\lambda$, while the Dirac operator D acts on $C^\infty(E_{L\otimes V_{\lambda+\rho_n}})$. By Okamoto and Ozeki, the laplacian of $\bar\partial$ under the hermitian metric of $\bar{T}^*$ given by a kählerian metric on M, has the expression

$$\frac{1}{2}\{\langle\lambda+2\rho,\lambda\rangle-\hat{\pi}(\Omega)\}$$

on $C^\infty(E_{\Lambda\mathfrak{p}_+\otimes V_\lambda})$ while the laplacian of D has the form

$$\langle\lambda+2\rho,\lambda\rangle-\hat{\pi}(\Omega)$$

on $C^\infty(E_{L\otimes V_{\lambda+\rho_n}})$((1.1)). Actually one can see* that there exists a K-module isomorphism $L\otimes V_{\rho_n}\xrightarrow{\sim}\Lambda\mathfrak{p}_+$ (not necessarily isometric with respect to natural metrics) such that for the induced isomorphism

$$\begin{array}{ccc} C^\infty(E_{L\otimes V_{\lambda+\rho_n}}) & \xrightarrow{\sim} & C^\infty(\Lambda\bar{T}^*\otimes E_{V_\lambda}) \\ D\big\downarrow & & \big\downarrow\bar\partial+2\vartheta \\ C^\infty(E_{L\otimes V_{\lambda+\rho_n}}) & \xrightarrow{\sim} & C^\infty(\Lambda\bar{T}^*\otimes E_{V_\lambda}) \end{array}$$

is commutative, where ϑ on $C^\infty(\Lambda\bar{T}^*\otimes E_{V_\lambda})$ is the adjoint of $\bar\partial$ with respect to the above metric of $\bar{T}^*$. By this isomorphism, $D^2=(\bar\partial+2\vartheta)^2=2(\bar\partial\vartheta+\vartheta\bar\partial)$ which gives the reason for the above coincidence (up to factor 2) of the laplacians. Seeing this relationship and the property (1.3) in §1 when Γ is given, the vanishing for the spaces of Γ-invariant harmonic forms valued in E_{V_λ} reduces to quite similar arguments as in [8(b), §2], and we have Proposition 1. (Notice that the set-up in [8(b)] is slightly different from ours in how to choose a positive root system P relating to the complex structure of M and in whether to take the contragredient of τ_λ or not.)

In order to compute the alternating sum

$$\chi(\Gamma\backslash M;\mathcal{O}_\lambda)=\sum_{q=0}^{n}(-1)^q\dim H^q(\Gamma\backslash M;\mathcal{O}_\lambda),$$

one can, of course, appeal to the holomorphic Lefschetz theorem in [1] again. However, by looking further into the isomorphism $L\otimes V_{\rho_n}\simeq\Lambda\mathfrak{p}_+$, one may make

* This fact is not needed for the subsequent argments. The proof may appear elsewhere.

use of the previous result. We recall that the $\mathfrak{k}$-modules $L^{\pm}$ were so defined that the set of weights of L^+ (resp. L^-) is $\{\rho_n - \langle Q\rangle; Q\subset P_n, |Q| \text{ is even (resp. odd)}\}$. The isomorphism is then divided into two parts:

$$L^+ \otimes V_{\rho_n} \cong \bigoplus_{q=\text{even}} \Lambda^{n-q}\mathfrak{p}_+$$

$$L^- \otimes V_{\rho_n} \cong \bigoplus_{q=\text{odd}} \Lambda^{n-q}\mathfrak{p}_+ \,.$$

Hence, denoting as in §2 by $\chi(\Gamma; \lambda)$ the difference $\dim H^+(\Gamma; Ev_{\lambda+\rho_n}) - \dim H^-(\Gamma; Ev_{\lambda+\rho_n})$ for Dirac spinors, we have

$$\chi(\Gamma; \lambda) = (-1)^n \chi(\Gamma\backslash M; \mathcal{O}_\lambda)\,,$$

which implies

$$\chi(\Gamma\backslash M; \mathcal{O}_\lambda) = (-1)^{q_\lambda} \sum v(\Gamma_\gamma\backslash G_\gamma)\Psi_\lambda(\gamma)$$

by Theorem 3, since the sign of $(-1)^{n-q_\lambda}$ is $j(\lambda)$. We thus have

Proposition 2. *Suppose $\lambda\in\mathcal{F}_0'$ satisfies the condition of Proposition* 1. *Then*

$$\dim H^{q_\lambda}(\Gamma\backslash M; \mathcal{O}_\lambda) = \sum v(\Gamma_\gamma\backslash G_\gamma)\Psi_\lambda(\gamma)$$

where the summation is as in Theorem 3.

Similar to §3, we now have the following

Corollary 1. *Let $\lambda\in\mathcal{F}_0'$ and assume $\omega(\lambda+\rho)\in\mathcal{E}_1(G)$. Then $N_{\omega(\lambda+\rho)}(\Gamma) = \dim H^{q_\lambda}(\Gamma\backslash M; \mathcal{O}_\lambda)$.*

Related to earlier works of several authors ([4], [5], [6(a)], [7], etc.), we shall now specify the case $q_\lambda=0$ for $\lambda\in\mathcal{F}_0'$, i.e., $\langle\lambda+\rho, \alpha\rangle>0$ ($\alpha\in P_k$) and $\langle\lambda+\rho, \alpha\rangle<0$ ($\alpha\in P_n$). The corresponding discrete class $\omega(\lambda+\rho)\in\mathcal{E}_2(G)$ is then called a 'holomorphic' discrete class, and using the characterization of holomorphic discrete classes in [3(a)], it is known that $N_{\omega(\lambda+\rho)}(\Gamma)$ is equal to the dimension of $H^0(\Gamma\backslash M; \mathcal{O}_\lambda)$, a space of automorphic forms for such a holomorphic discrete class $\omega(\lambda+\rho)$. Hence in this case, we have by Propositions 1, 2

Corollary 2. *Let $\lambda\in\mathcal{F}_0'$ be such that $\langle\lambda+\rho, \alpha\rangle<0$ for any $\alpha\in P_n$. Assume moreover that $\langle\lambda+2\rho_n, \alpha\rangle<0$ for any $\alpha\in P_n$. Then $H^q(\Gamma\backslash M; \mathcal{O}_\lambda)=0$ for $q>0$; hence $\dim H^0(\Gamma\backslash M; \mathcal{O}_\lambda)=\sum v(\Gamma_\gamma\backslash G_\gamma)\Psi_\lambda(\gamma)$. For a holomorphic discrete class $\omega(\lambda+\rho)\in\mathcal{E}_2(G)$ where λ satisfies the above condition, we thus have*

$$N_{\omega(\lambda+\rho)}(\Gamma) = \sum v(\Gamma_\gamma\backslash G_\gamma)\Psi_\lambda(\gamma)\,.$$

Proof. Since $P'=P_k\cup(-P_n)$ is again a positive root system, there exists a $\sigma\in W$ such that $\sigma P=P'$. It is easy to see that $\sigma\in W^1$ and $\mu\in D$ if we put $\sigma(\mu+\rho)=\lambda+\rho$. Then $\sigma\mu=\lambda+\rho-\sigma\rho=\lambda+2\rho_n$; hence the condition of Proposition 1 is, $\langle\lambda+2\rho_n, \alpha\rangle\neq 0$ for any $\alpha\in P_n$, which is equivalently, $\langle\lambda+2\rho_n, \alpha\rangle<0$ for any $\alpha\in P_n$ under our assumption. Thus Corollary 2 follows from Propositions 1, 2 and the above remark.

REMARK 1. The vanishing in the above Corollary is essentially contained in Matsushima and Murakami [7, Theorem 2] which states:

Assume $\lambda\in D$ *and put* $q_\lambda'=|\{\alpha\in P_n; \langle\lambda, \alpha\rangle>0\}|$. *Then* $H^q(\Gamma\backslash M; \mathcal{O}_\lambda)=0$ *for* $q<q_\lambda'$. *Especially, if* $\langle\lambda, \alpha\rangle>0$ *for any* $\alpha\in P_n$, *then*

$$H^q(\Gamma\backslash M; \mathcal{O}_\lambda)=0 \qquad \textit{for } q<n=\dim_C M\,.$$

For $\lambda\in\mathcal{F}_0'$ assume now that $\langle\lambda+\rho, \alpha\rangle<0$ for any $\alpha\in P_n$. By Serre duality, we have

$$H^q(\Gamma\backslash M; \mathcal{O}_\lambda)\cong H^{n-q}(\Gamma\backslash M; \mathcal{O}_{(-\kappa\lambda-2\rho_n)})\,,$$

where $\kappa\in W_G$ is the unique element such that $\kappa P_k=-P_k$. Matsushima-Murakami's condition $\langle-\kappa\lambda-2\rho_n, \alpha\rangle>0$ $(\alpha\in P_n)$ is equivalent to that $\langle\lambda+2\rho_n, \alpha\rangle<0$ $(\alpha\in P_n)$, which is the condition in Corollary 2. So the vanishing in Corollary 2 also follows from Matsushima-Murakami's theorem.

REMARK 2. For a holomorphic discrete class ω, the necessary and sufficient condition for $\omega\in\mathcal{E}_1(G)$ is given by Harish-Chandra [3(a)] and Trombi, Varadarajan [10, Theorem 9.2]. According to their results:

For $\lambda\in\mathcal{F}_0'$ such that $\langle\lambda+\rho, \alpha\rangle<0$ for any $\alpha\in P_n$, the following four are mutually equivalent.

(1) $\omega(\lambda+\rho)\in\mathcal{E}_1(G)$,

(2) $\langle\lambda+\rho, \alpha\rangle<\frac{1}{2}|\alpha|^2-\langle 2\rho_n, \alpha\rangle \qquad (\alpha\in P_n)$

(3) $|\langle\lambda+\rho, \alpha\rangle|>\frac{1}{2}\sum_{\beta\in P}|\langle\alpha, \beta\rangle| \qquad (\alpha\in P_n)$

(4) $\langle\lambda+2\rho, \alpha\rangle<0 \qquad (\alpha\in P_n)$

The equivalence of (1), (2), and (3) is in [10, Theorem 9.2]. It is clear that (4) follows from (3). Assume that (4) holds. Then $\langle\lambda+2\rho, \alpha\rangle<0$ for any $\alpha\in P'$ where $P'=(-P_k)\cup P_n$ is a new positive root system. Hence $\langle\lambda+2\rho+\rho', \alpha\rangle\leqq 0$ for any $\alpha\in P'$ where $\rho'=-\rho_k+\rho_n$, which implies

$$\begin{aligned}\langle\lambda+\rho, \alpha\rangle &\leqq -\langle 2\rho_n, \alpha\rangle\\ &<\frac{1}{2}|\alpha|^2-\langle 2\rho_n, \alpha\rangle \text{ for } \alpha\in P'\supset P_n\,.\end{aligned}$$

Thus (2) follows from (4); hence the equivalence follows.

There is also the following interesting equivalent condition. Consider the flag manifold G/T with a complex structure such that the holomorphic tangent space at the origin can be regarded as $\sum_{\alpha\in P}\mathfrak{g}^\alpha$. To a character λ of T, a holomorphic line bundle $\mathcal{L}_\lambda$ over G/T can be associated, Let $\mathcal{K}$ be the canonical line bundle of G/T ($\mathcal{K}=\mathcal{L}_{-2\rho}$). Under the natural invariant hermitian metrics on such line bundles, one can say whether the line bundle is positive or not in the sense of Kodaira. In this terminology, one can see, furthermore, that (4) holds if and only if

(5) the holomorphic line bundle $\mathcal{L}_\lambda \otimes \mathcal{K}^{-1}$ is positive in the sense of Kodaira.

This condition (5) is well known as a sufficient condition in order that $H^q(\Gamma\backslash G/T; \mathcal{O}(\mathcal{L}_\lambda))=0$ for $q>0$, where $\mathcal{O}(\mathcal{L}_\lambda)$ is the associated sheaf over $\Gamma\backslash G/T$ defined from $\mathcal{L}_\lambda$ similar to $\mathcal{O}_\lambda$.

REMARK 3. Let the situation be as in Remark 2. One can now compare the set-ups of several authors. M. Ise [5] obtained a sufficient condition in order that $H^q(\Gamma\backslash M; \mathcal{O}_\lambda)=0$ for $q>0$, by means of Kodaira's vanishing theorem, so under the condition (4). R. P. Langlands computed $N_{\omega(\lambda+\rho)}(\Gamma)$ for $\omega(\lambda+\rho)\in \mathcal{E}_1(G)$, or under the condition (2) ([6(a)]). By Remark 2, their set-ups are mutually equivalent. On the other hand, if (4) holds, λ satisfies the condition in Corollary 2, or the condition of Matsushima-Murakami. In general, the latter condition (i.e. that in Corollary 2) is actually weaker than those (1)~(5). This is easily checked, for example, in the case of $G=SU(3, 1)$. (In this case it is enough to consider $\lambda=-3\rho_n\in\mathcal{F}_0'$ which satisfies the condition of Corollary 2 but $\omega(\lambda+\rho)\notin\mathcal{E}_1(G)$.) Thus the multiplicity formula in Corollary 2 is valid for not necessarily integrable discrete classes.

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