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ON THE PATHWISE UNIQUENESS OF SOLUTIONS OF ONE-DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS

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Introduction

In this paper, we shall discuss a problem of the pathwise uniqueness for solutions of one-dimensional stochastic differential equations. Let $a(x)$ and $b(x)$ be bounded Borel measurable functions defined on R. We shall consider the following one-dimensional Itô's stochastic differential equation;

$$(1) \qquad dx_t = a(x_t)\,dB_t + b(x_t)\,dt.$$

K. Itô [1] proved that, if $a(x)$ and $b(x)$ are Lipschitz continuous, a solution is unique and it can be constructed on a given Brownian motion B_t. On the other hand, if $|a(x)|$ is bounded from below by a positive constant (i.e. uniformly positive), then a solution of (1) exists and it is unique in the law sense. This follows easily from a general result of one-dimensional diffusions (cf. [2]). However, though the distribution of $\{x., B.\}$ is unique, x_t is not always expressed as a measurable function of x_0 and $\{B_s, s\leqq t\}$. For example, if $a(x)=\text{sgn}\, x$, $a(0)=1$ and $x_0\equiv 0$, it is not difficult to see that $\sigma\{|x_s|\,;\, s\leqq t\}=\sigma\{B_s;\, s\leqq t\}$.

Here, we will show that, if $a(x)$ is uniformly positive and of bounded variation on any compact interval, then the pathwise uniqueness holds for (1). This implies, in particular, that x_t is expressed as a measurable function of x_0 and $\{B_s, s\leqq t\}$ (cf. [5]). In this direction, M. Motoo (unpublished) already proved that the pathwise uniqueness holds for (1) if $a(x)$ is uniformly positive and Lipschitz continuous and if $b(x)$ is bounded measurable. Also, T. Yamada and S. Watanabe [5] proved the pathwise uniqueness of (1) if $a(x)$ is Hölder continuous of exponent $\frac{1}{2}$ and $b(x)$ is Lipschitz continuous. Our above mentioned result may be interesting in a point that it applies for many discontinuous $a(x)$. It is still an open question whether only the uniform positivity of $a(x)$ implies the pathwise uniqueness.

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A precise meaning of the equation (1) is as follows: $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ stands for a probability space $(\Omega, \mathcal{F}, P)$ with an increasing family $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ of sub-σ-algebras of $\mathcal{F}$.

DEFINITION 1. By a solution of (1), we mean a quadruplet $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and a stochastic process $\mathfrak{X}_t=(x_t, B_t)$ defined on it such that

(i) with probability one, $\mathfrak{X}_t$ is continuous in t and $B_0=0$,

(ii) $\mathfrak{X}_t$ is an $\{\mathcal{F}_t\}$-adapted process and B_t is an $\{\mathcal{F}_t\}$-Brownian motion,

(iii) $\mathfrak{X}_t$ satisfies

$$x_t = x_0+\int_0^t a(x_s)\,dB_s+\int_0^t b(x_s)\,ds \quad a.s.\,,$$

where the integral by dB_s is understood in the sense of the stochastic integral of Itô.

DEFINITION 2. We shall say that the pathwise uniqueness holds for (1), for any two solutions $\mathfrak{X}_t=(x_t, B_t)$, $\mathfrak{Y}_t=(y_t, B'_t)$ defined on a same quadruplet $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$, $x_0=y_0$ and $B_t\equiv B'_t$ implies $x_t=y_t$.

REMARK 1. In Definition 2, it is sufficient to assume that $x_0=y_0=x$ for some constant $x\in R$.

REMARK 2. A definition of the pathwise uniqueness may be defined in a stronger way as follows; the pathwise uniqueness holds if $\mathfrak{X}_t=(x_t, B_t)$ is a solution on $(\Omega, \mathcal{F}, P; \mathcal{F}^1_t)$ and $\mathfrak{Y}_t=(y_t, B'_t)$ is a solution on $(\Omega, \mathcal{F}, P; \mathcal{F}^2_t)$ ($\mathcal{F}^1_t$ and $\mathcal{F}^2_t$ may be different) such that $x_0=y_0$ and $B_t\equiv B'_t$, then $x_t=y_t$. It is not difficult to show, using a result in [5], that this definition of the pathwise uniqueness is equivalent to Definition 2.

Lemma. *Let $(M_t, V_t)_{t\in[0,T]}$ be a pair of continuous real process defined on a probability space $(\Omega, \mathcal{F}, P)$. Suppose that the total variation $|||V(\omega)|||_T$ of $V_t(\omega)$ on $[0, T]$ has a finite expectation. Further, suppose M_t is a martingale satisfying the following conditions;*

(i) $$M_0 = 0 \quad a.s.\,,$$

(ii) *there exist positive constants m_1 and m_2 such that*

$$m_1M_t(\omega)\leqq V_t(\omega)\leqq m_2M_t(\omega) \quad a.s.\,, \tag{2}$$
$$\text{for} \quad (t, \omega)\in\{(t, \omega);\ t\in[0, T] \quad \text{and} \quad M_t(\omega)\geqq 0\}\,.$$

Then, $M_t=0$ a.s. for $0\leqq t\leqq T$.

Proof. For $y\in R$, let $N_1(y, \omega)$ be the number of $t\in[0, T]$ such that $V_t(\omega)=y$. By a theorem of Banach (cf. [4] pp. 280), we have

$$|||V(\omega)|||_T = \int_{-\infty}^{\infty} N_1(y, \omega)\,dy\,. \tag{3}$$

Obviously we may assume that $m_1<m_2$. For $y>0$, let $N_2(y, \omega)$ be the number of $\left[\frac{1}{m_2}y, \frac{1}{m_1}y\right]$-downcrossings of $M_t(\omega)$ on $[0, T]$. The condition (2) implies that

$$(4) \qquad N_1(y, \omega) \geqq N_2(y, \omega) \qquad \text{for} \quad y>0 .$$

For $y>0$, we define a sequence of stopping times $\{T_k\}$ in the following way;

$$T_0 = 0 ,$$

$$T_{2n+1} = \inf\left\{t \geqq T_{2n};\ M_t > \frac{1}{m_1}y\right\} \wedge {}^{1)}T \quad n = 0, 1, 2, \cdots ,$$

$$T_{2n+2} = \inf\left\{t \geqq T_{2n+1};\ M_t < \frac{1}{m_2}y\right\} \wedge T \quad n = 0, 1, 2, \cdots .$$

Then, for $n=1, 2, \cdots$, we can obtain the following inequality;

$$\left(\frac{1}{m_1}y - \frac{1}{m_2}y\right)(N_2(y, \omega) \wedge n+1)$$
$$\geqq \sum_{k=1}^{n} \{M_{T2k-1}(\omega) - M_{T2k}(\omega)\} + \{(M_T(\omega) - \frac{1}{m_1}y) \vee {}^{2)}0\} \chi^{3)}{}_{\{N_2(y,\omega)<n\}} .$$

Taking the expectation, we have

$$E[N_2(y) \wedge n] \geqq \frac{E[\{\left(M_T - \frac{1}{m_1}y\right) \vee 0\} \chi_{\{N_2(y)<n\}}]}{\left(\frac{1}{m_1} - \frac{1}{m_2}\right)y} - 1 .$$

Letting $n \to \infty$, we have

$$(5) \qquad E[N_2(y)] \geqq \frac{E[\left(M_T - \frac{1}{m_1}y\right) \vee 0]}{\left(\frac{1}{m_1} - \frac{1}{m_2}\right)y} - 1 .$$

Now, we assume that $P(M_T \neq 0)>0$. Then, there exist positive constants ε and δ such that

$$(6) \qquad E[\left(M_T - \frac{1}{m_1}y\right) \vee 0] > \varepsilon \qquad \text{for } 0<y<\delta .$$

The inequalities (5) and (6) provide us with the equality

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1) Let x and y be real numbers. $x \wedge y$ means $\min(x,y)$.

2) $x \vee y$ means $\max(x,y)$.

3) χ_A denotes the indicator function of a set A.

$$\int_0^\delta E[N_2(y)]\,dy = \infty\,.$$

But this is a contradiction since, by (3) and (4),

$$\int_0^\delta E[N_2(y)]\,dy \leqq \int_{-\infty}^\infty E[N_1(y)]\,dy < \infty\,.$$

Therefore we have

$$P(M_t = 0) = 1 \qquad \text{for} \quad 0 \leqq t \leqq T\,.$$

This completes the proof.

REMARK 3. In the above lemma, we may suppose the following condition instead of (2); there exist positive constants m_1 and m_2 such that

$$(7) \qquad m_1|M_t| \leqq |V_t| \leqq m_2|M_t| \qquad a.s. \quad \text{for} \quad 0 \leqq t \leqq T\,.$$

Now we will state our main result.

Theorem. *Let $a(x)$ and $b(x)$ be bounded Borel measurable. Suppose $a(x)$ is of bounded variation on any compact interval. Further, suppose there exists a constant $c>0$ such that*

$$(8) \qquad a(x) \geqq c \qquad \textit{for} \quad x \in R\,.$$

Then, the pathwise uniqueness holds for (1).

Proof. We assume that $|a(x)| \leqq M$ and $|b(x)| \leqq M$ for $x \in R$. Let $\mathfrak{X}_t = (x_t, B_t)$ and $\mathfrak{Y}_t = (y_t, B_t)$ be solutions of (1) such that $x_0 = y_0$ is a constant. For $N > |x_0|$, we define that

$$\tau_N = \begin{cases} \inf\{t \geqq 0;\ |x_t| = N\} & \\ \infty & \text{if } \{\ \} = \phi\,, \end{cases}$$

$$\eta_N = \begin{cases} \inf\{t \geqq 0;\ |y_t| = N\} & \\ \infty & \text{if } \{\ \} = \phi\,, \end{cases}$$

$$\gamma_N = \tau_N \wedge \eta_N\,.$$

Let, for $x \in R$,

$$f(x) = -2\int_0^x \frac{b(y)}{a^2(y)}\,dy\,, \qquad \varphi(x) = \int_0^x \exp\,[f(y)]\,dy\,.$$

By the time substitution and Cameron-Martin's formula (cf. [3]), there exists a constant $K_1 > 0$ depending only on c, M, N and t such that

$$(9) \qquad E\Big[\int_0^{t\wedge\gamma_N} g(x_s)\,ds\Big] \leqq K_1 \|g\|_{L^1([-N,N])} \qquad \text{for} \quad g \in L^1([-N, N])\,.$$

Since $\varphi'(x)$ is absolutely continuous and $\varphi''(x)$ is locally integrable, the inequality (9) assures us that Itô's formula applies to φ and we have

$$\varphi(x_{t\wedge\gamma_N}) = \varphi(x_0) + \int_0^{t\wedge\gamma_N} \varphi' a(x_s)\, dB_s \quad a.s..$$

Since φ is a homeomorphism R onto $I=(\varphi(-\infty), \varphi(\infty))$, we can define that

$$\sigma(x) = \varphi' a \circ \varphi^{-1}(x), \qquad h(x) = \int_0^x \frac{1}{\sigma(y)} dy \qquad \text{for} \quad x \in I.$$

Obviously σ is of bounded variation on any compact interval of I. Let $|||\sigma|||_N$ be the total variation of σ on $[\varphi(-N), \varphi(N)]$.

We can take an approximate sequence $\{\sigma_n(x)\}_{n=1,2\cdots}$ such that

(i) $\sigma_n(x) \in C^1(R)$ and $c \exp\left[-\frac{2MN}{c^2}\right] \leqq \sigma_n(x) \leqq M \exp\left[\frac{2MN}{c^2}\right]$ for $x \in R$,

(ii) $||\sigma - \sigma_n||_{L^1([\varphi(-N), \varphi(N)])} \leqq \frac{1}{n!}$ and $||\sigma_n'||_{L^1([\varphi(-N), \varphi(N)])} \leqq |||\sigma|||_N$.

Let

$$h_n(x) = \int_0^x \frac{1}{\sigma_n(y)} dy \qquad \text{for} \quad x \in I.$$

Since $h_n(x) \in C^2(R)$, we can apply Itô's formula to h_n and have

$$\begin{aligned} h_n(\varphi(x_{t\wedge\gamma_N})) &= h_n(\varphi(x_0)) + \int_0^{t\wedge\gamma_N} \frac{\sigma(\varphi(x_s))}{\sigma_n(\varphi(x_s))} dB_s \\ &\quad - \frac{1}{2} \int_0^{t\wedge\gamma_N} \frac{\sigma_n'(\varphi(x_s))\, \sigma^2(\varphi(x_s))}{\sigma_n^2(\varphi(x_s))} ds. \\ &= h_n(\varphi(x_0)) + L_t^n + W_t^n. \end{aligned}$$

It follows from (ii) that there exists a constant $K_2 > 0$ depending only on c, M and N such that

$$|h(x) - h_n(x)| \leqq K_2 \frac{1}{n!} \qquad \text{for} \quad x \in [\varphi(-N), \varphi(N)].$$

From this, we see that $h_n(\varphi(x_{t\wedge\gamma_N}))$ converges almost surely to $h(\varphi(x_{t\wedge\gamma_N}))$. There exists a constant $K_3 > 0$ depending only on c, M, N, and t such that

$$E[(L_t^n - B_{t\wedge\gamma_N})^2] \leqq K_3 \frac{1}{n!}.$$

Therefore L_t^n converges almost surely to $B_{t\wedge\gamma_N}$. Let

$$W_t = h(\varphi(x_{t\wedge\gamma_N})) - h(\varphi(x_0)) - B_{t\wedge\gamma_N}.$$

From the above results, W_t^n converges almost surely to W_t.

It is easy to see that there exists a constant $K_4>0$ depending only on c, M, and N such that

$$E[|||W^n|||_t]\leqq K_4 E[\int_0^{t\wedge\gamma_N}|\sigma_n'(\varphi(x_s))|\,ds]\,,$$

where $|||W^n|||_t$ is the total variation of W_s^n on $[0, t]$. Using the time substitution, we easily see that there exists a constant $K_5>0$ depending only on c, M, N, and t such that

$$E[\int_0^{t\wedge\gamma_N}|\sigma_n'(\varphi(x_s))|\,ds]\leqq K_5\|\sigma_n'\|_{L^1([\varphi(-N),\varphi(N)])}$$
$$\leqq K_5|||\sigma|||_N\,.$$

Hence it holds that

$$E[|||W|||_t]\leqq K_4K_5|||\sigma|||_N\,,$$

where $|||W|||_t$ is the total variation of W_s on $[0, t]$.
From the definition of $h(x)$, there exists positive constants m_1 and m_2 such that

$$m_1(x-y)\leqq h(x)-h(y)\leqq m_2(x-y)\qquad \text{for } y\leqq x \text{ and } x,\, y\in[\varphi(-N),\, \varphi(N)]\,.$$

Let

$$M_t = \int_0^{t\wedge\gamma_N}(\sigma(\varphi(x_s))-\sigma(\varphi(y_s)))\,dB_s\,,$$
$$V_t = h(\varphi(x_{t\wedge\gamma_N}))-h(\varphi(y_{t\wedge\gamma_N}))\,.$$

We can apply Lemma to $(M_t,\, V_t)$ and it follows that

$$P(\varphi(x_{t\wedge\gamma_N}) = \varphi(y_{t\wedge\gamma_N})) = 1\,.$$

Therefore we have

$$P(x_{t\wedge\gamma_N} = y_{t\wedge\gamma_N}) = 1\,.$$

Since $\lim_{N\to\infty}\gamma_N=\infty$ a.s., we obtain that $P(x_t=y_t)=1$ and the proof is complete.

REMARK 4. In Theorem, if $a(x)$ is continuous, we may assume that $a(x)$ is positive instead of (8).

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