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SPECTRAL AND SCATTERING THEORY FOR SECOND-ORDER DIFFERENTIAL OPERATORS WITH OPERATOR-VALUED COEFFICIENTS

YOSHIMI SAITŌ

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Introduction

Let us consider a differential operator

$$L = -\frac{d^2}{dr^2}+B(r)+C(r), \qquad r\in(0, \infty) = I, \tag{0.1}$$

where for each $r\in I$ $B(r)$ is a non-negative self-adjoint operator in a Hilbert space X with domain $\mathcal{D}(B(r))=D$ constant in r, and $C(r)$ is a symmetric operator with domain $\mathcal{D}(C(r))=D$ and $\text{s-}\lim_{r\to\infty} C(r)x=0$ for any $x\in D$. L acts on X-valued functions $f(r)$ on I. By defining the domain of L appropriately L can be regarded as a self-adjoint operator in $L^2(I,X,dr)$. In the present paper we shall develop a spectral and scattering theory for the self-adjoint operator L.

Jäger [3] constructs an eigenfunction expansion for L with $C(r)x=0(r^{-3/2-\varepsilon})$ $(\varepsilon>0)$ for any $x\in D$ as $r\to\infty$. He shows the existence of the "eigenoperator" $\Gamma(r, k)$ associated with L. $\Gamma(r,k)$ is a bounded, linear operator on X for each pair $(r,k)\in I\times(\boldsymbol{R}-\{0\})$ such that

$$-\frac{d^2}{dr^2}\Gamma(r, k)x+\Gamma(r, k)(B(r)+C(r))x = k^2\Gamma(r, k)x \tag{0.2}$$

holds for any $x\in D$. The "generalized Fourier transforms" $\mathcal{F}^{\pm}$ are defined by

$$\mathcal{F}^{\pm}f(k) = \int_I \Gamma^{\pm}(r, k)f(r)dr, \qquad \Gamma^{\pm}(r, k) = \pm ik\sqrt{\frac{2}{\pi}}\Gamma(r, \pm k). \tag{0.3}$$

$\mathcal{F}^{\pm}$ transform $L^2(I, X, dr)$ *into* $L^2((0, \infty), X, dk)$ and satisfy for any Borel set Δ in I

$$E(\Delta) = (\mathcal{F}^{\pm})^*\chi_{\sqrt{\Delta}}\,\mathcal{F}^{\pm}, \tag{0.4}$$

where $E(\lambda)$ is the resolution of the identity associated with L, $\chi_{\sqrt{\Delta}}$ is the characteristic function of $\sqrt{\Delta}=\{k>0/k^2\in\Delta\}$, and $(\mathcal{F}^{\pm})^*$ are the adjoints of $\mathcal{F}^{\pm}$

in $L^2((0, \infty), X, dk)$. These results imply that the spectrum of L is absolutely continuous on $(0, \infty)$. He has also shown that the spectrum of L is discrete on $(-\infty, 0)$. Jäger's methods are based on the fact that the principle of limiting absorption holds for the operator L. For the notion of the principle of limiting absorption see, for example, Eidus [2].

We shall treat in this paper the operator L with $C(r)x = 0(r^{-1-\varepsilon})$ $(\varepsilon > 0)$ for any $x \in D$ as $r \to \infty$. The other conditions imposed on $B(r)$ and $C(r)$ are the same as those in Jäger [3].

In [7] the author has justified the principle of limiting absorption for L with $C(r)x = 0(r^{-1-\varepsilon})$. The results in [7] will be summarized in §1. These will be used as main tools throughout in this paper. In §2 and §3 we shall give a spectral decomposition of L using the generalized Fourier transforms $\mathcal{F}^{\pm}$. In our case the existence of the eigenoperator is not shown. But with the aid of the results of Jäger [3] we can construct $\mathcal{F}^{\pm}$ such that (0.4) holds good.

§4 and §5 are devoted to investigating the properties of L using the techniques of stationary methods in perturbation theory. We shall make use of some results given in Kato and Kuroda [6]. L will be considered to be the perturbed operator of

$$L_0 = -\frac{d^2}{dr^2} + B(r). \tag{0.5}$$

In §4 we shall define and study two families of operators $\{G^{\pm}(\lambda)\}_{\lambda>0}$ and $\{H^{\pm}(\lambda)\}_{\lambda>0}$. Roughly speaking, $G^{\pm}(\lambda)$ and $H^{\pm}(\lambda)$ are defined by

$$\begin{cases} G^{\pm}(\lambda) = \lim\limits_{z \to \lambda \pm i0} (L - z)(L_0 - z)^{-1}, \\ H^{\pm}(\lambda) = \lim\limits_{z \to \lambda \pm i0} (L_0 - z)(L - z)^{-1}. \end{cases} \tag{0.6}$$

The results obtained in §4 will be used in the following sections. We shall show in §5 that the generalized Fourier transforms $\mathcal{F}^{\pm}$ corresponding to L are *orthogonal*, i.e., $\mathcal{F}^{\pm}$ transform $L^2(I, X, dr)$ *onto* $L^2((0, \infty), X, dk)$, under the assumption that the generalized Fourier transforms $\mathcal{F}_0^{\pm}$ associated with L_0 are orthogonal. We shall define the wave operators $W^{\pm} = W^{\pm}(L, L_0)$ and study their properties in §6. It will be shown that $W^{\pm}$ are *complete* and that we have

$$W^{\pm} = (\mathcal{F}^{\pm})^* \mathcal{F}^{\pm}. \tag{0.7}$$

As an application we shall consider the Schrödinger operator $-\Delta + q(x)$ in the whole $\boldsymbol{R}^n$ $(n \geq 3)$ in §7. In this case we have

$$\begin{cases} X = L^2(S^{n-1}) \\ B(r) = \dfrac{1}{r^2}\left\{-\Delta_n + \dfrac{(n-1)(n-3)}{4}\right\}, \\ C(r) = q(r\omega), \qquad (\omega \in S^{n-1}), \end{cases} \tag{0.8}$$

where S^{n-1} is the $(n-1)$-sphere and Λ_n is the Laplace-Beltrami operator on S^{n-1}. $q(x)$ is assumed to behave like $0(|x|^{-1-\varepsilon})$ $(\varepsilon>0)$. All the results obtained in the preceding sections are valid for the Schrödinger operator $-\Delta+q(x)$. In particular the spectrum of $-\Delta+q(x)$ on $(0, \infty)$ is absolutely continuous.

Throughout in this paper we shall follow Saitō [7] as to the notations. Here we recall some of them.

X — a Hilbert space with the norm $|\ \ |$ and inner product $(\ ,\ )$.

$H^{\beta}(J, X)$ — the Hilbert space $L^2(J, X, (1+|r|)^{\beta}dr)$ with the inner product

$$((f, g))_{\beta,J} = \int_J (f(r), g(r))(1+|r|)^{\beta}dr \tag{0.9}$$

and norm

$$||f||_{\beta} = [((f, f))_{\beta}]^{1/2}. \tag{0.10}$$

Here β is a real number and J is an open interval.

$C^m(J, Y)$—the set of all Y-valued functions on J having m strong continuous derivatives. Here m is a non-negative integer, J is as above, and Y is a subset of a topological vector space. In particular we set $C^m(J, \boldsymbol{C})=C^m(J)$.

$C_0^m(J, Y)$—the set of all $f(r)\in C^m(J, Y)$ with a compact carrier in J.

$C^{2,\beta}(J, X)(C_0^{2,\beta}(J, X))$—the linear space spanned by the set of all $\varphi\in C^2(J, X)$ having the form $\varphi(r)=\psi(r)x$, where $x\in D$, $\psi\in C^2(J)$ $(\psi\in C_0^2(J))$, and

$$\int_J \{|\psi'(r)|^2+|\psi(r)|^2(1+|B(r)x|^2)\}dr<\infty. \tag{0.11}$$

Here for each $r\in J$ $B(r)$ is a non-negative operator in X with domain $\mathcal{D}(B(r))=D$ constant in r.

$H^{1,B}(J, X)(H_0^{1,B}(J, X))$—the Hilbert space obtained by the completion of $C^{2,B}(J, X)(C_0^{2,B}(J, X))$ using the norm

$$||\varphi||_{B,J} = [\int_J \{|\varphi'(r)|^2+|B^{1/2}(r)\varphi(r)|^2+|\varphi(r)|^2\}dr]^{1/2}. \tag{0.12}$$

The inner product is given by

$$((f, g))_{B,J} = \int_J \{(f'(r), g'(r))+(B^{1/2}(r)f(r), B^{1/2}(r)g(r)) + (f(r), g(r))\}dr \tag{0.13}$$

$\mathcal{U}(J)$—the set of all linear, continuous functionals ℓ on $H_0^{1,B}(J, X)$. $\mathcal{U}(J)$ is a Banach space with the norm

$$|||\ell|||_J = \sup\{|\langle \ell, \varphi\rangle|\,;\ \varphi\in C_0^{2,B}(J, X),\ ||\varphi||_{B,J} = 1\}. \tag{0.14}$$

$\mathcal{V}_\alpha(J)$—the subspace of $\mathcal{V}(J)$ satisfying

$$(0.15) \qquad |||\ell|||_{\alpha,J} = \sup\{|<\ell, (1+|r|)^{\alpha/2}\varphi>|;\ \varphi\in C_0^{2,B}(J, X),\ ||\varphi||_{B,J} = 1\}<\infty.$$

$\mathcal{V}_\alpha(J)$ is a Banach space with the norm $|||\ \ |||_{\alpha,J}$. Here α is a positive number.

loc $H^\beta(\bar{I}, X)$—the set of all X-valued functions $f(r)$ on $I=(0, \infty)$ such that $f\in H^\beta((0, n), X)$ for every $n=1, 2, \cdots$. Here $\bar{I}$ means the closure of I, i.e., $\bar{I}=[0, \infty)$.

loc $H^{1,B}(\bar{I}, X)$(loc $H_0^{1,B}(\bar{I}, X)$)—the set of all X-valued function f on I such that $\psi_n f\in H^{1,B}(I, X)$ ($\psi_n f\in H_0^{1,B}(I, X)$) for every $n=1, 2,\cdots$, where $\psi_n\in C^1(I)$ satisfying $0\leqq_n\leqq 1$ and

$$(0.16) \qquad \psi_n(r) = \begin{cases} 1 & \text{for } 0<r\leqq n, \\ 0 & \text{for } r\geqq n+1. \end{cases}$$

1. Preliminaries

Let $I=(0, \infty)$. For each $r\in I$ $B(r)$ and $C(r)$ are operators in a Hilbert space X with its norm $|\ \ |$ and inner product $(\ ,\ )$. The following conditions are imposed on $B(r)$ and $C(r)$:

Assumption 1.1.[1] (B–1) For each $r\in I$ $B(r)$ is a non-negative, self-adjoint operator in X whose domain $\mathcal{D}(B(r))=D$ does not depend on r. We have $B(r)x\in C^0(I, X)$ for any $x\in D$.

(B–2) Let $x, y\in D$. Then $(B(r)x, y)\in C^2(I)$. For any compact interval I_0 in I there is a constant $c_1(I_0)$ such that

$$(1.1) \qquad \left|\frac{d^j}{dr^j}(B(s)x, y)\right| \leqq c_1(I_0)(|x|+|B^{1/2}(r)x|)(|y|+|B^{1/2}(r)y|)$$

for any $r, s\in I_0$ and $j=1, 2$.

(B–3) There exist constants $\rho>0$ and $\beta>1$ such that

$$(1.2) \qquad -\frac{d}{dr}(B(r)x, x)\geqslant\frac{\beta}{r}(B(r)x, x)$$

holds for any $x\in D$ and any $r\geqslant\rho$.

(B–4) The natural imbedding

$$(1.3) \qquad H_0^{1,B}((0, b), X) \rightarrow H^0((0, b), X)$$

is compact for each $b\in I$.

1) Assumption 1.1 is the same as Assumptions 1.1 and 1.2 in Saitō [7].

(C-1) For each $r\in I$ $C(r)$ is a symmetric operator with $\mathcal{D}(C(r))=D$. We have $C(r)x\in C^1(I, X)$ for any $x\in D$.

(C-2). For any compact interval I_0 in I there exists a constant $c_2(I_0)>0$ such that

$$\left|\frac{d}{dr}C(r)x\right|\leqq c_2(I_0)(|x|+|B^{1/2}(r)x|) \tag{1.4}$$

for any $x\in D$.

(C-3) There exist constants $c_0>0$ and $0<\varepsilon<1$ such that

$$|C(r)x|\leqq c_0(1+r)^{-1-\varepsilon}(|x|+|B^{1/2}(r)x|) \tag{1.5}$$

holds for any $x\in D$ and any $r\in I$.

Denote by $\mathfrak{A}_0$ the set of all X-valued functions $\varphi(r)$ on $\bar{I}$ satisfying the following (i)~(iii):

$$\begin{cases} \text{(i)} & \varphi(r)\in D \qquad (r\in I). \\ \text{(ii)} & \varphi\in C^2(I, X)\cap C^1(\bar{I}, X) \text{ and the carrier of } \varphi \text{ is compact in } \bar{I}. \\ \text{(iii)} & \varphi\in H_0^{1,B}(I, X) \text{ and } L_0\varphi\in H^0(I, X). \end{cases} \tag{1.6}$$

It is easy to see that we have

$$C_0^{2,B}(I, X)\subset\mathfrak{A}_0\subset H_0^{1,B}(I, X). \tag{1.7}$$

We define differential operators M and M_0 in $H^0(I, X)$ by

$$\begin{cases} \mathcal{D}(M_0) = \mathfrak{A}_0 \\ M_0\varphi = L_0\varphi = -\dfrac{d^2}{dr^2}\varphi+B(r)\varphi \end{cases} \tag{1.8}$$

and

$$\begin{cases} \mathcal{D}(M) = \mathfrak{A}_0 \\ M\varphi = L\varphi = L_0\varphi+C(r)\varphi = -\dfrac{d^2}{dr^2}\varphi+B(r)\varphi+C(r)\varphi. \end{cases} \tag{1.9}$$

obviously M_0 is symmetric and non-negative definite. It follows from (1.5) that

$$\begin{aligned} \|C(\cdot)\varphi\|_0^2 &= \int_I |C(r)\varphi(r)|^2dr \leqq \int_I c_0^2(|\varphi(r)|+|B^{1/2}(r)\varphi(r)|)^2dr \\ &\leqq 2c_0^2\int_I(|\varphi(r)|^2+|B^{1/2}(r)\varphi(r)|^2)dr \\ &\leqq 2c_0^2\|\varphi\|_B^2 = 2c_0^2((M_0\varphi, \varphi))_0+2c_0^2\|\varphi\|_0^2 \\ &\leqq c_0^2\left(\alpha\|M_0\varphi\|_0^2+\frac{1}{\alpha}\|\varphi\|_0^2\right)+2c_0^2\|\varphi\|_0^2 \text{ [2]} \qquad (\varphi\in\mathfrak{A}_0), \end{aligned} \tag{1.10}$$

2) Here and in the sequel we put $\|\ \|_{\beta,I}=\|\ \|_\beta$ and $\|\ \|_{B,I}=\|\ \|_B$ for simplicity.

where $\alpha>0$ is arbitrary and we have used integration by parts. Hence M is symmetric and bounded below. We denote by A_0 and A the Friedrichs extensions of M_0 and M, respectively. Then we have

$$(1.11)\qquad \begin{cases} C_0^{2,B}(I, X)\subset \mathcal{D}(A_0) = H_0^{1,B}(I, X)\cap \mathcal{D}(M_0^*)\subset H_0^{1,B}(I, X) \\ C_0^{2,B}(I, X)\subset \mathcal{D}(A) = H_0^{1,B}(I, X)\cap \mathcal{D}(M^*)\subset H_0^{1,B}(I, X). \end{cases}$$

where M_0^* and M^* are the adjoints of M_0 and M in $H^0(I, X)$, respectively.

In the remainder of this section we state some results on the differential equation $(L-k^2)v=f$ with a sort of radiation condition. These are proved in [7] and will be used in the following sections. First we give the next

DEFINITION 1.2. Let $\ell\in\mathcal{V}(I)$, $u\in H^{1,B}(I, X)$[3] and $k\in \boldsymbol{C}^+=\{k/k\in\boldsymbol{C},\ \mathrm{Im}\ k\geqslant 0,\ \mathrm{Re}\ k\neq 0\}$ be given. Then $v\in\mathrm{loc}\ H^{1,B}(\bar{I}, X)$ is called *the radiative function for* $\{L, k, \ell, u\}$, if the following three conditions hold:

(a) $v-u\in\mathrm{loc}\ H_0^{1,B}(\bar{I}, X)$.

(b) $v'-ikv\in H^{-1+\varepsilon}(I, X)$ (the "radiation condition").

(c) For all $\varphi\in C_0^{2,B}(I, X)$, we have

$$(1.12)\qquad ((v, (L-\bar{k}^2)\varphi))_0 = \langle \ell, \varphi\rangle.$$

We can prove under the assumptions (B–1)$\sim$(B–3) and (C–1)$\sim$(C–3) that the radiative function for given $\{k, \ell, u\}\in\boldsymbol{C}^+\times\mathcal{V}(I)\times H^{1,B}(I, X)$ is unique ([7], Therem 2.2). We can also prove under Assumption 1.1 that for given $\{k, \ell, u\}\in \boldsymbol{C}^+\times\mathcal{V}_{1+\varepsilon}(I)\times H^{1,B}(I, X)$ there exists a unique radiative function $v=v(\cdot, k, \ell, u)$ for $\{L, k, \ell, u\}$ which belongs to $H^{-1-\varepsilon}(I, X)\cap\mathrm{loc}\ H^{1,B}(\bar{I}, X)$, and that the mapping

$$(1.13)\qquad \Sigma : \boldsymbol{C}^+\times\mathcal{V}_{1+\varepsilon}(I)\times H^{1,B}(I, X)\ni\{k, \ell, u\} \\ \rightarrow v(\cdot, k, \ell, u)\in H^{-1-\varepsilon}(I, X)\cap\mathrm{loc}\ H^{1,B}(\bar{I}, X)$$

is continuous as a mapping from $\boldsymbol{C}^+\times\mathcal{V}_{1+\varepsilon}(I)\times H^{1,B}(I, X)$ into $H^{-1-\varepsilon}(I, X)$ and is also continuous as a mapping from $\boldsymbol{C}^+\times\mathcal{V}_{1+\varepsilon}(I)\times H^{1,B}(I, X)$ into loc $H^{1,B}(\bar{I}, X)$ ([7], Theorems 3.7 and 3.8). Under Assumption 1.1 an a priori estimate for radiative functions is obtained as follows ([7], Lemma 3.4): Let $v=v(\cdot, k, \ell)$ be the radiative function for $\{L, k, \ell, 0\}$, where k belongs to a compact set K in $\boldsymbol{C}^+$ and $\ell\in\mathcal{V}_{1+\varepsilon}(I)$. Then there exists a constant δ_0, depending only on K and L, such that

$$(1.14)\qquad ||v||_{-1-\varepsilon}+||v'-ikv||_{-1+\varepsilon}+||B^{1/2}v||_{-1+\varepsilon}\leqq\delta_0|||\ell|||_{1+\varepsilon}.$$

3) For the definition of $\mathcal{V}(I)$ and $H^{1,B}(I,X)$ see the list of the notations in the Introduction.

2. The resolvent kernel $G(r, s, k)$ for A

We are ready to define the resolvent kernel $G(r, s, k)$ for A, which satisfies

$$(L-k^2)G(r, s, k) = \delta(r-s), \tag{2.1}$$

where $\delta(r)$ is Dirac's distribution. To this end for $s\in\bar{I}$ and $x\in X$ we shall introduce a linear functional $\ell[s, x]$ on $H_0^{1,B}(I, X)$ by

$$\langle \ell[s, x], u\rangle = (x, u(s)) \qquad (u\in H_0^{1,B}(I, X)). \tag{2.2}$$

We can show that the inequalities

$$\begin{cases} |\varphi(s)-\varphi(t)|^2\leqq |s-t|\,\|\varphi\|_B^2, \\ |\varphi(s)|^2\leqq 2\|\varphi\|_B^2 \end{cases} \tag{2.3}$$

hold for any $\varphi\in C_0^{2,B}(I, X)$ and any $s, t\in\bar{I}$.[4] In fact we have

$$\begin{aligned} |\varphi(s)-\varphi(t)|^2 &= \left|\int_s^t \varphi(r)dr\right|^2\leqq |s-t|\left|\int_s^t |\varphi'(r)|^2dr\right| \\ &\leqq |s-t|\,\|\varphi\|_B^2 \end{aligned} \tag{2.4}$$

Next, putting $|\varphi(t)| = \min_{s\leqq r\leqq s+1} |\varphi(r)|$, we have

$$\begin{aligned} |\varphi(s)|^2 &\leqq 2\{|\varphi(s)-\varphi(t)|^2+|\varphi(t)|^2\} \\ &\leqq 2\{|s-t|\int_s^t |\varphi'(r)|^2dr+\int_s^{s+1} |\varphi(t)|^2dr\} \\ &\leqq 2\int_s^{s+1}\{|\varphi'(r)|^2+|\varphi(r)|^2\}dr \\ &\leqq 2\|\varphi\|_B^2. \end{aligned} \tag{2.5}$$

Since $C_0^{2,B}(I, X)$ is dense in $H_0^{1,B}(I, X)$, it follows from (2.3) that $H_0^{1,B}(I, X)$ is continuously imbedded in $C^0(\bar{I}, X)$. Hence $\ell[s, x]$ can be regarded as a linear functional on $H_0^{1,B}(I, X)$. Further, we can see from (2.3) that

$$\begin{cases} |||\ell[s, x]||| = \sup_{\|\varphi\|_B=1}|(x, \varphi(s)|\leqq |x|\cdot\sup_{\|\varphi\|_B=1}|\varphi(s)|\leqq\sqrt{2}\,|x|, \\ |||\ell[s, x]|||_{1+\varepsilon} \leqq \sqrt{2}(1+s)^{(1+\varepsilon)/2}|x|, \end{cases} \tag{2.6}$$

which implies that $l[s, x]\in\mathcal{V}_{1+\varepsilon}(I)$.

Lemma 2.1. *Let us assume Assumption* 1.1. *Let* $v=v(\cdot, s, k, x)$ *be the radiative function for* $\{L, k, \ell[s, x], 0\}$, *where* $k\in \boldsymbol{C}^+$, $s\in\bar{I}$ *and* $x\in X$. *Then for any* $R>0$ *there exists a constant* $\delta_2=\delta_2(R, k)>0$ *such that we have*

$$\|v\|_{B,(0,R)}\leqq\delta_2(1+s)^{(1+\varepsilon)/2}|x| \tag{2.7}$$

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4) See Jäger [3], p. 69.

and

$$\max_{0 \leqq r \leqq R} |v(r)| \leqq \sqrt{2}\, \delta_2 (1+s)^{(1+\varepsilon)/2} |x|. \tag{2.8}$$

$\delta_2(R, k)$ is bounded when R moves in a bounded set in I and k moves in a compact set in $\boldsymbol{C}^+$.

Proof. Put $I_1=(0, R)$ and $I_2=(0, R+1)$. It follows from Lemma 1.5 in [7] that we have

$$||v||_{B,I_1} \leqq c\{||v||_{0,I_2} + |||\ell[s, x]|||\} \tag{2.9}$$

with a constant $c=c(R, k)>0$ which is bounded when the pair (R, k) moves in a bounded set in $I \times \boldsymbol{C}$. Using (1.14) we have

$$||v||_{0,I_2} \leqq (1+R)^{(1+\varepsilon)/2} ||v||_{-1-\varepsilon} \leqq \delta_0 (1+R)^{(1+\varepsilon)/2} |||\ell[s, x]|||_{1+\varepsilon} \tag{2.10}$$

with a constant $\delta_0=\delta_0(k)$ which is bounded when k moves in a compact set in $\boldsymbol{C}^+$. Hence we obtain from (2.9), (2.10) and (2.6)

$$\begin{aligned} ||v||_{B,I} &\leqq c\{\delta_0(1+R)^{(1+\varepsilon)/2} |||\ell[s, x]|||_{1+\varepsilon} + |||\ell[s, x]|||\} \\ &\leqq c\{\delta_0(1+R)^{(1+\varepsilon)/2} \sqrt{2}(1+s)^{(1+\varepsilon)/2} |x| + \sqrt{2}\, |x|\} \\ &= \sqrt{2}\, c\{\delta_0(1+R)^{(1+\varepsilon)/2} + (1+s)^{(-1-\varepsilon)/2}\}(1+s)^{(1+\varepsilon)/2} |x| \\ &\leqq \sqrt{2}\, c\{\delta_0(1+R)^{(1+\varepsilon)/2} + 1\}(1+s)^{(1+\varepsilon)/2} |x|, \end{aligned} \tag{2.11}$$

which means (2.7) with $\delta_2 = \sqrt{2}\, c\{\delta_0(1+R)^{(1+\varepsilon)/2}+1\}$. (2.8) follows from (2.7) and (2.3). Q. E. D.

In view of the above lemma we can give the following definition.

DEFINITION 2.2. For each triple $(r, s, k) \in \bar{I} \times \bar{I} \times \boldsymbol{C}^+$ we define a bounded linear operator $G(r, s, k)$ on X by $G(r, s, k)x = v(r, s, k, x)$, where $v(r, s, k, x)$ is the radiative function for $\{L, k, \ell[s, x], 0\}$. $G(r, s, k)$ is said to be the *resolvent kernel* for A.

For the properties of the resolvent kernel $G(r, s, k)$ we can show almost the same results as given in §6 of Jäger [3].

Theorem 2.3. *Let us assume Assumption* 1.1. *Let $G(r,s,k)$ be the resolvent kernel for A. Then we have the following* (i)∼(v):

(i) *$G(\cdot, s, k)x \in$ loc $H_0^{1,B}(\bar{I}, X) \cap H^{-1-\varepsilon}(I, X)$ for $s \in \bar{I}$ and $x \in X$. $G(\cdot, s, k)x$ is continuous on $\bar{I} \times \boldsymbol{C}^+ \times X$ both in* loc *$H_0^{1,B}(\bar{I}, X)$ and in $H^{-1-\varepsilon}(I, X)$. $G(r, s, k)x$ is a continuous X-valued function on $\bar{I} \times \bar{I} \times \boldsymbol{C}^+ \times X$, too.*

(ii) *$G(0, r, k) = G(r, 0, k) = 0$ for any $r \in \bar{I}$.*

(iii) *For any $(s, k, x) \in \bar{I} \times \boldsymbol{C} \times X$ $G(\cdot, s, k)x \in C^2(I - \{s\}, D)$ and $G(\cdot, s, k)x$ satisfies the equation*

$$(2.12) \qquad (L-k^2)G(\cdot\,, s, k)x = 0 \qquad in\ I-\{s\}.$$

(iv) *Denote by* $R(z)=(A-z)^{-1}$ *the resolvent of* A. *Then we have*

$$(2.13) \qquad R(k^2)f(r) = \int_I G(r, s, k)f(s)ds,$$

where $k\in \boldsymbol{C}^+$, $\mathrm{Im}\ k>0$ *and* $f\in H^0(I, X)$ *with a compact carrier in* $\bar{I}$.

(v) *We have*

$$(2.14) \qquad G(r, s, k)^* = G(s, r, -\bar{k})$$

for any triple $(r, s, k)\in\bar{I}\times\bar{I}\times \boldsymbol{C}^+$, *where* $G(r, s, k)^*$ *is the adjoint of* $G(r, s, k)$ *in* X.

Proof. Since we have from (2.2) and (2.3)

$$\begin{aligned}(2.15) \qquad & |\langle \ell[s, x], (1+r)^{(1+\varepsilon)/2}\varphi\rangle - \langle \ell[t, x], (1+r)^{(1+\varepsilon)/2}\varphi\rangle| \\ &= |(1+s)^{(1+\varepsilon)/2}(x, \varphi(s)) - (1+t)^{(1+\varepsilon)/2}(y, \varphi(t))| \\ &\leqq \sqrt{2}\,|(1+s)^{(1+\varepsilon)/2} - (1+t)^{(1+\varepsilon)/2}|\,|x|\,||\varphi||_B + \sqrt{2}(1+t)^{(1+\varepsilon)/2}|x-y|\,||\varphi||_B \\ &\quad + (1+t)^{(1+\varepsilon)/2}|s-t|\,|y|\,||\varphi||_B \qquad (\varphi\in C_0^{2,B}(I, X)),\end{aligned}$$

we see that $\ell[s, x]$ is a $\mathcal{U}_{1+\varepsilon}(I)$-valued continuous function on $\bar{I}\times X$. Hence (i) follows from Theorem 3.7 in [7]. We obtain $G(0, r, k)=0$ from the fact that $G(\cdot\,, r, k)x\in \mathrm{loc}\ H_0^{1,B}(\bar{I}, X)$ for any $x\in X$. It follows from the uniqueness of the radiative solution that $G(r, 0, k)=0$. (iii) follows from the regularity theorem in Jäger [3] (Satz 3.1).

Next let us show (iv). It follows from (1.11) that we have

$$(2.16) \qquad \begin{cases} R(k^2)f\in H_0^{1,B}(I, X) \\ ((R(k^2)f, (L-\bar{k}^2)\varphi))_0 = ((f, \varphi))_0 \qquad (\varphi\in C_0^{2,B}(I, X)),\end{cases}$$

where $k\in \boldsymbol{C}^+$, $\mathrm{Im}\ k>0$ and $f\in H^0(I, X)$. Hence $R(k^2)f$ is the radiative function for $\{L, k, \ell[f], 0\}$. Further, assume that the carrier of f is compact in $\bar{I}$ and is contained in $[0, R]$. Then we shall show that

$$(2.17) \qquad V(r) = \int_0^R G(r, s, k)f(s)ds$$

is the radiative function for $\{L, k, \ell[f], 0\}$, too. In fact we can easily see for any $\varphi\in C_0^{2,B}(I, X)$

$$\begin{aligned}(2.18) \qquad ((V, (L-\bar{k}^2)\varphi))_0 &= \int_I dr(\int_0^R G(r, s, k)f(s)ds, (L-\bar{k}^2)\varphi(r)) \\ &= \int_0^R ds((G(\cdot\,, s, k)f(s), (L-\bar{k}^2)\varphi(\cdot)))_0 \\ &= \int_0^R (f(s), \varphi(s))ds\end{aligned}$$

$$= ((f, \varphi))_0.$$

The following estimate is sufficient to see that $V \in H^{-1-\varepsilon}(I, X)$:

$$\begin{aligned}(2.19) \quad & \int_I (1+r)^{-1-\varepsilon} |V(r)|^2 dr \\ &= \int_I (1+r)^{-1-\varepsilon} dr \int_0^R ds \int_0^R dt (G(r, s, k)f(s), G(r, t, k)f(t)) \\ &\leqq 2\int_0^R ds \int_0^R dt \int_I (1+r)^{-1-\varepsilon}\{|G(r, s, k)f(s)|^2 \\ &\qquad\qquad + |G(r, t, k)f(t)|^2\} dr \\ &= 4R \int_0^R \|G(\cdot\,, s, k)f(s)\|^2_{-1-\varepsilon} ds \\ &\leqq 4R\delta_0^2 \int_0^R |||\ell[s, f(s)]|||^2_{1+\varepsilon} ds \\ &\leqq 4R\delta_0^2 \int_0^R 2(1+s)^{1+\varepsilon} |f(s)|^2 ds < \infty,\end{aligned}$$

where we have made use of (1.14) and (2.6). Similarly we can show that $V' - ikV \in H^{-1+\varepsilon}(I, X)$. Therefore we have $V(r) = R(k^2)f(r)$ by the uniqueness theorem of the radiative function (Theorem 2.2 in [7]), which completes the proof of (iv).

Finally we shall prove (v). If $k \in \boldsymbol{C}^+$ and Im $k > 0$, then (2.14) follows from (2.13) and the fact that $R(k^2)^* = R(\bar{k}^2)$. Since $G(r, s, k)x$ is continuous on $\boldsymbol{C}^+$, (2.14) is also true for $k \in \boldsymbol{C}^+$, Im $k = 0$. Q.E.D.

Let $C_j(r)$, $j = 1, 2$, be operator-valued functions on I satisfying (C−1)∼(C−3) in Assumption 1.1. We set

$$(2.20) \quad \begin{cases} M_j = -\dfrac{d^2}{dr^2} + B(r) + C_j(r) \\ \mathscr{D}(M_j) = \mathfrak{A}_0 \end{cases}$$

and we denote by A_j the Friedrichs extensions of M_j, $j = 1, 2$. $G_j(r, s, k)$, $j = 1, 2$, are the resolvent kernels for A_j. We shall show a formula involving $G_1(r, s, k)$ and $G_2(r, s, k)$ which will be used in §3.

Lemma 2.4[5]. *Let* $C_1(r)$ *and* $C_2(r)$ *be as above. Let* $B(r)$ *satisfy* (B–1)−(B–4) *in Assumption* 1.1. *Let* $k \in \boldsymbol{R} - \{0\}$. *Then we have*

$$\begin{aligned}(2.21) \quad & (\{G_1(r, s, k) - G_2(r, s, -k)\}x, y) \\ &= \left(\frac{d}{dt}G_1(t, s, k)x, G_2(t, r, k)y\right) - \left(G_1(t, s, k)x, \frac{d}{dt}G_2(t, r, k)y\right) \\ &\quad + \int_0^t (\{C_2(\tau) - C_1(\tau)\}G_1(\tau, s, k)x, G_2(\tau, r, k)y)d\tau,\end{aligned}$$

5) Cf. Lemma 6.1 in Jäger [3].

where $x, y \in X$, r, $s \in I$, *and* $t > \max(r, s)$.

Proof. Let $\psi \in C^1(\boldsymbol{R})$ satisfying $0 \leqq \psi \leqq 1$ and

$$\psi(r) = \begin{cases} 1 & (r \leqq 0) \\ 0 & (r \geqq 1). \end{cases} \tag{2.22}$$

Set $\psi_n(\tau) = \psi_{n,t}(\tau) = \psi(n(\tau - t))$ and $I_0 = (0, t+1)$. Then we can see that

$$\psi_n(\cdot) G_j(\cdot, s, k,) x \in H_0^{1,B}(I_0, X), \tag{2.23}$$

where $j = 1, 2$, $s \in I$, $k \in \boldsymbol{C}^+$ and $x \in X$. Using integration by parts we rewrite the relation

$$((G_1(\cdot, s, k)x, (L_1 - \bar{k}^2)\varphi))_0 = (x, \varphi(s)) \tag{2.24}$$

to obtain

$$((G_1(\cdot, s, k)x, \varphi(\cdot)))_{B,I_0} + (((C_1(\cdot) - 1)G_1(\cdot, s, k)x, \varphi(\cdot)))_{0,I_0} = (x, \varphi(s)) \tag{2.25}$$

for any $\varphi \in C_0^{2,B}(I_0, X)$. Since both sides of (2.25) can be extended to bounded, anti-linear functionals on $H_0^{1,B}(I_0, X)$ for fixed $s \in I$, $k \in \boldsymbol{C}^+$, $x \in X$, (2.25) holds good for any $\varphi \in H_0^{1,B}(I_0, X)$. Hence we can put $\varphi = \psi_n G_2(\cdot, r, k)y$ in (2.25) to see

$$\begin{aligned} &((G_1(\cdot, s, k)x, \psi_n(\cdot)G_2(\cdot, r, k)y))_{B,I_0} \\ &\quad + (((C_1(\cdot) - 1)G_1(\cdot, s, k)x, \psi_n(\cdot)G_2(\cdot, r, k)y))_{0,I_0} \\ &\qquad = (x, \psi_n(s)G_2(s, r, k)y) \\ &\qquad = (G_2(r, s, -k)x, y) \end{aligned} \tag{2.26}$$

for r, $s \in (0, t)$, $x, y \in X$ and $k \in \boldsymbol{R} - \{0\}$, where we used Therem 2.3, (*v*). Similarly we obtain

$$\begin{aligned} &((\psi_n(\cdot)G_1(\cdot, s, k)x, G_2(\cdot, r, k)y))_{B,I_0} \\ &\quad + ((\psi_n G_1(\cdot, s, k)x, (C_2(\cdot) - 1)G_2(\cdot, r, k)y))_{0,I_0} \\ &\qquad = (G_1(r, s, k)x, y). \end{aligned} \tag{2.27}$$

Combining (2.26) and (2.27), we have

$$\begin{aligned} &(\{G_1(r, s, k) - G_2(r, s, -k)\}x, y) \\ &\quad = \int_0^\infty \psi_n(\tau)(\{C_2(\tau) - C_1(\tau)\}G_1(\tau, s, k)x, G_2(\tau, r, k)y)d\tau \\ &\quad + \int_0^\infty \psi_n'(\tau)\Big\{\Big(G_1(\tau, s, k)x, \frac{d}{d\tau}G_2(\tau, r, k)y\Big) \\ &\quad - \Big(\frac{d}{d\tau}G_1(\tau, s, k)x, G_2(\tau, r, k)y\Big)\Big\}d\tau. \end{aligned} \tag{2.28}$$

Noting that

$$(2.29) \qquad \begin{cases} \psi_n(\tau) \to \begin{cases} 1 & \text{for } \tau \leqq t \\ 0 & \text{for } \tau > t, \end{cases} \\ \psi_n'(\tau) \to -\delta(\tau - t), \end{cases}$$

as $n \to \infty$, we obtain (2.21) from (2.28). Q.E.D.

3. The spectral representation for A

This section is devoted to constructing a spectral decomposition for A by means of a "generalized Fourier transform". We start with the results of Jäger [3]. The following theorem on the spectrum of A has been proved in Jäger [3], §4.

Theorem 3.1. *Let us assume Assumption* 1.1. (i) *Then A is bounded below with the lower bound $\kappa_0 \leqq 0$.* (ii) *On $(\kappa_0, 0)$ the continuous spectrum of A is absent. The negative eigenvalues, if they exist, are of finite multiplicity and are discrete in the sense that they form an isolated set having no limit point other than the origin* 0. (iii) *We have $\sigma_e(A) \subset (0, \infty)$, where $\sigma_e(A)$ means the essential spectrum of A. There exists no positive eigenvalue of A. If in addition there exists $x_0 \in D$ which satisfies*

$$(3.1) \qquad \begin{cases} |x_0| = 1, \\ s-\lim_{r \to \infty} B(r)x_0 = 0, \end{cases}$$

then we have $\sigma_e(A) = [0, \infty)$.

If we assume instead of (1.5) that we have for any $x \in D$ and any $r \in I$

$$(3.2) \qquad |c(r)x| \leqq \tilde{c}_0(1+|r|)^{-3/2-\varepsilon_0}(|x| + |B^{1/2}(r)x|)$$

with constants $\tilde{c}_0 > 0$, $\varepsilon_0 > 0$, then an eigenfunction expansion for A has been obtained by Jäger [3], §6. Jäger's results are summarized as follows:

Theorem 3.2. *Let us assume* (B–1) (B–4) *and* (C–1), (C–2) *in Assumption* 1.1 *and* (3.2). *Then for $k \in \boldsymbol{R} - \{0\}$, $r \in \bar{I}$ and $x \in X$ the strong limit*

$$(3.3) \qquad \Gamma(r, k)x = s-\lim_{t \to \infty} e^{-itk} G(t, r, k)x$$

exists. $\Gamma(r, k)$ is a bounded linear operator on X for each $r \in \bar{I}$ and $k \in \boldsymbol{R} - \{0\}$. The adjoint $\Gamma^*(r, k)$ of $\Gamma(r, k)$ satisfies

$$(3.4) \qquad \begin{cases} \Gamma^*(\cdot, k)x \in C^0(\bar{I}, X) \cap \mathrm{loc}\, H_0^{1,B}(\bar{I}, X), \\ \Gamma^*(\cdot, k)x \in C^2(I, D), \\ (L - k^2)\Gamma^*(r, k)x = 0 \qquad (r \in I), \\ \Gamma^*(0, k)x = 0, \end{cases}$$

for $x\in X$ *and* $k\in \boldsymbol{R}-\{0\}$. *Set*

$$(3.5)\qquad \begin{cases} \Gamma^{\pm}(r, k) = \pm\sqrt{\dfrac{2}{\pi}}ik\Gamma(r, \pm k) \\ \mathcal{G}^{\pm}f(k) = \displaystyle\int_I \Gamma^{\pm}(r, k)f(r)dr, \end{cases}$$

where $f\in C_0^0(\bar{I}, X)$.[6] *Then* $\mathcal{G}^{\pm}f\in L_2((0, \infty), X, dk)$ *and* $\mathcal{G}^{\pm}$ *have unique extentions* $\mathcal{F}^{\pm}$ *to* $H^0(I, X)$ *which are bounded linear operators on* $H^0(I, X)$ *into* $L_2((0, \infty), X, dk)$. *Denote the resolution of the identity for* A *by* $E(\lambda)$. *We have for* $0<\lambda_1<\lambda_2\leqq\infty$ *and* $f, g\in H^0(I, X)$

$$(3.6)\qquad ((E((\lambda_1, \lambda_2))f, g))_0 = \int_{\lambda_1<k^2<\lambda_2} ((\mathcal{F}^{\pm}f)(k), (\mathcal{F}^{\pm}g)(k))\, dk.$$

Hence the spectrum of A *is absolutely continuous on* $(0, \infty)$.

Let us assume (C–3) in place of (3.2) again. For each $n=1, 2, \cdots$ we take $\psi_n\in C^1(I)$ satisfying $0\leqq\psi_n\leqq 1$ and

$$(3.7)\qquad \psi_n(r) = \begin{cases} 1 & \text{for } 0<r\leqq n, \\ 0 & \text{for } r\geqq n+1. \end{cases}$$

A_n, $n=1, 2,\cdots$, denote the Friedrichs extensions of the operators M_n which are defined by

$$(3.8)\qquad \begin{cases} \mathcal{D}(M_n) = \mathfrak{A}_0 \\ M_n = -\dfrac{d^2}{dr^2}+B(r)+C_n(r), \qquad C_n(r) = \psi_n(r)C(r), \end{cases}$$

respectively, We can easily see that for each $n=1, 2, \cdots$, $C_n(r)$ satisfies (3.2) and that the sequence $C_n(r)$ satisfies Assumption 4.1 in [7]. For each $n=1, 2, \cdots$, $G_n(r, s, k)$ is the resolvent kernel for A_n. $\Gamma_n(r, k)$, $\Gamma_n^{\pm}(r, k)$, $\mathcal{G}_n^{\pm}$, $\mathcal{F}_n^{\pm}$ and $E_n(\lambda)$ are defined as in Theorem 3.2.

Lemma 3.3. *We have*

$$(3.9)\qquad (\{G_n(r, s, k)-G_m(r, s, -k)\}x, y) = 2ik(\Gamma_n(s, k)x, \Gamma_m(r, k)y) + \int_0^{\infty}(\{C_m(\tau)-C_n(\tau)\}G_n(\tau, s, k)x, G_m(\tau, r, k)y)d\tau \qquad (n, m = 1, 2, \cdots),$$

where $k\in\boldsymbol{R}-\{0\}$, $r, s\in\bar{I}$ *and* $x, y\in X$.

---

6) $C_0^0(\bar{I}, X)$ denotes the set of all X-valued, continuous functions on $\bar{I}=[0, \infty)$ with a compact carrier in $\bar{I}$.

Proof. We obtain from Lemma 2.4

$$(3.10)\qquad \begin{aligned}&(\{G_n(r, s, k)-G_m(r, s, -k)\}x, y)\\ &\quad= 2ik(G_n(t, s, k)x, G_m(t, r, k)y)\\ &\qquad+\int_0^t(\{C_m(\tau)-C_n(\tau)\}G_n(\tau, s, k)x, G_m(\tau, r, k)y)d\tau\\ &\qquad+\Big\{\Big(\frac{d}{dt}G_n(t, s, k)x-ikG_n(t, s, k)x, G_m(t, r, k)y\Big)\\ &\qquad-\Big(G_n(t, s, k)x, \frac{d}{dt}G_m(t, r, k)y-ikG_m(t, r, k)y\Big)\Big\}\\ &\quad= K_1(t)+K_2(t)+K_3(t).\end{aligned}$$

First it follows from (1.5) that

$$(3.11)\qquad \begin{aligned}&|(\{C_n(\tau)-C_m(\tau)\}G_n(\tau, s, k)x, G_m(\tau, r, k)y)|\\ &\leqq 2(1+\tau)^{-1-\varepsilon}c_0(|G_n(\tau, s, k)x|+|B^{1/2}(\tau)G_n(\tau, s, k)x|)|G_m(\tau, r, k)y|\\ &\leqq 4(1+\tau)^{-1-\varepsilon}c_0\{2|G_n(\tau, s, k)x|^2+2|B^{1/2}(\tau)G_n(\tau, r, k)x|^2\\ &\qquad\qquad+|G_m(\tau, r, k)y|^2\\ &\in L^1(I, d\tau),\end{aligned}$$

where we have noted that $G_n(\cdot, s, k)x$, $G_m(\cdot, r, k)y\in H^{-1-\varepsilon}(I, X)$ and $B^{1/2}(\cdot)G_n(\cdot, s, k)x\in H^{-1+\varepsilon}(I, X)$ by (1.14). Hence we have

$$(3.12)\qquad \lim_{t\to\infty} K_2(t)=\int_0^\infty(\{C_n(\tau)-C_m(\tau)\}G_n(\tau, s, k)x, G_m(\tau, r, k)y)d\tau$$

Since $G_n'(\cdot, s, k)x-ikG_n(\cdot, s, k)x$, $G_m'(\cdot, r, k)y-ikG_m(\cdot, r, k)y\in H^{-1+\varepsilon}(I, X)$ and $G_n(\cdot, s, k)x$, $G_m(\cdot, r, k)y\in H^{-1-\varepsilon}(I, X)$, we have $\lim_{j\to\infty} K_3(t_j)=0$ along some sequence $\{t_j\}$, $t_j\to\infty$. Finally we see from (3.3)

$$(3.13)\qquad \begin{aligned}\lim_{t\to\infty} K_1(t)&=\lim_{t\to\infty} 2ik(e^{-itk}G_n(t, s, k)x, e^{-itk}G_m(t, r, k)y)\\ &=2ik(\Gamma_n(s, k)x, \Gamma_m(r, k)y).\end{aligned}$$

Thus we obtain (3.9). Q.E.D.

Now we are in a position to show that $\mathcal{G}_n^{\pm}f$ is a Cauchy sequence in loc $L_2(\bar{I}, X, dk)$ for any $f\in C_0^0(\bar{I}, X)$. Then $E(\lambda)$, the resolution of the identity for A, will be represented by unique extensions $\mathcal{F}^{\pm}$ of $\mathcal{G}^{\pm}=\lim_{n\to\infty}\mathcal{G}_n^{\pm}$.

Theorem 3.4. *Let us assume Assumption* 1.1.

(i) *Then for any $f\in C_0^0(\bar{I}, X)$ there exist*

$$(3.14)\qquad \mathcal{G}^{\pm}f=\lim_{n\to\infty}\mathcal{G}_n^{\pm}f\qquad in\ L_2((\alpha, \beta), X, dk),$$

where α and β are arbitrary numbers such that $0<\alpha<\beta<\infty$. For $0<\lambda_1<\lambda_2<$

∞ *and* $f, g \in C_0^0(\bar{I}, X)$ *we have*

$$((E((\lambda_1, \lambda_2))f, g)) = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{G}^{\pm} f(k), \mathcal{G}^{\pm} g(k)) dk, \tag{3.15}$$

where $E(\lambda)$ *is the resolution of the identity for* A. *Hence* $\mathcal{G}^{\pm}$ *are bounded, linear operators from* $C_0^0(\bar{I}, X)$ *contained in* $H^0(I, X)$ *into* $L_2((0, \infty), X, dk)$.

(ii) *Let* $\mathcal{F}^{\pm}$ *be unique extensions of* $\mathcal{G}^{\pm}$ *to* $H^0(I, X)$, *respectively. Then for* $\Delta = (\lambda_1, \lambda_2)$, $0 \leqq \lambda_1 < \lambda_2 \leqq \infty$ *we have*

$$E(\Delta) = (\mathcal{F}^{\pm})^* \chi_{\sqrt{\Delta}} \mathcal{F}^{\pm}, \tag{3.16}$$

where $\chi_{\sqrt{\Delta}}$ *is the characteristic function of* $(\sqrt{\lambda_1}, \sqrt{\lambda_2})$ *and* $(\mathcal{F}^{\pm})^*$ *are the adjoints of* $\mathcal{F}^{\pm}$ *acting from* $L_2((0, \infty); X, dk)$ *into* $H^0(I, X)$, *respectively.*

The following corollary directly follows from (3.16).

Corollary 3.5. *The spectrum of* A *is absolutely continuous on* $(0, \infty)$.

For the proof of Theorem 3.4. we need

Lemma 3.6. *We have*

$$((E((\lambda_1, \lambda_2))f, g))_0 = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \frac{kdk}{\pi i} \int_I \int_I drds(\{G(r, s, k) - G(r, s, -k)f(s), g(r)), \tag{3.17}$$

where $0 < \lambda_1 < \lambda_2 < \infty$ *and* $f, g \in H^0(I, X)$ *with compact carriers.*

Proof. Let us start with the well-known relation[7)]

$$((E(\lambda_1, \lambda_2))f, g))_0 = \lim_{\eta \downarrow 0} \lim_{\mu \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1+\eta}^{\lambda_2-\eta} ((\{R(\lambda + i\mu) - R(\lambda - i\mu)\} f, g))_0 d\lambda. \tag{3.18}$$

Using (2.13), $R(\lambda + i\mu)f$ can be represented by the resolvent kernel $G(r, s, k)$. Hence (3.18) becomes

$$\begin{aligned} &((E((\lambda_1, \lambda_2))f, g))_0 \\ &= \lim_{\eta \downarrow 0} \lim_{\mu \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1+\eta}^{\lambda_2-\eta} ((\int_I \{G(\cdot, s, \sqrt{\lambda + i\mu}) - G(\cdot, s, \sqrt{\lambda - i\mu})\} f(s) ds, g))_0 d\lambda,^{8)} \\ &= \lim_{\eta \downarrow 0} \lim_{\mu \downarrow 0} \frac{1}{2\pi i} \int_{\lambda_1+\eta}^{\lambda_2-\eta} \int_0^R \int_0^R (\{G(r, s, \sqrt{\lambda + i\mu} - G(r, s, \sqrt{\lambda - i\mu}\} f(s), g(r)) dsdrd\lambda, \end{aligned} \tag{3.19}$$

7) See, for example, Dunford & Schwartz [1], p. 1202.

8) Here and in the sequel by $\sqrt{z}$ is meant the branch of the square root of z with $\mathrm{Im}\sqrt{z} \geq 0$.

where R is taken so large that the carriers of f and g are contained in $[0, R]$. Since $G(\cdot, s, \sqrt{\lambda \pm i\mu})f(s)$ is the radiative function for $\{L, \lambda \pm i\mu, \ell[s, f(s)], 0\}$, we can apply Theorem 3.7 in [7] to show that $G(r, s, \sqrt{\lambda + i\mu})f(s)$ is uniformly bounded for $(r, s, \lambda, \mu) \in [0, R] \times [0, R] \times [\lambda_1, \lambda_2] \times [0, 1]$ and

$$(3.20) \qquad \lim_{\eta \downarrow 0} G(r, s, \sqrt{\lambda \pm i\mu})f(s) = G(r, s, \pm\sqrt{\lambda})f(s) \quad \text{in } X.$$

(3.17) follows from (3.19) and (3.20). Q.E.D.

Proof of Theorem 3.4. Since we have

$$(3.21) \qquad \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{G}_n^{\pm} f - \mathcal{G}_m^{\pm} f|^2 dk = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{G}_n^{\pm} f(k)|^2 dk + \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{G}_m^{\pm} f(k)|^2 dk$$
$$- \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{G}_n^{\pm} f(k), \mathcal{G}_m^{\pm} f(k)) dk - \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{G}_m^{\pm} f(k), \mathcal{G}_n^{\pm} f(k)) dk,$$

in order to show that $\{\mathcal{G}_n^{\pm} f\}_{n=1}^{\infty}$ are Cauchy sequences in $L_2((\sqrt{\lambda_1}, \sqrt{\lambda_2}), X, dk)$ it suffices to show

$$(3.22) \qquad \lim_{n, m \to \infty} \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{G}_n^{\pm} f(k), \mathcal{G}_m^{\pm} f(k) dk = ((E(\lambda_1, \lambda_2))f, f))_0.$$

Recalling (3.9) of Lemma 3.3 and (3.5), we have

$$(3.23) \qquad \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{G}_n^{\pm} f(k), \mathcal{G}_m^{\pm} f(k)) dk = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} dk \int_I \int_I drds (\Gamma_n^{\pm}(s, k)f(s), \Gamma_m^{\pm}(r, k)f(r))$$
$$= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \frac{2k^2}{\pi} dk \int_0^R \int_0^R drds (\Gamma_n(s, \pm k)f(s), \Gamma_m(r, \pm k)f(r))$$
$$= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \pm \frac{k}{\pi i} dk \int_0^R \int_0^R drds [(\{G_n(r, s, \pm k) - G_m(r, s, \mp k)\} f(s), f(r))$$
$$+ \int_0^{\infty} (\{C_n(\tau) - C_m(\tau)\} G_n(\tau, s, \pm k)f(s), G_m(\tau, r, \pm k)f(r)) d\tau]$$
$$= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} \pm \frac{kdk}{i\pi} \int_0^R \int_0^R drds [K_{n,m}^{(1)}(r, s, R) + K_{n,m}^{(2)}(r, s, k)],$$

where we take $R > 0$ such that $[0, R]$ contains the carrier of f. Let us show that $\lim_{n, m \to \infty} K_{n,m}^{(2)} = 0$ and $|K_{n,m}^{(2)}(r, s, k)|$ is uniformly bounded for $(r, s, k) \in [0, R] \times [0, R] \times [\sqrt{\lambda_1}, \sqrt{\lambda_2}]$ and $n, m = 1, 2 \cdots$. Using (1.5) we estimate $K_{n,m}^{(2)}(r, s, k)$ as follows:

$$(3.24) \qquad |K_{n,m}^{(2)}| \leq c_0 \int_I |\psi_n(\tau) - \psi_m(\tau)| (1+\tau)^{-1-\varepsilon} (|G_n(\tau, s, \pm k)f(s)|$$
$$+ |B^{1/2}(\tau) G_n(\tau, s, \pm k)f(s)|) \times |G_m(\tau, r, \pm k)f(r)| d\tau$$
$$\leq 8c_0 \int_I |\psi_n(\tau) - \psi_m(\tau)| (1+\tau)^{-1-\varepsilon} \{|G_n(\tau, s, \pm k)f(s) -$$

$$
\begin{aligned}
&-G(\tau, s, \pm k)f(s)|^2+|G_m(\tau, r, \pm k)f(r)-G(\tau, r, \pm k)f(r)|^2 \\
&+|G(\tau, s, \pm k)f(s)|^2+|G(\tau, r, \pm k)f(r)|^2 \\
&+|B^{1/2}(\tau)G_n(\tau, s, \pm k)f(s)|^2\}d\tau \\
\leqq\ & 8c_0[\{||G_n(\cdot, s, \pm k)f(s)-G(\cdot, s, \pm k)f(s)||^2_{-1-\varepsilon} \\
&+||G_m(\cdot, r, \pm k)f(r)-G(\cdot, r, \pm k)f(r)||^2_{-1-\varepsilon}\} \\
&+\int_{\xi(m,n)}^{\infty}(1+\tau)^{-1-\varepsilon}\{|G(\tau, s, \pm k)f(s)|^2+|G(\tau, r, \pm k)f(r)|^2\}d\tau \\
&+\xi(m, n)^{-2\varepsilon}||B^{1/2}(\cdot)G_n(\cdot, s, \pm k)f(s)||^2_{-1+\varepsilon}] \\
=\ & Q_1+Q_2+Q_3,
\end{aligned}
$$

where we put $\xi(m, n)=\min(m, n)$. We see from (2.15) that $\ell[s, f(s)]$ is a $\mathcal{U}_{1+\varepsilon}(I)$-valued, continuous function on $[0, R]$ such that

(3.25) $$|||\ell[s, f(s)]|||_{1+\varepsilon}\leqq\sqrt{2}(1+s)^{(1+\varepsilon)/2}|f(s)|.$$

Hence, applying Theorem 4.2 in [7], we see

(3.26) $$Q_3\leqq c(\xi(m, n))^{-2\varepsilon}\sqrt{2}(1+s)^{(1+\varepsilon)/2}|f(s)|\to 0 \qquad (m, n\to\infty)$$

uniformly on $(s, k)\in[0, R]\times[\sqrt{\lambda_1}, \sqrt{\lambda_2}]$, where c is a positive constant. Putting $v_n=G_n(\cdot, s, \pm k)f(s)$ and $\ell_n(s, k)\equiv\ell[s, f(s)]$ (or $v_m=G_m(\cdot, r, \pm k)f(s)$ and $\ell_m(r, k)\equiv\ell[r, f(r)]$) in (ii) of Theorem 4.3 in [7], we obtain

(3.27) $$\begin{cases} ||G_n(\cdot, s, \pm k)f(s)-G(\cdot, s, \pm k)f(s)||_{-1-\varepsilon}\to 0, & n\to\infty, \\ ||G_m(\cdot, r, \pm k)f(r)-G(\cdot, r, \pm k)f(r)||_{-1-\varepsilon}\to 0, & m\to\infty \end{cases}$$

uniformly on $(r, s, k)\in[0, R]\times[0, R]\times[\sqrt{\lambda_1}, \sqrt{\lambda_2}]$, which implies the uniform convergence of $Q_1\to 0$. It follows from (1.14) and (3.25) that Q_2 is uniformly bounded for $(r, s, k)\in[0, R]\times[0, R]\times[\sqrt{\lambda_1}, \sqrt{\lambda_2}]$ and $n, m=1, 2,\cdots$, and $Q_2\to 0$ as $n, m\to\infty$. Thus we have shown that $K^{(2)}_{n,m}(r, s, k)$ is uniformly bounded and $K^{(2)}_{n,m}(r, s, k)\to 0$ as $n, m\to\infty$. For $K^{(1)}_{n,m}(r, s, k)$ we obtain from (3.27)

(3.28) $$\begin{aligned}&\int_0^R\int_0^R K^{(1)}_{n,m}(r, s, k)\,drds \\ &\qquad\to\int_0^R\int_0^R(\{G(r, s, \pm k)-G(r, s, \mp k)\}f(s), f(r))drds\end{aligned}$$

as $n, m\to\infty$ uniformly for $k\in[\sqrt{\lambda_1}, \sqrt{\lambda_2}]$. Therefore with the aid of Lemma 3.6 we have

(3.29) $$\begin{aligned}&\lim_{n,m\to\infty}\int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}}(\mathcal{G}^{\pm}_n f(k), \mathcal{G}^{\pm}_m f(k))dk \\ &=\int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}}\frac{kdk}{i\pi}\int_0^R\int_0^R drds\,(\{G(r, s, k)-G(r, s, -k)\}f(s), f(r)) \\ &=((E(\lambda_1, \lambda_2)f, f))_0,\end{aligned}$$

which implies the congergence of $\{\mathcal{G}_n^{\pm}f\}$.

We define $\mathcal{G}_{\pm}f$ for $f \in C_0^0(\bar{I}, X)$ and $0<\alpha<\beta<\infty$ by

$$\mathcal{G}^{\pm}f = \lim_{n\to\infty} \mathcal{G}_n^{\pm}f \qquad \text{in } L_2((\alpha, \beta), X, dk) \tag{3.30}$$

Then putting $m=n$ in (3.22), we have

$$\begin{aligned} ((E(\lambda_1, \lambda_2))f, f))_0 &= \lim_{n\to\infty}\int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{G}_n^{\pm}f(k)|^2 dk \\ &= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} |\mathcal{G}^{\pm}f(k)|^2 dk, \end{aligned} \tag{3.31}$$

which implies that $\mathcal{G}^{\pm}f \in L_2((0, \infty), X, dk)$ and

$$((E((\lambda_1, \lambda_2))f, g))_0 = \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{G}^{\pm}f(k), \mathcal{G}^{\pm}g(k))dk, \tag{3.32}$$

for $f, g \in C_0^0(\bar{I}, X)$ and $0 \leqq \lambda_1 < \lambda_2 \leqq \infty$. Thus (i) of Theorem 3.4 has been proved. (ii) follows directly from (i). Q.E.D.

4. The operators $G_n^{\pm}(\lambda)$ and $H_n^{\pm}(\lambda)$

In this and the following two sections we study the property of A from the standpoint of perturbation theory. Stationary methods[9] are useful for our purpose. In particular we shall make use of the results of Kato and Kuroda [6]. This section is a preliminary one. We shall define a family of operators $G_n^{\pm}(\lambda)$, $H_n^{\pm}(\lambda)$ ($\lambda>0$, $n=1, 2,\cdots$) and study the properties of them.

Let $C_n(r)$, L_n, A_n, $\mathcal{F}_n^{\pm}$ etc. be as in §3. Let $L_0 = -\frac{d^2}{dr^2} + B(r)$ and A_0 be as in §1. Starting with L_0, we can define the resolvent $R_0(z)$, the resolvent kernel $G_0(r, s, k)$, and the generalized Fourier transforms $\mathcal{F}_0^{\pm}$. In this section we put $C(r) = C_\infty(r)$, $L=L_\infty$, $A=A_\infty$, $R(z)=R_\infty(z)$ etc.

Let us set

$$\boldsymbol{C}_1 = \boldsymbol{C} - (-\infty, 0], \text{ i.e., } \boldsymbol{C}_1 = \{z \in \boldsymbol{C} / \sqrt{z} \in \boldsymbol{C}^+\} \tag{4.1}$$

and

$$\boldsymbol{C}_{1,+} = \{z \in \boldsymbol{C}_1 / \text{Im } z \geqslant 0\}, \ \boldsymbol{C}_{1,-} = \{z \in \boldsymbol{C}_1 / \text{Im } z \leqq 0\}. \tag{4.2}$$

Then $R_n(z, f) = R_n(\cdot, z, f)$ is defined as the radiative function for $\{L_n, \sqrt{z}, \ell[f], 0\}$, where $z \in \boldsymbol{C}_1$, $f \in H^{1+\varepsilon}(I, X)$ and $n=0, 1, 2,\cdots, \infty$. By the regularity theorem of Jäger [3] (Satz 3.1) $C_m(r)R_n(r, z, f)$ is well-defined for any $f \in H^{1,B}(I, X) \cap H^{1+\varepsilon}(I, X)$, where $n, m = 0, 1, 2,\cdots, \infty$.

9) For the literature of stationary methods see Kato [5] and Kato and Kuroda [6].

Lemma 4.1. *Let us assume Assumption* 1.1. *Let K be a compact set in* $\boldsymbol{C}_1$. *Let* $f \in H^{1,B}(I, X) \cap H^{1+\varepsilon}(I, X)$ *and* $z \in K$.

(i) *Then there exists a constant* $\delta_3 > 0$ *such that*

$$\|C_m R_n(z, f)\|_{1+\varepsilon} \leqq \delta_3 \|f\|_{1+\varepsilon} \qquad (n, m = 0, 1, 2, \cdots, \infty), \tag{4.3}$$

where δ_3 *depends only on K.*

(ii) $C_m R_n(z, f)$ *is an* $H^{1+\varepsilon}(I, X)$*-valued continuous function on both* $\boldsymbol{C}_{1,+}$ *and* $\boldsymbol{C}_{1,-}$.

The proof is easy from Theorem 4.2 in [7] and we omit it.

It follows from Lemma 4.1 that the operator $C_m R_n(z, \cdot)$ from $H^{1,B}(I, X) \cap H^{1+\varepsilon}(I, X)$ into $H^{1+\varepsilon}(I, X)$ is uniquely extended to a bounded linear operator $Q_{m,n}(z)$ from $H^{1+\varepsilon}(I, X)$ into $H^{1+\varepsilon}(I, X)$. $Q_{m,n}(z)f$ is an $H^{1+\varepsilon}(I, X)$-valued continuous function on both $\boldsymbol{C}_{1,+}$ and $\boldsymbol{C}_{1,-}$.

DEFINITION 4.2. For each $n=1, 2, \cdots, \infty$ and each $z \in \boldsymbol{C}_1$ bounded linear operators $G_n(z)$, $H_n(z)$ from $H^{1+\varepsilon}(I, X)$ into $H^{1+\varepsilon}(I, X)$ are defined by

$$\begin{cases} G_n(z) = 1 + Q_{n,0}(z), \\ H_n(z) = 1 - Q_{n,n}(z). \end{cases} \tag{4.4}$$

In particular for $\lambda > 0$ we put

$$G_n^{\pm}(\lambda) = G_n(\lambda \pm i0) \quad \text{and} \quad H_n^{\pm}(\lambda) = H_n(\lambda \pm i0). \tag{4.5}$$

Lemma 4.3. *Let* $G_n(z)$ *and* $H_n(z)$, $n=1, 2, \cdots, \infty$, *be as above.*

(i) *Then we have for* $z \in \boldsymbol{C} - \boldsymbol{R}$ *and* $f \in H^{1+\varepsilon}(I, X)$

$$G_n(z)f = (A_n - z)R_0(z)f \text{ and } H_n(z)f = (A_0 - z)R_n(z)f. \tag{4.6}$$

(ii) *For any* $n=1, 2, \cdots, \infty$ *and any* $f \in H^{1+\varepsilon}(I, X)$, $G_n(z)f$ *and* $H_n(z)f$ *are* $H^{1+\varepsilon}(I, X)$*-valued, continuous functions on both* $\boldsymbol{C}_{1,+}$ *and* $\boldsymbol{C}_{1,-}$.

(iii) *Let* $z \in \boldsymbol{C}_1$. *Then we have*

$$G_n(z)H_n(z) = H_n(z)G_n(z) = 1 \qquad \text{on } H^{1+\varepsilon}(I, X)\,. \tag{4.7}$$

In particular we obtain for $\lambda > 0$

$$G_n^{\pm}(\lambda)H_n^{\pm}(\lambda) = H_n^{\pm}(\lambda)G_n^{\pm}(\lambda) = 1. \tag{4.8}$$

(iv) *For any* $f \in H^{1+\varepsilon}(I, X)$ *we have*

$$\begin{cases} \lim_{n\to\infty} G_n(z)f = G_\infty(z)f \\ \lim_{n\to\infty} H_n(z)f = H_\infty(z)f \end{cases} \tag{4.9}$$

in $H^{1+\varepsilon}(I, X)$ *uniformly on any compact set in* $\boldsymbol{C}_1$

Proof. Since we have

$$(4.10)\qquad R_n(z, f) = R_n(z)f \qquad (z\in \boldsymbol{C}-\boldsymbol{R},\ f\in H^{1+\varepsilon}(I, X)),$$

it follows that

$$(4.11)\qquad \begin{aligned} G_n(z)f &= \{1+C_nR_0(z)\}f = \{1+((A_n-z)-(A_0-z))R_0(z)\}f \\ &= (A_n-z)R_0(z)f \qquad (f\in H^{1+\varepsilon}(I, X)). \end{aligned}$$

Similarly we have $H_n(z)=(L_0-z)R_n(z)$ on $H^{1+\varepsilon}(I, X)$. (ii) follows from (ii) of Lemma 4.1. If $z\in \boldsymbol{C}-\boldsymbol{R}$, (4.7) follows from (4.6). Taking account of (ii), we can see that (4.7) is true for $z=\lambda>0$. (iv) follows from (ii) of Theorem 4.3 in [7]. Q. E. D

Next we show some formulas involving $G_n(z)$, $H_n(z)$ and the radiative solutions $R_n(z, f)$. Let $\lambda>0$ and $n=0, 1, 2,\cdots, \infty$. Then, according to Kato and Kuroda [6], we introduce a *bilinear form* $e_n(\lambda;\ \cdot,\ \cdot)$ *on* $H^{1+\varepsilon}(I, X)\times H^{1+\varepsilon}(I, X)$ by

$$(4.12)\qquad e_n(\lambda; f, g) = \frac{1}{2\pi i}((R_n(\lambda+i0, f)-R_n(\lambda-i0, f), g))_0.$$

Since $R_n(\lambda\pm i0, f)\in H^{-1-\varepsilon}(I, X)$, the right-hand side of (4.8) is well-defined. We can easily see that $e_n(\cdot\,;\ \cdot,\ \cdot)$ is a continuous function on $(0, \infty)\times H^{1+\varepsilon}(I, X)\times H^{1+\varepsilon}(I, X)$. It follows from (3.18), (4.12) and the continuity in z of the radiative function $R_n(z, f)$ that we have $e(\cdot\,; f, g)\in L^1((0, \infty); d\lambda)$ and

$$(4.13)\qquad ((E_n(\Delta)f, g))_0 = \int_\Delta e_n(\lambda; f, g)\, d\lambda,$$

where Δ is a Borel set in $(0, \infty)$ and $f, g\in H^{1+\varepsilon}(I, X)$. The bilinear form $e_n(\lambda; f, g)$ is called the *spectral form* for $E_n(\lambda)$.

Lemma 4.4. (i) *Let* $z\in \boldsymbol{C}_1$, $f\in H^{1+\varepsilon}(I, X)$ *and* $n=1, 2,\cdots, \infty$. *Then we have*

$$(4.14)\qquad R_n(z, f) = R_0(z, H_n(z)f),$$

and

$$(4.15)\qquad R_0(z, f) = R_n(z, G_n(z)f)$$

in $H^{-1-\varepsilon}(I, X)$. *In particular we have for* $\lambda>0$

$$(4.16)\qquad \begin{cases} R_n(\lambda\pm i0, f) = R_0(\lambda\pm i0, H_n^{\pm}(\lambda)f) \\ R_0(\lambda\pm i0, f) = R_n(\lambda\pm i0, G_n^{\pm}(\lambda)f) \end{cases} \quad in\ H^{-1-\varepsilon}(I, X).$$

(ii) *Let* $\lambda>0$ *and let* $n=1, 2, 3,\cdots, \infty$. *Then the relations*

$$(4.17) \qquad \begin{cases} e_n(\lambda;\, G_n^{\pm}(\lambda)f,\, G_n^{\pm}(\lambda)g) = e_0(\lambda;\, f,\, g) \\ e_0(\lambda;\, H_n^{\pm}(\lambda)f,\, G_n^{\pm}(\lambda)g) = e_n(\lambda;\, f,\, g) \end{cases}$$

hold for any pair of $(f, g)\in H^{1+\varepsilon}(I, X)\times H^{1+\varepsilon}(I, X)$.

Proof. Let us prove (i). The both sides of (4.14) are $H^{-1-\varepsilon}(I, X)$-valued continuous functions on $H^{1+\varepsilon}(I, X)$, and hence it suffices to show (4.14) for $f\in H^{1,B}(I, X)\cap H^{1+\varepsilon}(I, X)$. For $\varphi\in C_0^{2,B}(I, X)$ we have

$$(4.18) \qquad \begin{aligned} &((R_0(z,\, H_n(z)f),\, (L_n-\bar{z})\varphi))_0 \\ &= ((R_0(z,\, H_n(z)f),\, (L_0-\bar{z}+C_n)\varphi))_0 \\ &= ((H_n(z)f,\, \varphi))_0+((C_nR_0(z,\, H_n(z)f),\, \varphi))_0 \\ &= ((G_n(z)H_n(z)f,\, \varphi))_0 \\ &= ((f,\, \varphi))_0, \end{aligned}$$

i.e., $R_0(z, H_n(z)f)$ is the radiative function for $\{L_n, \sqrt{z}, l[f], 0\}$. It follows from the uniqueness of the radiative function (Theorem 2.2 in [7]) that we have $R_0(z, H_n(z)f)=R_n(z, f)$. In a similar way we can prove (4.15).

Next let us prove (ii). For $z\in \boldsymbol{C}-\boldsymbol{R}$ and $f, g\in H^{1+\varepsilon}(I, X)\cap H^{1,B}(I, X)$ we have

$$(4.19) \qquad \begin{aligned} &((\{R_n(z)-R_n(\bar{z})\}G_n(z)f,\, G_n(z)g))_0 \\ &= ((R_n(z)G_n(z)f,\, G_n(z)g))_0-((G_n(z)f,\, R_n(z)G_n(z)g))_0 \\ &= ((R_0(z)f,\, \{1+C_nR_0(z)\}g))_0-((\{1+C_nR_0(z)\}f,\, R_0(z)g))_0 \\ &= ((\{R_0(z)-R_0(\bar{z})\}f,\, g))_0 \end{aligned}$$

Here we have made use of (4.6) and the relation $Q_{n,0}(z)f=C_nR_0(z)f$. Letting $z\to\lambda+i0$ in (4.19), we obtain

$$(4.20) \qquad e_n(\lambda;\, G_n^{+}(\lambda)f,\, G_n^{+}(\lambda)g) = e_0(\lambda;\, f,\, g).$$

The both sides of (4.20) are bounded forms on $H^{1+\varepsilon}(I, X)\times H^{1-\varepsilon}(I, X)$, and hence (4.20) holds for all $f, g\in H^{1+\varepsilon}(I, X)$. The other relations in (4.17) are obtained similarly. Q.E.D.

Finally we shall show some formulas involving $\mathcal{F}_n^{\pm}$, $\mathcal{F}_0^{\pm}$, $G_n^{\pm}(\lambda)$ and $H_n^{\pm}(\lambda)$.

Lemma 4.5. (i) *Let* $k>0$ *and* $n=1, 2, \cdots (n\neq\infty)$. *Let* $f\in C_0^0(\bar{I}, X)$. *Then we have*

$$(4.21) \qquad \mathcal{F}_0^{\pm}(H_n^{\pm}(k^2)f)(k) = \mathcal{F}_n^{\pm}f(k)$$

and

$$(4.22) \qquad \mathcal{F}_n^{\pm}(G_n^{\pm}(k^2)f)(k) = \mathcal{F}_0^{\pm}f(k).$$

(ii) *Let* $n=1, 2, \cdots, \infty$ *and let* Δ *be a Borel set in* $(0, \infty)$. *Let* $f, g\in H^{1+\varepsilon}(I, X)$. *Then we have*

$$((\mathcal{F}_n^{\pm}\chi_{\sqrt{\Delta}}\mathcal{F}_0^{\pm}f, g))_0 = \int_{\Delta} e_n(\lambda; G_n^{\pm}(\lambda)f, g)d\lambda = \int_{\Delta} e_0(\lambda; f, H_n^{\pm}(\lambda)g)d\lambda, \tag{4.23}$$

where $\chi_{\sqrt{\Delta}}$ *is the characteristic function of* $\sqrt{\Delta}=\{k>0/k^2\in\Delta\}$.

Proof. First let us prove (i). We prove (4.21) only, since (4.22) can be shown similarly. Note that

$$R_n(z, g) = \int_I G_n(r, s, \sqrt{z})g(s)ds \qquad (n = 0, 1, 2, \cdots, \infty,\ z\in \boldsymbol{C}_1) \tag{4.24}$$

hold for $g\in H^0(I, X)$ with a compact carrier in $\bar{I}$, and that the carrier of $H_n^{\pm}(\lambda)f$ is compact in $\bar{I}$ for $n=1, 2, \cdots$, and $f\in C_0^0(\bar{I}, X)$, as can be seen from the fact that $C_n(r)=0$ for $r\geqq n+1$. Then from the definition of $\mathcal{F}_n^{\pm}$ and (4.16) of Lemma 4.4 we see

$$\begin{aligned} \mathcal{F}_n^{\pm}f(k) &= \int_I \Gamma_n^{\pm}(r, k)f(r)dr \\ &= \pm\sqrt{\frac{2}{\pi}}ik \lim_{t\to\infty}[e^{\pm itk}\int_I G_n(t, r, \pm k)f(r)dr] \\ &= \pm\sqrt{\frac{2}{\pi}}ik \lim_{t\to\infty}[e^{\pm itk}R_n(t, k^2\pm i0, f)] \\ &= \pm\sqrt{\frac{2}{\pi}}ik \lim_{t\to\infty}[e^{\pm itk}R_0(t, k^2\pm i0, H_n^{\pm}(k^2)f)] \\ &= \pm\sqrt{\frac{2}{\pi}}ik \lim_{t\to\infty}[e^{\pm itk}\int_I G_0(t, r, \pm k)(H_n^{\pm}(k^2)f)(r)dr] \\ &= \pm\sqrt{\frac{2}{\pi}}ik \int_I \Gamma_0(r, \pm k)(H_n^{\pm}(k^2)f)(r)dr \\ &= \mathcal{F}_0^{\pm}(H_n^{\pm}(k^2)f)(r). \end{aligned} \tag{4.25}$$

Next let us prove (ii). Obviously we may assume that $\Delta=(\lambda_1, \lambda_2)$, $0<\lambda_1<\lambda_2<\infty$. Since we obtain from (4.5) and Theorems 4.2 and 4.3 in [7] that for fixed $f, g\in H^{1+\varepsilon}(I, X)$ $e_n(\lambda; G_n(\lambda)f, g)$ is uniformly bounded for $\lambda\in\Delta$ and $n=1, 2, \cdots$, and $e_n(\lambda; G_n^{\pm}(\lambda)f, g)\to e_\infty(\lambda; G_\infty^{\pm}(\lambda)f, g)$ as $n\to\infty$, we may assume that $n\neq\infty$. Both sides of (4.23) are bounded bilinear forms on $H^{1+\varepsilon}(I, X)\times H^{1+\varepsilon}(I, X)$, and hence we may assume that $f, g\in C_0^0(\bar{I}, X)$. Thus it suffices to show (4.23) for $n\neq\infty$, $\Delta=(\lambda_1, \lambda_2)$, $0<\lambda_1<\lambda_2<\infty$, and $f, g\in C_0^0(\bar{I}, X)$.

Using (4.12) and the continuity of $((\{R_n(\lambda+i\mu)-R_n(\lambda-i\mu)\}(G_n^{\pm}(\lambda)f), g))_0$ on $\{(\lambda, \mu)/\lambda_1\leqq\lambda\leqq\lambda_2, 0\leqq\mu\leqq1\}$, we have

$$(4.26)\qquad \int_{\Delta} e_n(\lambda;\, G_n^{\pm}(\lambda)f,\, g)d\lambda$$
$$= \int_{\Delta} \frac{1}{2\pi i}((R_n(\lambda+i0,\, G_n^{\pm}(\lambda)f) - R_n(\lambda-i0,\, G_n^{\pm}(\lambda)f,\, g))_0 d\lambda$$
$$= \lim_{\mu\downarrow 0} \frac{1}{2\pi i}\int_{\Delta} I(\lambda,\, \mu)\, d\lambda,$$

where $I(\lambda, \mu) = ((\{R_n(\lambda+i\mu) - R_n(\lambda-i\mu)\}(G_n^{\pm}(\lambda)f),\, g))_0$. $I(\lambda, \mu)$ is calculated as follows:

$$(4.27)\qquad I(\lambda,\, \mu) = \int_0^{\infty} (\mathcal{F}_n^{\pm}\{R_n(\lambda+i\mu) - R_n(\lambda-i\mu)\}(G_n^{\pm}(\lambda)f)(k),\, \mathcal{F}_n^{\pm}g(k))dk$$
$$+ \sum_j ((\{R_n(\lambda+i\mu) - R_n(\lambda-i\mu)\}(G_n^{\pm}(\lambda)f),\, \varphi_{n,j}))_0((\varphi_{n,j},\, g))_0$$
$$= \int_0^{\infty} \frac{2i\mu}{(k^2-\lambda)^2+\mu^2}(\mathcal{F}_n^{\pm}(G_n^{\pm}(\lambda)f)(k),\, \mathcal{F}_n^{\pm}g(k))dk$$
$$+ \sum_j \frac{2i\mu}{(\lambda_{nj}-\lambda)^2+\mu^2}((G_n^{\pm}(\lambda)f,\, \varphi_{n,j}))_0((\varphi_{n,j},\, g))_0$$
$$= I_1(\lambda,\, \mu) + I_2(\lambda,\, \mu),$$

where $-\infty < \lambda_{n,1} \leqq \lambda_{n,2} \leqq \cdots \leqq \lambda_{n,j} \leqq \cdots \leqq 0$ are the eigenvalues of A_n and for each j $\varphi_{n,j}$ is the normalized eigenfunction of A_n associated with $\lambda_{n,j}$. Here we have made use of the relation $(\mathcal{F}_n^{\pm})^* \mathcal{F}_n^{\pm} f + \sum_j (f,\, \varphi_{n,j})\varphi_{n,j} = f$ for $f \in H^0(I, X)$ (Theorems 3.1 and 3.4 in §3). Since we can easily show that $I_2(\lambda, \mu) \to 0$, $\mu \to 0$, uniformly in $\lambda \in [\lambda_1, \lambda_2]$, we have

$$(4.28)\qquad \int_{\Delta} e_n(\lambda;\, G_n^{\pm}(\lambda)f,\, g)d\lambda$$
$$= \lim_{\mu\downarrow 0}\int_{\Delta} \frac{1}{\pi}\int_I \frac{\mu}{(k^2-\lambda)^2+\mu^2}(\mathcal{F}_n^{\pm}(G_n^{\pm}(\lambda)f)(k),\, \mathcal{F}_n^{\pm}g(k))dk d\lambda$$

Noting that

$$(4.29)\qquad \int_{\Delta}\int_I \frac{1}{\pi}\frac{\mu}{(k^2-\lambda)^2+\mu^2}\,|(\mathcal{F}_n^{\pm}(G_n^{\pm}(\lambda)f)(k),\, \mathcal{F}_n^{\pm}g(k)|\, dk d\lambda$$
$$\leqq \frac{1}{\pi\mu}\int_{\Delta} \|G_n^{\pm}(\lambda)f\|_0 d\lambda \|g\|_0 < \infty,$$

we can change the order of integration to obtain

$$(4.30)\qquad \int_{\Delta} e_n(\lambda;\, G_n^{\pm}(\lambda)f,\, g)d\lambda$$
$$= \int_0^{\infty} \lim_{\mu\downarrow 0}\int_{\Delta} \frac{1}{\pi}\frac{\mu}{(k^2-\lambda)^2+\mu^2}(\mathcal{F}_n^{\pm}(G_n^{\pm}(\lambda)f)(k),\, \mathcal{F}_n^{\pm}g(k))d\lambda dk$$
$$= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{F}_n^{\pm}(G_n^{\pm}(k^2)f)(k),\, \mathcal{F}_n^{\pm}g(k))dk$$

$$= \int_{\sqrt{\lambda_1}}^{\sqrt{\lambda_2}} (\mathcal{F}_n^{\pm} f(k), \mathcal{F}_n^{\pm} g(k)) dk = (((\mathcal{F}_n^{\pm})^* \chi_{\sqrt{\Delta}} \mathcal{F}_0^{\pm} f, g))_0,$$

where we have made use of (i) and the well-known relation[10] for a continuous function $h(\lambda)$

(4.31) $$\frac{1}{\pi} \lim_{\mu \downarrow 0} \int_{\alpha}^{\beta} \frac{\mu}{(a-\lambda)^2+\mu^2} h(\lambda) d\lambda = \begin{cases} 0, & \text{if } a \notin (\alpha, \beta) \\ h(a), & \text{if } a \in (\alpha, \alpha). \end{cases}$$

From (4.8) and (4.17) we can easily see that $e_n(\lambda; G_n^{\pm}(\lambda)f, g) = e_0(\lambda; f, H_n^{\pm}(\lambda)g)$. Thus we have proved (4.23) completely. Q.E.D.

5. The orthogonality of $\mathcal{F}^{\pm}$

The purpose of this section is to prove the orthogonality of the operators $\mathcal{F}^{\pm}$ which have been constructed in §3 under the assumption that $\mathcal{F}_0^{\pm}$ are orthogonal. By the orthogonality of $\mathcal{F}^{\pm}$ we mean the relation $\mathcal{F}^{\pm}(\mathcal{F}^{\pm})^* = 1$ on $L^2((0, \infty), X, dk)$ or, equivalently, that $\mathcal{F}^{\pm}$ transform $H^0(I, X)$ onto $L^2((0, \infty), X, dk)$. For this we shall make use of the spectral representation for the absolutely continuous part of a spectral measure with values in the set of orthogonal projections in a Hilbert space which has been given in Kato and Kuroda [6], §1.

Put $\mathfrak{X} = H^{1+\varepsilon}(I, X)$ and define the spectral form e for $E(\lambda)$, the resolution of the identity associated with A, by

(5.1) $$e(\lambda; f, g) = \frac{1}{2\pi i} ((R(\lambda + i0, f) - R(\lambda - i0, f), g))_0$$

for each $\lambda > 0$ as in §4, where $R(\lambda \pm i0, f)$ is the radiative function for $\{L, \pm\sqrt{\lambda}, \ell[f], 0\}$. Let $\mathfrak{N}(\lambda)$ be the set of all f with $e(\lambda; f) = e(\lambda; f, f) = 0$. Then the quotient space $\mathfrak{X}/\mathfrak{N}(\lambda)$ is a pre-Hilbert space with the inner product induced by $e(\lambda; \cdot, \cdot)$. Its completion is denoted by $\mathfrak{X}(\lambda)$ and the inner product and norm in $\mathfrak{X}(\lambda)$ are denoted by $(\ ,\ )_\lambda$ and $\|\ \ \|_\lambda$. We denote by $J(\lambda)$ the canonical map of $\mathfrak{X}$ onto $\mathfrak{X}/\mathfrak{N}(\lambda) \subset \mathfrak{X}(\lambda)$. Let Δ be a Borel set in $(0, \infty)$. Then a vector field $F = \{F(\lambda)\}_{\lambda \in \Delta} \in \mathfrak{X}(\Lambda) = \prod_{\lambda \in \Delta} \mathfrak{X}(\lambda)$ is said to be e-measurable if there is a sequence h_n of quasi-simple functions on Δ to $\mathfrak{X}$ such that

(5.2) $$\lim_{n \to \infty} \|F(\lambda) - J(\lambda) h_n(\lambda)\|_\lambda = 0 \qquad \text{for a.e. } \lambda \in \Delta.$$

Here we mean by a quasi-simple function h a function of the form (finite sum)

(5.3) $$h(\lambda) = \Sigma \alpha_k(\lambda) f_k,$$

where $f_k \in \mathfrak{X}$ and $\alpha_k(\lambda)$ is a measurable, bounded scalar function on Δ. $\mathcal{M}(\Delta)$

10) See, for example, Titchmarsh [8], p. 31.

is the set of all e-measurable elements $F\in\mathcal{X}(\Delta)$ such that $\|F\|^2_{\mathcal{M}(\Delta)}=\int_\Delta\|F(\lambda)\|_\lambda^2d\lambda<\infty$. $\mathcal{M}(\Delta)$ is known to be a Hilbert space with the inner product

$$(5.4)\qquad (F_1, F_2)_{\mathcal{M}(\Delta)}=\int_\Delta(F_1(\lambda), F_2(\lambda))_\lambda d\lambda$$

(Proposition 1.9 in Kato and Kuroda [6]). Then we see that there is a unitary map $\Pi(\Delta)$ from $E(\Delta)H^0(I, X)$ onto $\mathcal{M}(\Delta)$ which satisfies the following (a) and (b) (Theorem 1.11 in Kato and Kuroda [6]):

(a) We have

$$(5.5)\qquad \Pi(\Delta)\alpha(E)u=\alpha\Pi(\Delta)u,$$

for $u\in E(\Delta)H^0(I, X)$ and a measurable, bounded scalar function $\alpha(\ )$ on Δ, where

$$(5.6)\qquad \alpha(E)=\int_\Delta\alpha(\lambda)dE(\lambda).$$

(b) For each $f\in\mathcal{X}=H^{1+\varepsilon}(I, X)$ we have

$$(5.7)\qquad \Pi(\Delta)E(\Delta)f=\{J(\lambda)f\}.$$

Let $e_0(\lambda;\cdot,\cdot)$ be the spectral form for E_0 as in §4. In a similar way, starting with $e_0(\lambda,\cdot,\cdot)$ and $\mathcal{X}_0=H^{1+\varepsilon}(I, X)$, we define the null set $\mathcal{N}_0(\lambda)$, the quotient space $\mathcal{X}_0/\mathcal{N}_0(\lambda)$, the Hilbert space $\mathcal{X}_0(\lambda)$ with the inner product $(\ )_{\lambda,0}$, the canonical map $J_0(\lambda)$, the e_0-measurability, the Hilbert space $\mathcal{M}_0(\Delta)$ with the inner product $(\ ,\ )_{\mathfrak{M}_0(\Delta)}$ and the unitary map $\Pi_0(\Delta)$. Thus we have obtained spectral representations for $E(\lambda)$ and $E_0(\Delta)$. We denote by $\mathcal{M}'(\Delta)$ (or $\mathcal{M}_0'(\Delta)$) the set of all $F(\lambda)\in\mathcal{M}(\Delta)$ (or $\mathcal{M}_0(\Delta)$) of the foım $F(\lambda)=\{J(\lambda)h\}_{\lambda\in\Delta}$(or $F(\lambda)=\{J_0(\lambda)h\}_{\lambda\in\Delta}$), where h is a quasi-simple function. $\mathcal{M}'(\Delta)$ and $\mathcal{M}_0'(\Delta)$ are dense in $\mathcal{M}(\Delta)$ and $\mathcal{M}_0(\Delta)$, respectively (Proposition 1.10 in Kato and Kuroda [6]). We put $G^\pm(\lambda)=G^\pm_\infty(\lambda)$ and $H^\pm(\lambda)=H^\pm_\infty(\lambda)$, where $G^\pm_\infty(\lambda)$ and $H^\pm_\infty(\lambda)$ are as in §4. Define the operator $G^\pm(\Delta)$ on $\mathcal{M}_0'(\Delta)$ into $\mathcal{M}(\Delta)$ and the operator $H^\pm(\Delta)$ on $\mathcal{M}'(\Delta)$ into $\mathcal{M}_0(\Delta)$ as follows:

$$(5.8)\qquad \begin{cases} G^\pm(\Delta)\colon \{J_0(\lambda)h\}_{\lambda\in\Delta}\to\{J(\lambda)G^\pm(\lambda)h\}_{\lambda\in\Delta} \\ H^\pm(\Delta)\colon \{J(\lambda)h\}_{\lambda\in\Delta}\to\{J_0(\lambda)H^\pm(\lambda)h\}_{\lambda\in\Delta}. \end{cases}$$

From (ii) of Lemma 4.4 we obtain

$$(5.9)\qquad \begin{cases} \|J(\lambda)G^\pm(\lambda)h\|_\lambda^2=e(\lambda; G^\pm(\lambda)h)=e_0(\lambda; h)=\|J_0(\lambda)h\|_{\lambda,0}^2, \\ \|J_0(\lambda)H^\pm(\lambda)h\|_{\lambda,0}^2=e_0(\lambda; H^\pm(\lambda)h)=e(\lambda; h)=\|J(\lambda)h\|_\lambda^2, \end{cases}$$

and hence we have

$$(5.10)\qquad \begin{cases} \|G^{\pm}(\Delta)J_0(\cdot)h\|_{\mathcal{M}(\Delta)} = \|J_0(\cdot)h\|_{\mathcal{M}_0(\Delta)}, \\ \|H^{\pm}(\Delta)J(\cdot)h\|_{\mathcal{M}_0(\Delta)} = \|J(\cdot)h\|_{\mathcal{M}(\Delta)}. \end{cases}$$

DEFINITION 5.1. We define the operators $\tilde{G}^{\pm}(\Delta)$ and $\tilde{H}^{\pm}(\Delta)$ as unique extensions $G^{\pm}(\Delta)$ and $H^{\pm}(\Delta)$, respectively. $\tilde{G}^{\pm}(\Delta)$ is an isometric operator on $\mathcal{M}_0(\Delta)$ into $\mathcal{M}(\Delta)$. $\tilde{H}^{\pm}(\Delta)$ is an isometric operator on $\mathcal{M}(\Delta)$ into $\mathcal{M}_0(\Delta)$.

We shall show that $\tilde{G}^{\pm}(\Delta)$ and $\tilde{H}^{\pm}(\Delta)$ are really unitary operators.

Lemma 5.2. *Let Δ be a Borel set in $(0, \infty)$. Then $\tilde{G}^{\pm}(\Delta)$ is a unitary operator from $\mathcal{M}_0(\Delta)$ onto $\mathcal{M}(\Delta)$ and $\tilde{H}^{\pm}(\Delta)$ is a unitary operator from $\mathcal{M}(\Delta)$ onto $\mathcal{M}_0(\Delta)$. Further, we have*

$$(5.11)\qquad \begin{cases} \tilde{G}^{\pm}(\Delta)\tilde{H}^{\pm}(\Delta) = 1 & \text{on } \mathcal{M}(\Delta), \\ \tilde{H}^{\pm}(\Delta)\tilde{G}^{\pm}(\Delta) = 1 & \text{on } \mathcal{M}_0(\Delta). \end{cases}$$

Proof. It follows from (iii) of Lemma 4.3 that

$$(5.12)\qquad \begin{aligned} G^{\pm}(\Delta)H^{\pm}(\Delta)\{J(\lambda)h\}_{\lambda\in\Delta} &= G^{\pm}(\Delta)\{J_0(\lambda)H^{\pm}(\lambda)h\}_{\lambda\in\Delta} \\ &= \{J(\lambda)G^{\pm}(\lambda)H^{\pm}(\lambda)h\}_{\lambda\in\Delta} = \{J(\lambda)h\}_{\lambda\in\Delta} \end{aligned}$$

holds for any quasi-simple function h and for any $\lambda > 0$. Similarly we have

$$(5.13)\qquad H^{\pm}(\Delta)G^{\pm}(\Delta)\{J_0(\lambda)h\}_{\lambda\in\Delta} = \{J_0(\lambda)h\}_{\lambda\in\Delta}.$$

(5.11) is obtained from (5.12) and (5.13). Hence $\tilde{G}^{\pm}(\Delta)$ and $\tilde{H}^{\pm}(\Delta)$ are unitary operators. Q.E.D.

Lemma 5.3. *Let Δ be a Borel set in $(0, \infty)$ and let $\mathcal{M}_0(\Delta)$, $\mathcal{M}(\Delta)$, $\Pi_0(\Delta)$, $\Pi(\Delta)$ be as above. Then for $f, g \in H^0(I, X)$ we have*

$$(5.14)\qquad \begin{aligned} (((\mathcal{F}^{\pm})^*\chi_{\sqrt{\Delta}}\mathcal{F}_0^{\pm}f, g))_0 &= (\tilde{G}^{\pm}(\Delta)\Pi_0(\Delta)E_0(\Delta)f, \Pi(\Delta)E(\Delta)g)_{\mathcal{M}(\Delta)} \\ &= (\Pi_0(\Delta)E_0(\Delta)f, \tilde{H}^{\pm}(\Delta)\Pi(\Delta)E(\Delta)g)_{\mathcal{M}_0(\Delta)}. \end{aligned}$$

where $\chi_{\sqrt{\Delta}}$ is the characteristic function of $\sqrt{\Delta} = \{k > 0 / k^2 \in \Delta\}$, and $\mathcal{F}^{\pm}$ and $\mathcal{F}_0^{\pm}$ are as above.

Proof. Let f_m and g_m be sequences such that $f_m, g_m \in H^{1+\varepsilon}(I, X)$ and, $f_m \to f$, $g_m \to g$ in $H^0(I, X)$. Then it follows from Lemma 4.5 that

$$(5.15)\qquad \begin{aligned} (((\mathcal{F}^{\pm})^*\chi_{\sqrt{\Delta}}\mathcal{F}_0^{\pm}f_m, g_m))_0 &= \int_\Delta e(\lambda; G^{\pm}(\lambda)f_m, g_m)d\lambda \\ &= \int_\Delta (J(\lambda)G^{\pm}(\lambda)f_m, J(\lambda)g_m)_\lambda d\lambda \\ &= (G^{\pm}(\Delta)\{J_0(\lambda)f_m\}_{\lambda\in\Delta}, \{J(\lambda)g_m\}_{\lambda\in\Delta})_{\mathcal{M}(\Delta)} \\ &= (\tilde{G}^{\pm}(\Delta)\Pi_0(\Delta)E_0(\Delta)f_m, \Pi(\Delta)E(\Delta)g_m)_{\mathcal{M}(\Delta)}, \end{aligned}$$

where we have made use of (5.7). Let $m\to\infty$ in (5.15). Then the left-hand side of (5.15) tends to $(((\mathcal{F}^{\pm})^{*}\chi_{\sqrt{\Delta}}\mathcal{F}_{0}^{\pm}f, g))_{0}$ and the right-hand side tends to $(\tilde{G}^{\pm}(\Delta)\Pi_0(\Delta)E_0(\Delta)f, \Pi(\Delta)E(\Delta)g)_{\mathcal{M}(\Delta)}$. From the unitarity of the mappings $\tilde{G}^{\pm}(\Delta)$ and $\tilde{H}^{\pm}(\Delta)$ we obtain $(\tilde{G}^{\pm}(\Delta)\Pi_0(\Delta)E_0(\Delta)f, \Pi(\Delta)E(\Delta)g)_{\mathcal{M}(\Delta)}=(\Pi_0(\Delta)E_0(\Delta)f, \tilde{H}^{\pm}(\Delta)\Pi(\Delta)E(\Delta)g)_{\mathcal{M}_0(\Delta)}$. Q.E.D.

Now we can show the orthogonality of $\mathcal{F}^{\pm}$ assuming the orthogonality of $\mathcal{F}_0^{\pm}$.

Theorem 5.4. *Let us assume Assumption* 1.1. *Let $\mathcal{F}^{\pm}$ and $\mathcal{F}_0^{\pm}$ be as above. Suppose that $\mathcal{F}_0^{+}$ is orthogonal. Then $\mathcal{F}^{+}$ is also orthogonal. Similar results hold for $\mathcal{F}^{-}$.*

Proof. Let $\Delta=(0, \infty)$ in Lemma 5.4. We put $\tilde{G}^{+}((0, \infty))=\tilde{G}^{\pm}$, $\tilde{H}^{\pm}((0, \infty))=\tilde{H}^{\pm}$, $\mathcal{M}((0, \infty))=\mathcal{M}$, $\mathcal{M}_0((0, \infty))=\mathcal{M}_0$, $\Pi((0, \infty))=\Pi$, $\Pi_0((0, \infty))=\Pi_0$ etc. It suffices to show that $(\mathcal{F}^{+})^{*}\mathcal{F}^{+}f_0=0$ and $f_0\in E_0(0, \infty)H^0(I, X)$ imply $f_0=0$ in $H^0(I, X)$, because $\mathcal{F}_0^{+}$ transforms $H^0(I, X)$ onto $L_2((0, \infty); X, dk)$ and $\mathcal{F}_0^{+}E_0(0, \infty)f=\mathcal{F}_0^{+}f$. It follows from Lemma 5.4 that

$$(\tilde{G}^{\pm}\Pi_0 f_0, \Pi g)_{\mathcal{M}} = 0 \tag{5.16}$$

for any $g\in E((0, \infty))H^0(I, X)$. Since $\Pi(E((0, \infty))H^0(I, X))=\mathcal{M}$ we have $\tilde{G}^{\pm}\Pi_0 f_0=0$. Hence we obtain from the unitarity of $\tilde{G}^{\pm}$ $\Pi_0 f_0=0$ in $\mathcal{M}_0$, which implies $f_0=0$ in $H^0(I, X)$ by the unitarity of Π_0. Q.E.D.

6. The wave operators $W^{\pm}(\Delta)$

In this section we shall investigate the wave operators $W^{\pm}(\Delta)$ for A which will be defined according to Kato and Kuroda [6][11]. We shall see the ranges of $W^{\pm}(\Delta)$ to be equal to $E(\Delta)H^0(I, X)$. We also discuss the invariance of the wave operators $W^{\pm}(\Delta)$.

To make use of the results of Kato and Kuroda [6], §5 and §7, we consider the Cayley transforms U and U_0 of A and A_0, respectively. Set

$$\begin{cases} U = (A-i)(A+i)^{-1} = \int_0^{2\pi} e^{i\theta}F(d\theta), \\ U_0=(A_0-i)(A_0+i)^{-1} = \int_0^{2\pi} e^{i\theta}F_0(d\theta), \end{cases} \tag{6.1}$$

where $F(\theta)$ and $F_0(\theta)$ are the resolutions of the identity on $(0, 2\pi)$ associated with U and U_0, and we have

11) In this section we assume Assumption 1.1 only. We do not assume the orthogonality of $\mathcal{F}_0^{\pm}$.

$$(6.2)\qquad \begin{cases} F(\theta) = E(\lambda(\theta)), & \lambda(\theta) = \dfrac{-\sin\theta}{1-\cos\theta}, \\ F_0(\theta) = E_0(\lambda(\theta)), \end{cases}$$

$E(\lambda)$ and $E_0(\lambda)$ being the resolutions of the identity on $(-\infty, \infty)$ associated with A and A_0. Let us define $R_U(\zeta)$ and $R_{U_0}(\zeta)$, $\zeta\in \boldsymbol{C}$, $|\zeta|\neq 1$ by

$$(6.3)\qquad \begin{cases} R_U(\zeta) = (1-\zeta U^*)^{-1} = U(U-\zeta)^{-1}, \\ R_{U0}(\zeta) = (1-\zeta U_0^*)^{-1} = U_0(U_0-\zeta)^{-1}. \end{cases}$$

Then we have by simple calculations

$$(6.4)\qquad \begin{cases} R_U(\zeta) = \dfrac{1}{1-\zeta}\left\{1+\dfrac{2i\zeta}{1-\zeta}R\left(\dfrac{1+\zeta}{1-\zeta}i\right)\right\} \\ R_{U_0}(\zeta) = \dfrac{1}{1-\zeta}\left\{1+\dfrac{2i\zeta}{1-\zeta}R_0\left\{\dfrac{1+\zeta}{1-\zeta}i\right\}\right\}, \end{cases}$$

and hence if we write $\zeta=re^{i\theta}$, $\zeta'=r^{-1}e^{i\theta}$, we obtain for $f\in H^{1+\varepsilon}(I, X)$

$$(6.5)\qquad \begin{cases} \lim\limits_{r\uparrow 1} R_U(\zeta)f = \dfrac{1}{1-e^{i\theta}}f+\dfrac{1}{i(1-\cos\theta)}R(\lambda(\theta)+i0, f) \\ \lim\limits_{r\uparrow 1} R_U(\zeta')f = \dfrac{1}{1-e^{i\theta}}f+\dfrac{1}{i(1-\cos\theta)}R(\lambda(\theta)-i0, f) \end{cases}$$

in $H^{-1-\varepsilon}(I, X)$. We obtain quite similar relations for $R_{U_0}(\zeta)$ and $R_{U_0}(\zeta')$. Since we have

$$(6.6)\qquad \begin{cases} \lim\limits_{r\uparrow 1} \dfrac{1}{2\pi}(([R_U(\zeta)-R_U(\zeta')]u, v))_0 = \dfrac{d}{d\theta}((F(\theta)u, v))_0, \\ \lim\limits_{r\uparrow 1} \dfrac{1}{2\pi}(([R_{U_0}(\zeta)-R_{U_0}(\zeta')]u, v))_0 = \dfrac{d}{d\theta}((F_0(\theta)u, v))_0 \end{cases}$$

for $u, v\in H^0(I, X)$ and a.e. $\theta\in(\pi, 2\pi)$[12] ((5.4) in Kato and Kuroda [6]), the bilinear forms e_U and e_{U_0} on $H^{1+\varepsilon}(I, X)\times H^{1+\varepsilon}(I, X)$ defined by

$$(6.7)\qquad \begin{cases} e_U(\theta; f, g) = \dfrac{1}{2\pi}\lim\limits_{r\uparrow 1}(([R_U(\zeta)-R_U(\zeta')]f, g))_0 \\ \qquad = \dfrac{1}{1-\cos\theta}e(\lambda(\theta); f, g) \\ e_{U0}(\theta; f, g) = \dfrac{1}{2\pi}\lim\limits_{r\uparrow 1}(([R_{U_0}(\zeta)-R_{U_0}(\zeta')]f, g))_0 \\ \qquad = \dfrac{1}{1-\cos\theta}e_0(\lambda(\theta); f, g) \end{cases}$$

12) Note that $\lambda(\theta)$ maps $(\pi, 2\pi)$ onto $(0, \infty)$.

are the spectral forms for $F(\theta)$ and $F_0(\theta)$ for each $\theta\in(\pi, 2\pi)$, respectively. Put $\mathfrak{X}_U=\mathfrak{X}_{U_0}=H^{1+\varepsilon}(I, X)$. Then we construct, starting with e_U(or e_{U_0}), the Hilbert space $\mathfrak{X}_U(\theta)$ (or $\mathfrak{X}_{U_0}(\theta)$) with the inner product $(\ ,\ )_{\theta,U}$ (or $(\ ,\ )_{\theta,U_0}$), the Hilbert space $\mathcal{M}_U(\Gamma)$ (or $\mathcal{M}_{U_0}(\Gamma)$) for a Borel set Γ in $(\pi, 2\pi)$ with the inner product $(\ ,\ )_{\mathcal{M}_U(\Gamma)}$ (or $(\ ,\ )_{\mathcal{M}_{U_0}(\Gamma)}$) and the unitary map $\Pi_U(\Gamma)$ (or $\Pi_{U_0}(\Gamma)$) as in §5.

For each $\zeta\in\boldsymbol{C}$, $|\zeta|\neq1$, we put

$$(6.8)\qquad \begin{cases} G_U(\zeta) = R_U(\zeta)^{-1}R_{U_0}(\zeta) \\ H_U(\zeta) = R_{U_0}(\zeta)^{-1}R_U(\zeta). \end{cases}$$

Then by the definition of $R_U(\zeta)$ and $R_{U_0}(\zeta)$ we have

$$(6.9)\qquad \begin{cases} G_U(\zeta) = (1-\zeta U^*)(1-\zeta U_0^*)^{-1} = 1-\zeta(U^*-U_0^*)R_{U_0}(\zeta) \\ \qquad\quad = 1+\dfrac{2i\zeta}{1-\zeta}R(i)CR_0\Big(\dfrac{1+\zeta}{1-\zeta}i\Big) \\ H_U(\zeta) = 1-\dfrac{2i\zeta}{1-\zeta}R_0(i)CR\Big(\dfrac{1+\zeta}{1-\zeta}i\Big), \end{cases}$$

where $R(z)=(A-z)^{-1}$, $R_0(z)=(A_0-z)^{-1}$. We define the operators $G_U^{\pm}(\theta)$ and $H_U^{\pm}(\theta)$, $\theta\in(\pi, 2\pi)$ by

$$(6.10)\qquad \begin{cases} G_U^{\pm}(\theta)f = f+\dfrac{2ie^{i\theta}}{1-e^{i\theta}}R(i)CR_0(\lambda(\theta)\pm i0, f) \\ H_U^{\pm}(\theta)f = f-\dfrac{2ie^{i\theta}}{1-e^{i\theta}}R_0(i)CR(\lambda(\theta)\pm i0, f), \end{cases}$$

which transform $H^{1+\varepsilon}(I, X)$ onto itself. In fact we can see from (6.8) that the relations

$$(6.11)\qquad G_U^{\pm}(\theta)H_U^{\pm}(\theta) = H_U^{\pm}(\theta)G_U^{\pm}(\theta) = 1$$

hold on $H^{1+\varepsilon}(I, X)$. We can also see from (6.8) that we have the relations

$$(6.12)\qquad \begin{cases} e_U(\theta;\ G_U^{\pm}(\theta)f,\ G_U^{\pm}(\theta)g) = e_{U_0}(\theta; f, g) \\ e_{U_0}(\theta;\ H_U^{\pm}(\theta)f,\ H_U^{\pm}(\theta)g) = e_U(\theta; f, g) \end{cases}$$

for $\theta\in(\pi, 2\pi)$ and $f, g\in H^{1+\varepsilon}(I, X)$. Thus, as in Definition 5.1, the unitary operators $\tilde{G}_U^{\pm}(\Gamma)$ and $\tilde{H}_U^{\pm}(\Gamma)$ are induced by $G_U^{\pm}(\theta)$ and $H_U^{\pm}(\theta)$, respectively, i.e., $\tilde{G}_U^{\pm}(\Gamma)$ (or $\tilde{H}_U^{\pm}(\Gamma)$) are defined by unique extensions of the operators

$$(6.13)\qquad \begin{cases} G_U^{\pm}(\Gamma):\ \{J_{U_0}(\theta)f\}_{\theta\in\Gamma}\to\{J_U(\theta)G_U^{\pm}(\theta)f\}_{\theta\in\Gamma} \\ (\text{or } H_U^{\pm}(\Gamma):\ \{J_U(\theta)f\}_{\theta\in\Gamma}\to\{J_{U_0}(\theta)H_U^{\pm}(\theta)f\}_{\theta\in\Gamma}), \end{cases}$$

$J_U(\theta)$ (or $J_{U_0}(\theta)$) being the canonical map on $\mathfrak{X}_U$ (or $\mathfrak{X}_{U_0}$) into $\mathfrak{X}_U(\theta)$ (or $\mathfrak{X}_{U_0}(\theta)$). $\tilde{G}_U^{\pm}(\Gamma)$ transform $\mathcal{M}_{U_0}(\Gamma)$ onto $\mathcal{M}_U(\Gamma)$ and $\tilde{H}_U^{\pm}(\Gamma)$ transform $\mathcal{M}_U(\Gamma)$

onto $\mathcal{M}_{U_0}(\Gamma)$.

DEFINITION 6.1 Let Δ be a Borel set in $(0, \infty)$. Put $\Gamma(\Delta)=\{\theta\in(\pi, 2\pi)/\lambda(\theta)\in\Delta\}$. Then the *wave operators* $W^{\pm}(\Delta)$ on $H^0(I, X)$ are defined by

$$W^{\pm}(\Delta)u = \Pi_U^{-1}(\Gamma(\Delta))\tilde{G}_U^{\pm}(\Gamma(\Delta))\Pi_{U_0}(\Gamma(\Delta))F_0(\Gamma(\Delta))u, \tag{6.14}$$

where $\tilde{G}_U^{\pm}(\Gamma)$ are defined as above.

Using the unitarity of $\tilde{G}^{\pm}(\Gamma)$ and $\tilde{H}^{\pm}(\Gamma)$ we can show

Theorem 6.2. *Let us assume Assumption* 1.1. *Let $W^{\pm}(\Delta)$ be as above. Then $W^{\pm}(\Delta)$ are partial isometries with initial set $E_0(\Delta)H^0(I, X)$ and final set $E(\Delta)H^0(I, X)$. In particular $W^{\pm}(I)$ are complete. $W^{\pm}(\Delta)$ have the intertwining property $\alpha(E)W^{\pm}(\Delta)=W^{\pm}(\Delta)\alpha(E_0)$ for any bounded measurable function α on Δ.*

The proof is almost the same as the proofs of Theorems 2.3 and 2.4 in Kato and Kuroda [6], and hence we omit it.

Now we turn to the problem of the invariance of $W^{\pm}(\Delta)$. Since $\mathcal{X}=\mathcal{X}_0=H^{1+\varepsilon}(I, X)$ is a Hilbert space, we can apply Theorem 7.1 in Kato and Kuroda [6] to our case.

Theorem 6.3. *Let us assume Assumption* 1.1. *Let α be a real-valued function on $(0, \infty)$ such that*

$$\lim_{t\to\infty}\sum_{k=0}^{\infty}\left|\int_0^{2\pi}e^{-ik\theta-it\alpha(\lambda(\theta))}\eta(\theta)d\theta\right|^2 = 0 \tag{6.15}$$

for every $\eta\in L^2(0, 2\pi)$. Then we have

$$\operatorname*{s-lim}_{t\to\pm\infty} e^{it\alpha(E)}e^{-it\alpha(E_0)}E_0(\Delta)u = W^{\pm}(\Delta)u \tag{6.16}$$

for all $u\in H^0(I, X)$, where $\lambda(\theta)=-\sin\theta/(1-\cos\theta)$ as given in (6.2). *In particular, we have*

$$W^{\pm}(\Delta) = \operatorname*{s-lim}_{t\to\pm\infty} e^{itA}e^{-itA_0}E_0(\Delta) \tag{6.17}$$

on $H^0(I, X)$.

Finally we represent $W^{\pm}(\Delta)$ using $\mathcal{F}^{\pm}$ and $\mathcal{F}_0^{\pm}$

Theorem 6.4. *Let us assume Assumption* 1.1. *Let Δ be a Borel set in $(0, \infty)$. Then we have on $H^0(I, X)$*

$$(\mathcal{F}^{\pm})^*\chi_{\sqrt{\Delta}}\mathcal{F}_0^{\pm} = W^{\pm}(\Delta). \tag{6.18}$$

For the proof we need

Lemma 6.5. *Let $G_U^{\pm}(\theta)$, $\theta\in(\pi, 2\pi)$ be as in* (6.10) *and let* $G^{\pm}(\lambda)$, $\lambda>0$ *be as in Definition* 4.2. *Then we have for* $f\in C_{0,}^{2,B}(I, X)$

$$G_U^{\pm}(\theta)f = (A-i)^{-1}G^{\pm}(\lambda(\theta))(A_0-i)f. \tag{6.19}$$

Proof. We have by an easy computation

$$\begin{aligned} G_U(\zeta)f &= U^{-1}(U-\zeta)(U_0-\zeta)^{-1}U_0f \\ &= (A-i)^{-1}G\Big(\frac{1+\zeta}{1-\zeta}i\Big)(A_0-i)f, \end{aligned} \tag{6.20}$$

where $\zeta=re^{i\theta}$, $r\neq 1$. Since $(A_0-i)f\in H^{1+\varepsilon}(I, X)$, we obtain (6.19) from (6.20), letting $r\uparrow 1$ and $r\downarrow 1$. Q.E.D.

Proof of Theorem 6.4. Let $f, g\in C_0^{2,B}(I, X)$. Then it follows from Lemma 5.3 that

$$\begin{aligned} &(((\mathcal{F}^{\pm})^*\chi_{\sqrt{\Delta}}\mathcal{F}_0^{\pm}f, g))_0 \\ &= (((\mathcal{F}^{\pm})^*\chi_{\sqrt{\Delta}}\mathcal{F}_0^{\pm}(A_0-i)f, (A+i)^{-1}g))_0 \\ &= (\tilde{G}^{\pm}(\Delta)\Pi_0(\Delta)E_0(\Delta)(A_0-i)f, \Pi(\Delta)E(\Delta)(A+i)^{-1}g)_{\mathcal{M}(\Delta)} \\ &= \int_\Delta e(\lambda; G^{\pm}(\lambda)(A_0-i)f, (A+i)^{-1}g)d\lambda \\ &= \int_\Gamma e(\lambda(\theta); G^{\pm}(\lambda(\theta))(A_0-i)f, (A+i)^{-1}g)\frac{1}{1\text{-}\cos\theta}d\theta \qquad (\Gamma=\Gamma(\Delta)) \\ &= \int_\Gamma e_U(\theta; G^{\pm}(\lambda(\theta))(A_0-i)f, (A+i)^{-1}g)d\theta, \end{aligned} \tag{6.21}$$

where we have made use of $\dfrac{d\lambda}{d\theta}=\dfrac{1}{1\text{-}\cos\theta}$ and (6.7). On the other hand we see from (6.7) and Lemma 6.5

$$\begin{aligned} &e_U(\theta; G^{\pm}(\lambda(\theta))(A_0-i)f, (A+i)^{-1}g) \\ &= \frac{1}{2\pi}\lim_{r\uparrow 1}(([R_U(\zeta)-R_U(\zeta')]G^{\pm}(\lambda(\theta))(A_0-i)f, (A+i)^{-1}g))_0 \\ &= \frac{1}{2\pi}\lim_{r\uparrow 2}(([R_U(\zeta)-R_U(\zeta')](A-i)^{-1}G^{\pm}(\lambda(\theta))(A_0-i)f, g))_0 \\ &= \frac{1}{2\pi}\lim_{r\uparrow 1}(([R_U(\zeta)-R_U(\zeta')]G_U^{\pm}(\theta)f, g))_0 \\ &= e_U(\theta; G_U^{\pm}(\theta)f, g). \end{aligned} \tag{6.22}$$

(6.21) and (6.22) are cooperated to give

$$\begin{aligned} &(((\mathcal{F}^{\pm})^*\chi_{\sqrt{\Delta}}\mathcal{F}_0^{\pm}f, g))_0 \\ &= \int_\Gamma e_U(\theta; G_U^{\pm}(\theta)f, g)d\theta \end{aligned} \tag{6.23}$$

$$
\begin{aligned}
&= \int_\Gamma (G_U^\pm(\Gamma) J_{U_0}(\theta) f, J_U(\theta) g)_{\lambda, U} d\theta \\
&= (\tilde{G}_U^\pm(\Gamma) \Pi_{U_0}(\Gamma) F_0(\Gamma) f, \Pi_U(\Gamma) F(\Gamma) g)_{\mathcal{M}_U}(\Gamma) \\
&= ((F(\Gamma) \Pi_U^{-1}(\Gamma) \tilde{G}_U^\pm(\Gamma) \Pi_{U_0}(\Gamma) F_0(\Gamma) f, g))_0 \\
&= ((W^\pm(\Delta) f, g))_0.
\end{aligned}
$$

Here we used (6.13) and (6.14). Thus we have proved (6.23) on $C_0^{2,B}(I, X)$. Since both $(\mathcal{F}^\pm)^* \chi_{\sqrt{\Delta}} \mathcal{F}_0^\pm$ and $W^\pm(\Delta)$ are bounded operators, (6.23) holds on the whole $H^0(I, X)$. Q.E.D.

7. The Schrödinger operator in R^n ($n \geqq 3$)

The results obtained in the preceding sections can be applied to the Schrödinger operator in the whole space R^n, $n \geqq 3$.

We set in this section

$$X = L^2(S^{n-1}), \tag{7.1}$$

where S^{n-1} is the $(n-1)$-sphere. Then there is a unitary operator V from $L^2(R^n)$ onto $H^0(I, X)$ defined by

$$V : L^2(R^n) \ni F(y) \to r^{(n-1)/2} F(r\omega) \in H^0(I, X), \tag{7.2}$$

where $y \in R^n$, $r = |y|$ and $\omega = \dfrac{y}{r} \in S^{n-1}$.

Let us consider the Laplace operator in R^n

$$-\Delta = -\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}. \tag{7.3}$$

We obtain

$$V(-\Delta)V^{-1} = -\frac{d^2}{dr^2} + \frac{1}{r^2}\left(-\Lambda_n + \frac{(n-1)(n-3)}{4}\right), \tag{7.4}$$

where Λ_n is the Laplace-Beltrami operator on S^{n-1}. We put

$$\begin{cases} B(r) = \dfrac{1}{r^2}\left(-\Lambda_n + \dfrac{(n-1)(n-3)}{4}\right), \\ \mathcal{D}(B(r)) = \mathcal{D}(\Lambda_n) = D. \end{cases} \tag{7.5}$$

Then it follows from (7.4) that we have for $\varphi \in C_0^{2,B}(I, X)$

$$||\varphi||_B = ||V^{-1}\varphi||_1, \tag{7.6}$$

$||\ \ ||_1$ being the norm of $\mathcal{D}^1_{L^2}(R^n)$, i.e.,

$$(7.7)\qquad ||F||_1 = \int_{R^n}\left\{\sum_{j=1}^{n}\left|\left|\frac{\partial F}{\partial x_j}\right|\right|^2 + |F(x)|^2\right\}dx.^{13)}$$

Hence (7.6) holds on $H_0^{1,B}(I, X)$. It is easy to see that $B(r)$ satisfies (B.1)~(B.4) of Assumption 1.1. (B.4) is implied by the compactness of the imbedding $\mathcal{D}^1_{L^2}(\Omega)\to L^2(\Omega)$ for any bounded domain Ω in $\boldsymbol{R}^n$. As is well known, the spectrum of the non-negative self-adjoint operator $-\Lambda_n$ is discrete. Let λ_0 be an eigenvalue of $-\Lambda_n$ and let $x_0=x_0(\omega)$ be a normalized eigenfunction associated with λ_0. Then we have

$$(7.8)\qquad B(r)x_0 = \frac{1}{r^2}\left\{\lambda_0+\frac{(n-1)(n-3)}{4}\right\}x_0\to 0 \qquad (r\to\infty)$$

in $L^2(S^{n-1})$. Thus we have seen that the condition (3.1) is satisfied.

Define a symmetric operator T_0 by

$$(7.9)\qquad \begin{cases} \mathcal{D}(T_0) = C_0^\infty(\boldsymbol{R}^n), \\ T_0F = -\Delta F. \end{cases}$$

It is well-known[14] that T_0 is essentially self-adjoint with a unique extension H_0.

Lemma 7.1. *Let H_0 be as above and let A_0 be as in Definition* 1.1. *Then we have $VH_0V^{-1}=A_0$.*

Proof. Noting that $\mathcal{D}(VT_0V^{-1})=VC_0^\infty(\boldsymbol{R}^n)\subset\mathfrak{A}_0$, we can easily see that $M_0\supset VT_0V^{-1}$, where M_0 is as given in (1.9). Since $\tilde{T}_0=H_0$[15], VT_0V^{-1} is essentially self-adjoint in $H^0(I, X)$. Therefore M_0 is essentially self-adjoint in $H^0(I, X)$. Thus we obtain

$$(7.10)\qquad VH_0V^{-1} = V\tilde{T}_0V^{-1} = \widetilde{VT_0V^{-1}} = \tilde{M}_0 = A_0,$$

which completes the proof. Q.E.D.

Denote by $q(y)$ a real-valued function on $\boldsymbol{R}^n$. $q(y)$ is assumed to satisfy the following conditions:

(Q) *$q(y)$ is a real-valued function which is continuously differentiable. Further, $q(y)$ behaves like $O(|y|^{-1-\varepsilon})(0<\varepsilon<1)$ at infinity, i.e., there exist $\rho>0$, $C>0$ and $0<\varepsilon<1$ such that*

$$(7.11)\qquad |q(y)|\leqq c|y|^{-1-\varepsilon} \qquad (|y|\geqq\rho).$$

We define a symmetric operator T by

13) As usual $\mathcal{D}^1_{L^2}(\boldsymbol{R}^n)$ denotes the Hilbert space obtained by the completion of $C_0^\infty(\boldsymbol{R}^n)$ in the norm $||\ \ ||_1$.

14) See Kato [4].

15) $\tilde{T}$ means the closure of T.

(7.12)
$$\begin{cases} \mathcal{D}(T) = C_0^\infty(\boldsymbol{R}^n) \\ TF = -\Delta F + q(y)F = T_0 F + q(y)F \end{cases}$$

Since $q(y)\times$ is a bounded, linear operator and T_0 is essentially self-adjoint, T is also essentially self-adjoint[16]. We put $\tilde{T}=H$. On the other hand we define an operatorvalued function $C(r)$ on I by

(7.13)
$$\begin{cases} \mathcal{D}(C(r)) = C, \\ C(r)f = q(r\omega)f(r). \end{cases}$$

We can easily see from (Q) that $C(r)$ satisfies (C–1)∼(C–3) in Assumption 1.1. Let M be as in (1.10). Then, proceeding as in the proof of Lemma 7.1, we have the following

Lemma 7.2. *Let A be as in Definition* 1.2. *Then we have* $VHV^{-1}=A$.

Let $G_0(r, s, k)$, $\Gamma_0^\pm(r, k)$ and $\mathcal{F}_0^\pm$ be as in §4. In order to show the orthogonality of $\mathcal{F}_0^\pm$, we have to calculate $\Gamma_0^\pm(r, k)$. Denote by $g_0(y, y', z)(y, y'\in \boldsymbol{R}^n$, $z\in \boldsymbol{C}-\boldsymbol{R})$ the Green kernel for H_0. Then we have

(7.14)
$$g_0(y, y', z) = \frac{iz^{(n/4)-(1/2)}}{4(2\pi|y-y'|)^{(n/2)-1}} H^{(1)}_{(n/2)-1}(\sqrt{z}\,|y-y'|)^{17)},$$

where $H_\nu^{(1)}(t)$ is the Hankel function of the first kind. $G_0(z, r, k)$ is represented by $g_0(y, y', z)$ as follows:

(7.15)
$$\begin{aligned} G_0(s, r, k)x(\omega) &= s^{(n-1)/2}r^{(n-1)/2}\int_{S^{n-1}} g_0(s\omega, r\omega', k^2+i0)x(\omega')d\omega' \\ &= \frac{i}{4}s^{(n-1)/2}r^{(n-1)/2}k^{(n/2)-1}\int_{S^{n-1}} \frac{H^{(1)}_{(n/2)-1}(k|s\omega-r\omega'|)}{(2\pi|s\omega-r\omega'|)^{(n/2)-1}}x(\omega')d\omega', \end{aligned}$$

where $k>0$ and $x(\omega)\in L^2(S^{n-1})$. Thus by an easy computation we obtain from (3.3)

(7.16)
$$\begin{aligned} \Gamma_0(r, k)x(\omega) &= \lim_{s\to\infty} e^{-iks}G_0(s, r, k) \\ &= -e^{-(\pi/4)(n-1)i}\sqrt{\frac{\pi}{2}}\,\frac{1}{ik}\,\frac{k^{(n-1)/2}r^{(n-1)/2}}{(2\pi)^{n/2}}\int_{S^{n-1}} e^{-ikr(\omega,\omega')}x(\omega')d\omega', \end{aligned}$$

where $k>0$ and we used the asymptotic formula

(7.17)
$$H_\nu^{(1)}(t)\sim\sqrt{\frac{2}{\pi t}}e^{i(t(-\pi/4)(2\nu+1))} \qquad (t\to\infty)^{18)}.$$

16) See Kato [5], p. 287 – p. 293.
17) See Titchmarsh [9], p. 79.
18) See, for example, Watson [10], p. 197.

Hence we have

$$(7.18)\qquad \Gamma_0^+(r, k)x(\omega) = -e^{-(\pi/4)(n-1)i}\frac{k^{(n-1)/2}r^{(n-1)/2}}{(2\pi)^{n/2}}\int_{S^{n-1}} e^{-ikr(\omega,\omega')}x(\omega')d\omega'.$$

On the other hand we have

$$(\Gamma_0(r, -k)x_1(\cdot), x_2(\cdot)) = \lim_{s\to\infty}(e^{isk}G_0(s, r, -k)x_1(\cdot), x_2(\cdot))$$
$$= \lim_{s\to\infty}(x_1(\cdot), e^{-isk}G_0(r, s, k)x_2(\cdot)).$$

As in (7.16) we have

$$(7.19)\qquad \lim_{s\to\infty} e^{-isk}G_0(r, s, k)x_2(\omega)$$
$$= -e^{-(\pi/4)(n-1)i}\sqrt{\frac{\pi}{2}}\frac{1}{ik}\frac{r^{(n-1)/2}k^{(n-1)/2}}{(2\pi)^{r/2}}\int_{S^{n-1}} e^{-ikr(\omega,\omega')}x_2(\omega')d\omega'$$

whence follows for $k>0$

$$(7.20)\qquad \Gamma_0^-(r, k)x(\omega) = -\sqrt{\frac{2}{\pi}}ik\Gamma_0(r, -k)x(\omega)$$
$$= -e^{\pi/4(n-1)i}\frac{r^{(n-1)/2}k^{(n-1)/2}}{(2\pi)^{n/2}}\int_S e^{ikr(\omega,\omega')}x(\omega')d\omega'.$$

We see from (7.16) and (7.20) that $\mathcal{F}_0^{\pm}$ are essentially Fourier transforms, and hence we have

Lemma 7.3. *$\mathcal{F}_0^{\pm}$ are orthogonal transforms.*

By Lemmas 7.1~7.3 we can apply to H the results obtained in the preceding sections.

Theorem 7.4. *Let $q(x)$ be a real-valued function on $\boldsymbol{R}^n$, $n \geqq 3$, satisfying the conditions* (Q). *Let H be as above.*

(i) *Then H is bounded below. We have $\sigma_e(H)=(0, \infty)$ and the negative eigenvalues, if they exsit, are of finite multiplicity. The spectrum of H on* (0, ∞) *is absolutely continuous.*

(ii) *Denote by $E_H(\lambda)$ the resolution of the identity associated with H. Then there exist unitary operators $\mathcal{F}_H^{\pm}(=\mathcal{F}^{\pm}V)$ from $E_H((0, \infty))L^2(\boldsymbol{R}^n)$ onto $L^2((0, \infty)$, $L^2(S^{n-1})$, $dk)$ such that we have for a Borel set Δ in* (0, ∞)

$$(7.21)\qquad \begin{cases} E_H(\Delta) = (\mathcal{F}_H^{\pm})^*\chi_{\sqrt{\Delta}}\mathcal{F}_H^{\pm}, \\ \mathcal{F}_H^{\pm}(\mathcal{F}_H^{\pm})^* = 1. \end{cases}$$

(iii) *Let Δ be as above. Then there exist the wave operators $W_H^{\pm}(\Delta)$ which are partial isometries with initial set $E_{H_0}(\Delta)L^2(\boldsymbol{R}^n)$ and finial set $E_H(\Delta)L^2(\boldsymbol{R}^n)$,* *where* $E_{H_0}(\Delta)$ *denotes the resolution of the identity associated with* H_0. $W_H^{\pm}(\Delta)$ *have the intertwining property. We have*

$$W_H^{\pm}(\Delta) = (\mathcal{F}_H^{\pm})^* \chi_{\sqrt{\Delta}} \mathcal{F}_{H_0}^{\pm}, \tag{7.22}$$

where $\mathcal{F}_{H_0}^{\pm} = \mathcal{F}_0^{\pm} V$ *which are unitary operators from* $E_{H_0}((0, \infty))L^2(\boldsymbol{R}^n)$ *onto* $L^2((0, \infty), L^2(S^{n-1}), dk)$. *For a real valued function* α *on* $(0, \infty)$ *which satisfies* (6.15) *for every* $\eta \in L^2(0, 2\pi)$ *we have*

$$W_H^{\pm}(\Delta)F = \text{s-}\lim_{t \to \pm\infty} e^{it\alpha(E_H)} e^{-it\alpha(E_{H_0})} E_{H_0}(\Delta) F \tag{7.23}$$

for every $F \in L^2(\boldsymbol{R}^n)$. *In particular we have*

$$W_H^{\pm}(\Delta) = \text{s-}\lim_{t \to \pm\infty} e^{itH} e^{-itH_0} E_{H_0}(\Delta). \tag{7.24}$$

OSAKA CITY UNIVERSITY

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