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NOTE ON KRULL-REMAK-SCHMIDT-AZUMAYA'S THEOREM

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The Krull-Remak-Schmidt's theorem is very important in the ring theory and many algebraists tried to generalize it. Especially, Azumaya [1] succeeded to extend this theorem in a case of infinite direct sums.

In this note, we shall further generalize Azumaya's theorem. Let R be a ring with unit, and let all modules in this note be unitary R-modules. We assume that an R-module M has decompositions: $M=\sum_{\alpha\in I}\oplus M_\alpha=\sum_{\beta\in J}\oplus N_\beta$, where M_α and N_β are completely indecomposable modules for every $\alpha\in I$ and $\beta\in J$, (namely, $\mathrm{End}_R(M_\alpha)$, $\mathrm{End}_R(N_\beta)$ are local rings). In this case we shall consider the following condition.

(*) *For any subset I' in I (resp. J' in J), there exists a one-to-one mapping φ of I' into J (resp. J' into I), such that $M_\alpha\approx N_{\varphi(\alpha)}$ for all $\alpha\in I'$ (resp. $N_\beta\approx M_{\varphi(\beta)}$ for all $\beta\in J'$) and $M=\sum_{\alpha'\in I'}\oplus N_{\varphi(\alpha')}\oplus\sum_{\alpha\in I-I'}\oplus M_\alpha$ (resp. $M=\sum_{\beta'\in J'}\oplus N_{\beta'}\oplus\sum_{\alpha\in I-\varphi(J')}\oplus M_\alpha$).*

Then we have a problem whether the condition (*) is satisfied for any two decompositions of M or not.

Aaumaya [1] showed that if I' is finite, (*) is satisfied. In a case of infinite set I', P. Crawley and B. Jónsson gave an example in which (*) was not satisfied. On the other hand, M. Harada and Y. Sai [3] considered a category related to $\{M_\alpha\}_{\alpha\in I}$ and gave a necessary and sufficient condition for (*) being satisfied for any objects in the category

In this note, we shall give a complete answer that problem for any two direct decompositions: $M=\sum_{\alpha\in I}\oplus M_\alpha=\sum_{\beta\in J}\oplus N_\beta$.

We shall refer the reader to [3] for the notations and definitions.

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Let $\{M_\alpha\}_{\alpha\in I}$ be a family of completely indecomposable modules and $\{f_i\}_{i=1}^\infty$ any sequence of non isomorphic R-homomorphisms of M_{α_i} to $M_{\alpha_{i+1}}$ in $\{M_\alpha\}_{\alpha\in I}$. We call $\{M_\alpha\}_{\alpha\in I}$ a *T-nilpotent system*, if there exists n which depends on the sequence and any element m in M such that $f_n f_{n-1}\cdots f_1(m)=0$

for any set $\{M_{\alpha_i}, f_i\}_i$, and we call $\{M_\alpha\}_{\alpha\in I}$ a *semi-T-nilpotent system*, if the above condition is satisfied only for any sequence $\{f_i\}_{i=1}^{\infty}$ such that $\alpha_i \neq \alpha_j$ for $i \neq j$ (cf. [3], [4]).

We shall show the following theorem.

Theorem. *The following conditions* (i), (ii) *are equivalent for* $M=\sum_{\alpha\in I}\oplus M_\alpha = \sum_{\beta\in J}\oplus N_\beta$ *with infinite sets* I *and* J, *where* M_α*'s and* N_β*'s are completely indecomposable modules.*

(i) $\{M_\alpha\}_{\alpha\in I}$ *is a semi-T-nilpotent system.*

(ii) M *satisfies the condition* (*) *for any decomposition of* M.

In the above case, we have

(iii) *Every direct summand of* M *is also a directsum of completely indecomposable modules which are isomorphic to some* M_α.

(*cf.* condition III In [3].)

In order to prove the theorem, we give several lemmas and definitions.

Let I be a well ordered set and $M=\sum_{\alpha\in I}\oplus M_\alpha$, then $S_M=\mathrm{End}_R(M)$ is equal to the ring of column summable matrices, whose entries consist of elements in $\mathrm{Hom}_R(M_\tau, M_\sigma)$, namely for $f\in S_M$ and $x_\tau\in M_\tau$, $f=(b_{\sigma\tau})$ and $b_{\sigma\tau}(x_\tau)=0$ for almost all $\sigma\in I$. Let $\mathfrak{A}$ be the set of all endomorphisms $(b_{\sigma\tau})$ of M such that $b_{\sigma\tau}$ is a non-isomorphic R-homomorphism of M_τ to M_σ for every σ, τ. Then $\mathfrak{A}$ is a two-sided ideal of S_M and $\mathfrak{A}$ is independent of a decomposition of M. (*cf.* [3] Corollary to Lemma 5).

Lemma 1. *Let* M *and* $\mathfrak{A}$ *be as above. If* $\mathfrak{A}$ *is the radical of* S_M, *then Condition* (ii) *for any subset* I', J' *and any decomposition of* M, *and* (iii) *of the theorem are satisfied.* (*cf,* [3] Corollary 2 to Theorem 7.)

Lemma 2. *If* $\{M_\alpha\}_{\alpha\in I}$ *is a T-nilpotent system, then* $\mathfrak{A}$ *is the radical of* S_M.

Proof. We take the induced additive category $\mathfrak{B}$ from $\{M_\alpha\}_{\alpha\in I}$ and $\mathfrak{J}$ the ideal of $\mathfrak{B}$ in [3]. From [3] Lemma 10 and $\mathfrak{A}=\mathfrak{J}\,(\mathrm{End}_R(M))$, we obtain the lemma.

Lemma 3. *For every element* f *in* $\mathfrak{A}$, $1-f$ *is monomorphic.*

Proof. We obtain the lemma from [3] the proof of Proposition 10.

Lemma 4. *Let* $M=\sum_{\alpha\in K}\oplus\sum_{\beta\in I_\alpha}\oplus M_{\alpha\beta}$ *be a directsum of completely indecomposable modules,* I_α *be an inifinite set for all* $\alpha\in K$ *and* $M_{\alpha\beta}\approx M_{\alpha\beta'}$ $M_{\alpha\beta}\not\approx M_{\alpha'\beta'}$ *for* $\alpha\neq\alpha'$. *If* $\{M_{\alpha\beta}\}_{\substack{\alpha\in K\\ \beta\in I_\alpha}}$ *is a semi-T-nilpotent system, then* $\{M_{\alpha\beta}\}_{\substack{\alpha\in K\\ \beta\in I_\alpha}}$ *is a T-nilpotent system.*

Proof. Let $\{f_\alpha\}_{i=1}^{\infty}$ be any sequence of non-isomorphic R-homomorphism of $M_{\alpha_i\beta_i}$ to $M_{\alpha_{i+1}\beta_{i+1}}$ ($M_{\alpha_i\beta_i}$ may be equal to $M_{\alpha_j\beta_j}$ for $i\neq j$). From the assumption of infinite set I_α, we may assume that $\beta_i\neq\beta_j$ for all $i\neq j$. Therefore, we obtain the lemma, since $\{M_{\alpha\beta}\}$ is a semi-T-nilpotent system.

Lemma 5. *Let M, $\{M_{\alpha\beta}\}$ be as above, but I_α be a finite set for all $\alpha\in K$. Then $\mathfrak{A}$ is the radical of S_M.*

Proof. Let f be any element in $\mathfrak{A}$. $1-f$ is monomorphic by Lemma 3. Therefore, we have only to show that $1-f$ is an epimorphism, that is, f is a quasi-regular element in S_M. From the assumption, we obtain

$$M=\sum_{|I_\alpha|=1}\oplus M_{\alpha_1}\oplus\sum_{|I_\beta|=2}\oplus(M_{\beta_1}\oplus M_{\beta_2})\oplus\cdots\oplus\sum_{|I_\gamma|=n}\oplus(M_{\gamma_1}\oplus\cdots\oplus M_{\gamma_n})\oplus\cdots$$

where $|I_\alpha|$ is the cardinal number of I_α. Put $N_{\alpha(n)}=M_{\alpha(n)_1}\oplus\cdots\ominus M_{\alpha(n)n}$, $M=\sum_{n=1}^{\infty}\oplus\sum_{\alpha(n)}\oplus N_{\alpha(n)}$, and we consider the following sequence

$$N_{\alpha(n)}\xrightarrow{\ i\ }M\xrightarrow{\ 1-f\ }M\xrightarrow{\ p_{\alpha(n)}\ }N_{\alpha(n)}$$

where i is an inclusion and $p_{\alpha(n)}$ is a projection of M to $N_{\alpha(n)}$. Then we have $p_{\alpha(n)}\,(1\text{-}f)\,i=1_{N\alpha(n)}(mod\ \mathfrak{A})$, therefore $p_{\alpha(n)}\,(1\text{-}f)\,i$ is an isomorphism by [3] Corollary 1 to Theorem 7 and we put $\mathrm{Im}\,((1\text{-}f)\,i)=N'_{\alpha(n)}$, which is a direct summand of M and a submodule of Im (1-f), then we have a decomposition

$$*(\alpha(n)):M=N'_{\alpha(n)}\oplus\sum_{\alpha'(n')\neq\alpha(n)}\oplus N_{\alpha'(n')}.$$

We shall show that the submodule $M_0=\sum_{n=1}^{\infty}\sum_{\alpha(n)}N'_{\alpha(n)}$ is a directsum $M=\sum_{n=1}^{\infty}\oplus\sum_{\alpha(n)}\oplus N'_{\alpha(n)}$. Let x be an element in M_0 and $x=\sum_{i=1}^{t}x_{\alpha_i(n_i)}$, where $x_{\alpha_i(n_i)}\in N'_{\alpha_i(n_i)}$. From $*(\alpha_1(n_1))$ and $*(\alpha_2(n_2))$, we have $M=N'_{\alpha_1(n_1)}\oplus\sum_{\alpha(n)\neq\alpha(n_1)}\oplus N_{\alpha(n)}=N'_{\alpha_2(n_2)}\oplus\sum_{\alpha(n)\neq\alpha_2(n_2)}\oplus N_{\alpha(n)}$. We apply [3] Corollary 1 to Theorem 7 to these decomposition, by $M_{\alpha_1(n)j_1}\not\approx M_{\alpha_2(n_2)j_2}$ we have

$$*(\alpha_1(n_1),\ \alpha_2(n_2)):M=N'_{\alpha_1(n_1)}\oplus N'_{\alpha_2(n_2)}\oplus\sum_{\alpha(n)\neq\alpha_1(n_1),\,\alpha_2(n_2)}\oplus N_{\alpha(n)}.$$

Using $*(\alpha_3(n_3))$ again, we have

$$*(\alpha_1(n_1),\ \alpha_2(n_2),\ \alpha_3(n_3)):M=N'_{\alpha_1(n_1)}\oplus N'_{\alpha_2(n_2)}\oplus N'_{\alpha_3(n_3)}\oplus\sum_{\alpha(n)\neq\alpha_i(n_i)i=1,2,3}\oplus N_{\alpha(n)}.$$

Repeating this argument, we have $M_0=\sum_{n=1}^{\infty}\oplus\sum_{\alpha(n)}\oplus N'_{\alpha(n)}$.

We assume $M\neq M_0$, that is, there exists some $\alpha_1(n_1)$ such that $N_{\alpha_1(n_1)}$ is not contained in M_0. Let x be an element in $M_{\alpha_1(n_1)j_1}$ which is not contained in M_0.

By $*(\alpha_1(n_1))$ we have $x=x_1+\cdots+x_t$, where $x_1\in N'_{\alpha_1(n_1)}$, $x_k\in M_{\alpha_k(n_k)j_k}$ for $\alpha_k(n_k)\neq\alpha_1(n_1)$, $k\geqq 2$. Then there exists $k\geq 2$, such that x_k is not contained in M_0, and x_k is a homomorphic image of some homomorphism f_1 of $M_{\alpha_1(n_1)j_1}$ to $M_{\alpha_k(n_k)j_k}$ and we may assume $\alpha_k(n_k)=\alpha_2(n_2)$. By $*(\alpha_1(n_1), \alpha_2(n_2))$ and the same argument, there is $x_3\in M_{\alpha_3(n_3)j_3}$, $f_2\in \mathrm{Hom}_R(M_{\alpha_2(n_2)j_2}, M_{\alpha_3(n_3)j_3})$ such that $x_3=f_2(x_2)=f_2f_1(x)\notin M_0$ and $\alpha_3(n_3)\neq\alpha_1(n_1)$, $\alpha_2(n_2)$. Repeating this process, we have a sequence $\{f_i\}$ of non-isomorphisms f_i such that $f_nf_{n-1}\cdots f_1(x)\neq 0$ for any n, and $f_i\in \mathrm{Hom}_R(M_{\alpha_i(n_i)j_i}, M_{\alpha_{i+1}(n_{i+1})j_{i+1}})$, $\alpha_k(n_k)\neq\alpha_j(n_j)$ for $k\neq j$, which contradicts the assumption of semi-T-nilpotent system.

Proof of Theorem (ii)⇒(i) is proved in [3] Lemma 9. (i)⇒(ii) Let $M=\sum_{\alpha\in K}\oplus\sum_{\beta\in I_\alpha}\oplus M_{\alpha\beta}\oplus\sum_{\sigma\in L}\oplus\sum_{\rho\in J_\sigma}\oplus M_{\sigma\rho}$, where $M_{\alpha\beta}$ and $M_{\sigma\rho}$ are completely indecomposable modules for all α,β,σ,ρ, I_α is a finite set for all $\alpha\in K$, J_σ is an infinite set for all $\sigma\in L$, and $M_{\nu\mu}\approx M_{\nu\mu'}$, $M_{\nu\mu}\not\approx M_{\nu'\mu'}$ for $\nu\neq\nu'$. Put $M_1=\sum_{\alpha\in K}\oplus\sum_{\beta\in I_\alpha}\oplus M_{\alpha\beta}$, $M_2=\sum_{\sigma\in L}\oplus\sum_{\rho\in J_\sigma}\oplus M_{\sigma\rho}$, $M=M_1\oplus M_2$ and

$$f=\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22}\end{pmatrix}\in\mathfrak{A}\subset\begin{pmatrix}\mathrm{Hom}_R(M_1, M_1) & \mathrm{Hom}_R(M_2, M_1) \\ \mathrm{Hom}_R(M_1, M_2) & \mathrm{Hom}_R(M_2, M_2)\end{pmatrix}$$

Then 1-f is a unit in S_M, since 1-f_{ii} is a unite in $\mathrm{Hom}_R(M_i, M_i)$ by lemmas 4 and 5, (cf. [3] the proof of Lemma 8). (ii) is clear from Lemma 1.

In the following, we shall consider the several results given by M. Harada in [4], which are related to semi-T-nilpotent system. We use definitions in [4].

Let $\{M_\alpha\}_{\alpha\in I}$ be a family of completely indecomposable right R-modules. We define an additive category $\mathfrak{B}$ in the same way as in [3] and [4]. The objects of $\mathfrak{B}$ consist of some directsum of M'_α s and the morphisms of $\mathfrak{B}$ consist of all R-homomorphisms, Furthermore, we consider the ideal $\mathfrak{J}$ of $\mathfrak{B}$ (see [3], [4]). Then we know from Theorem 7 in [3] that $\mathfrak{B}/\mathfrak{J}$ is a C_3-completely reducible abelian category. Let M, N be objects in $\mathfrak{B}$ and i be an inclusion of N to M. If i is isomorphic modulo $\mathfrak{J}$, then we call Im i a dense submodule in M (see [4]).

Corollary 1. *Let $M=\sum_{\alpha\in I}\oplus M_\alpha$, then the following conditions are equivalent.*

1. *$\{M_\alpha\}_{\alpha\in I}$ is a semi-T-nilpotent system.*
2. *Condition* (ii) *of Theorem.*
3. *$\mathfrak{A}$ is the radical of S_M.*
4. *Every dense submodule of M coincides with M.*

It is clear from Theorem and [4], Theorem 2.

Corollary 2. ([4], Proposition 2) *Let M and N be in $\mathfrak{B}$ and $\bar{M}\supseteq\bar{N}$, then there exists a submodule N in M satisfying the following conditions.*

1. *N_0 is an object in* $\mathfrak{B}$: $N_0=\sum_{\alpha\in I'}\oplus M_\alpha$.

2. *N_{0J} is a direct summand of M for any finite subset J of I', (if $\{M_\alpha\}_{\alpha\in I'}$ is a semi-T-nilpotent system, J needs not be finite).*

3. $\bar{N}_0=\bar{N}$

Furthermore, if $N=\operatorname{Im} e$ and e is an idempotent in S_M, then we can choose N_0 in N.

Corollary 3. ([4], Lemma 2) *Assume that $M=\sum_{\alpha\in I}\oplus M_\alpha=N_1\oplus N_2$. If either N_i is finitely generated or a dense submodule of N is semi-T-nilpotent system, then N_i is in $\mathfrak{B}$.*

Corollary 4. *$M=\sum_{i=1}^{\infty}\oplus M_i$, where M_i is a comcompletely indecomposable module such that $[M_i, M_j]=0$ for $i<j$. Then M satisfies the condition (*).*

Proof. It is clear that the assumption implies that $\{M_i\}$ is a semi-T-nilpotent system.

Proposition 13 in [3] is a special case of Corollary 4.

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