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PRINCIPAL CIRCLE ACTIONS ON A PRODUCT OF SPHERES

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0. Introduction

A smooth circle action $\varphi : S^1 \times X \rightarrow X$ on a smooth manifold X is called *principal* if the isotropy subgroup

$$I(x) = \{z \in S^1 \mid \varphi(z, x) = x\}$$

consists of the identity element alone for each point x of X. For a principal smooth circle action on a smooth manifold X, the orbit space M is a smooth manifold, the natural projection $\pi : X \rightarrow M$ is a smooth principal S^1-bundle, and in addition the manifold M is orientable if and only if the manifold X is orientable.

Two principal smooth circle actions (φ, X) and (φ', X') are called to be *equivalent* if there is an equivariant diffeomorphism of (φ, X) onto (φ', X'). A principal smooth circle action (φ, X) on a closed oriented smooth manifold X is called to *bord* if there is a principal smooth circle action (Φ, W) on a compact oriented smooth manifold W and there is an equivariant orientation preserving diffeomorphism of (φ, X) onto $(\Phi, \partial W)$, the boundary of W.

In this paper we consider principal smooth circle actions on a closed orientable smooth manifold which is cohomologically a product of spheres. We show that any principal circle action on a manifold which is cohomologically a product $S^{2m+1} \times S^{2n+1}$ of odd dimensional spheres bords but on a certain manifold which is cohomologically $S^{2m} \times S^{2n+1}$ $(n \geqq m)$ there is a principal circle action which does not bord. And the cohomology rings of orbit manifolds show that there are infinitely many (topologically) distinct principal circle actions on $S^{2m+1} \times S^{2n+1} (m \neq n)$. We can also show that the Pontrjagin classes of orbit manifolds well distinguish some of the circle actions on a product of spheres.

1. Cobordism of principal circle actions

Let E be a topological space whose integral cohomology group $H^*(E)$ is isomorphic to an integral cohomology group $H^*(S^{2m+1} \times S^{2n+1})$ of a product of odd dimensional spheres with $0 \leqq m \leqq n$. Let $\pi : E \to M$ be a principal S^1-bundle over an orientable closed smooth manifold M. Then,

Lemma 1.1.

(1) *The integral cohomology ring $H^*(M)$ of M is isomorphic to one of the truncated polynomial rings given under*:

(a) $\boldsymbol{Z}[c, x]/(x^2, c^{n+1})$, *where* $\deg c = 2$ *and* $\deg x = 2m+1$,

(b) $\boldsymbol{Z}[c, y]/(y^2, c^{n+1}, kc^{m+1}, yc^{m+1})$, *where* $\deg c = 2$ *and* $\deg y = 2n+1$

and k is a positive integer. Here the element c corresponds to the Euler class of the principal S^1-bundle $\pi : E \to M$.

(2) *The each odd dimensional Stiefel-Whitney class of M vanishes.*

Proof. By the Thom-Gysin sequence ([5] p. 60, Theorem 21) for the principal S^1-bundle $\pi : E \to M$, $H^{2m-1}(M)=0$ and $H^{2m}(M)$ is an infinite cyclic group generated by c^m. Then $H^{2n+2}(M)=0$ by the universal coefficient theorem and the Poincaré duality of M. Now the ring structure of $H^*(M)$ is obtained from the Thom-Gysin sequence by a routine calculation. Next, let $V_i \in H^i(M; \boldsymbol{Z}_2)$ be a class characterized by the equation

$$Sq^i \alpha = \alpha \cup V_i \quad \text{for all} \quad \alpha \in H^{\dim M - i}(M; \boldsymbol{Z}_2),$$

and let $V = V_0 + V_1 + \cdots + V_i + \cdots$, then $SqV = W(M)$, the total Stiefel-Whitney class of M by the Wu's formula ([5] p. 55, Theorem 17). Then $W_{2i+1}(M)=0$ follows from the ring structure of $H^*(M; \boldsymbol{Z}_2)$ and a property of the Steenrod operations ([6] p. 5, Lemma 2.5). q.e.d.

Theorem 1. *Let E be an orientable closed smooth manifold. Assume that the integral cohomology group of E is isomorphic to one of a product $S^{2m+1} \times S^{2n+1}$ of odd dimensional spheres. Then any principal smooth circle action on E bords as an orientable principal smooth circle action.*

Proof. Let $\pi : E \to M$ be a principal S^1-bundle associated with a given principal smooth circle action on E. Denote by $\bar{c}$ the modulo 2 reduction of the Euler class c of the principal S^1-bundle $\pi : E \to M$. Then the circle action on E bords as an orientable principal smooth circle action if and only if all bordism Stiefel-Whitney numbers vanish

$$\langle W_{i_1}(M) \cdots W_{i_r}(M) \bar{c}^k, [M]_2 \rangle = 0,$$

and all bordism Pontrjagin numbers vanish

$$\langle P_{i_1}(M) \cdots P_{i_r}(M) c^k, [M] \rangle = 0,$$

where $[M]_2$ is the modulo 2 reduction of the fundamental class $[M]$ of M ([3]

p. 49, Theorem 17.5). But the orbit manifold M is odd dimensional and each odd dimensional Stiefel-Whitney class of M vanishes by Lemma 1.1. Hence all bordism Stiefel-Whitney numbers and all bordism Pontrjagin numbers of $\pi : E \to M$ vanish. Therefore this principal smooth circle action bords as an orientable principal smooth circle action. q.e.d.

2. Principal circle actions on a product of spheres

For a sequence $a=(a_0, \cdots, a_m)$ of integers, we define the circle action φ_a on C^{m+1} by

$$\varphi_a(z, (u_0, \cdots, u_m)) = (z^{a_0}u_0, \cdots, z^{a_m}u_m),$$

and denote by $S^{2m+1}(a_0,\cdots, a_m)$ the unit sphere S^{2m+1} in C^{m+1} with this action φ_a.

Let $a=(a_0,\cdots, a_m)$, $b=(b_0,\cdots, b_n)$ be sequences of integers. We also define the circle action $\varphi_{a,b}$ on $S^{2m+1}\times S^{2n+1}$ by

$$\varphi_{a,b}(z, (\vec{u}, \vec{v}))=(\varphi_a(z, \vec{u}), \varphi_b(z, \vec{v}))$$

where $\vec{u}=(u_0,\cdots, u_m)$, $\vec{v}=(v_0,\cdots, v_n)$, and denote by

$$S^{2m+1}(a_0,\cdots, a_m)\times S^{2n+1}(b_0,\cdots, b_n)$$

the product $S^{2m+1}\times S^{2n+1}$ with the action $\varphi_{a,b}$. Then the circle action $\varphi_{a,b}$ is principal if and only if each a_i is relatively prime to each b_j. When the circle action $\varphi_{a,b}$ is principal, the orbit manifold is denoted by

$$M(a_0,\cdots, a_m; b_0,\cdots,b_n).$$

In particular, $M(a_0; b_0,\cdots, b_n)$ is naturally diffeomorphic to the lens space obtained from S^{2n+1} by the identification $\vec{v}=\varphi_b(\lambda, \vec{v})$ for all $\lambda\in C$, $\lambda^{a_0}=1$. The cohomology ring of $M(a_0,\cdots, a_m; b_0,\cdots, b_n)$ is determined as follows:

Theorem 2. *Suppose $0\leqq m\leqq n$. Then the integral cohomology ring of $M(a_0,\cdots, a_m; b_0,\cdots, b_n)$ is isomorphic to*

(i) $Z[c, x]/(x^2, c^{n+1})$, *where* $\deg c=2$ *and* $\deg x=2m+1$, *if $m=n$ or if $a_i=0$ for some i,*

(ii) $Z[c, y]/(y^2, c^{n+1}, kc^{m+1}, yc^{m+1})$, *where* $\deg c=2$, $\deg y=2n+1$ *and $k=\prod_i a_i$ if $m<n$ and $\prod_i a_i\neq 0$. Here the element c corresponds to the Euler class of the principal S^1-bundle*

$$\pi : S^{2m+1}(a_0,\cdots, a_m)\times S^{2n+1}(b_0,\cdots, b_n)\to M(a_0,\cdots, a_m; b_0,\cdots, b_n).$$

By virtue of Lemma 1.1, it is sufficient to determine the $(2m+2)$-dimensional cohomology group of $M(a_0,\cdots, a_m; b_0,\cdots, b_n)$, and furthermore if

$m=n$ the cohomology ring is determined by Lemma 1.1 already.

Denote by ξ_n the canonical complex line bundle over the complex projective n-space $P^n(C)$ obtained from $S^{2n+1}\times C$ by the identification $(\vec{u}, \rho)=(\lambda\vec{u}, \lambda\rho)$ for all $\lambda\in C$, $|\lambda|=1$ ([5] p. 75). Then there is a mapping

$$p : M(a_0,\cdots, a_m; \underbrace{1,\cdots, 1}_{(n+1)\text{times}}) \to P^n(C)$$

given by the following commutative diagram:

$$\begin{array}{ccc} S^{2m+1}(a_0,\cdots, a_m)\times S^{2n+1}(1,\cdots, 1) & \xrightarrow{p_2} & S^{2n+1}(1,\cdots, 1) \\ \Big\downarrow \pi & & \Big\downarrow \pi_0 \\ M(a_0,\cdots, a_m; 1,\cdots, 1) & \xrightarrow{p} & P^n(C) \end{array}$$

where p_2 is the projection to the second factor and π_0 is the projection of the principal S^1-bundle associated with the canonical complex line bundle ξ_n.

Lemma 2.1.

(i) *The natural projection*

$$p: M(a_0,\cdots, a_m; 1,\cdots, 1) \to P^n(C)$$

is a sphere bundle associated with the complex $(m+1)$*-plane bundle*

$$\xi_n^{a_0}\oplus\cdots\oplus\xi_n^{a_m}$$

where ξ^a *is the a-fold tensor product of a complex line bundle* ξ *for* $a\geqq 0$ *and the* $(-a)$*-fold tensor product of the conjugate line bundle* $\bar{\xi}$ *of* ξ *for* $a<0$.

(ii) *For* $M=M(a_0,\cdots, a_m; 1,\cdots, 1)$, *we have*

$$H^{2m+2}(M)\cong\begin{cases} Z/(\prod_i a_i)\cdot Z & \text{if } m<n, \\ 0 & \text{if } m\geqq n. \end{cases}$$

Proof. (i) is proved easily from the fact that the total space $E(\xi_n^a)$ of the complex line bundle ξ_n^a can be represented as the space obtained from $S^{2n+1}\times C$ by the identification $(\vec{u}, \rho)=(\lambda\vec{u}, \lambda^a\rho)$ for all $\lambda\in C$, $|\lambda|=1$. Next the Euler class of the complex $(m+1)$-plane bundle $\zeta=\xi_0^{a_0}\oplus\cdots\oplus\xi_n^{a_m}$ is

$$e(\zeta) = (\prod_{i=0}^{m} a_i)\cdot e(\xi_n)^{m+1}.$$

Then, by the Thom-Gysin sequence for the complex $(m+1)$-plane bundle ζ, there is an exact sequence;

$$H^0(P^n(\boldsymbol{C})) \xrightarrow{h} H^{2m+2}(P^n(\boldsymbol{C})) \xrightarrow{p^*} H^{2m+2}(M) \xrightarrow{p_*} H^1(P^n(\boldsymbol{C}))$$

where the homomorphism h is given by $h(x)=x\cdot e(\zeta)$. And this implies (ii). q.e.d.

Lemma 2.2. *We have*

$$H^{2m+2}(M(a_0,\cdots, a_m; b_0,\cdots, b_n)) \cong Z/(\prod_i a_i)\cdot Z$$

for $m<n$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
S^{2m+1}(a)\times S^{2n+1}(b) \arrow[r, "i_1"] \arrow[d, "\pi"] & S^{2m+1}(a)\times S^{4n+3}(b, c) \arrow[d, "\pi"] & S^{2m+1}(a)\times S^{2n+1}(c) \arrow[l, "i_2"'] \arrow[d, "\pi"] \\
M(a; b) \arrow[r, "f_1"] & M(a; b, c) & M(a; c) \arrow[l, "f_2"']
\end{tikzcd}$$

where $a=(a_0,\cdots, a_m)$, $b=(b_0,\cdots, b_n)$, $c=(c_0,\cdots, c_n)$, $i_1((\vec{u}, \vec{v}))=(\vec{u}, (\vec{v}, 0))$, $i_2((\vec{u}, \vec{v}))=(\vec{u}, (0, \vec{v}))$ and f_1, f_2 are induced mappings. Then f_1, f_2 induce isomorphisms of $(2m+2)$-dimensional cohomology groups if $m<n$, and we have $H^{2m+2}(M(a_0,\cdots, a_m; b_0,\cdots, b_n))\cong H^{2m+2}(M(a_0,\cdots, a_m; c_0,\cdots, c_n))$. Thus Lemma 2.2 follows from this isomorphism and Lemma 2.1 (ii). q.e.d.

The proof of Theorem 2 completes.

Corollary. *There are infinitely many (topologically) distinct principal smooth circle actions on* $S^{2m+1}\times S^{2n+1}$ *for each* $m\neq n$.

This follows directly from Lemma 2.2.

3. Pontrjagin classes of orbit manifolds

For a given principal smooth circle action on E in our examples, the Pontrjagin classes of the orbit manifold M can be expressed by the Euler class c of the principal S^1-bundle $\pi : E\rightarrow M$.

Let E be a smooth submanifold of an N-dimensional euclidean space $\boldsymbol{R}^N$. For each point p of E, the tangent space $\tau_p(E)$ of E at p can be canonically imbedded into the tangent space $\tau_p(\boldsymbol{R}^N)$ of $\boldsymbol{R}^N$ at p. If we denote by $\nu_p(E)$ the orthogonal complement of $\tau_p(E)$ in $\tau_p(\boldsymbol{R}^N)$, then $\nu(E)=\bigcup_{p\in E}\nu_p(E)$ is the normal bundle of E in $\boldsymbol{R}^N$. Let t be an isometry of $\boldsymbol{R}^N$ such that $t(E)\subset E$. Then the differential $(dt)_p$ of t at p in E maps $\tau_p(E)$ onto $\tau_{t(p)}(E)$, and $\nu_p(E)$ onto $\nu_{t(p)}(E)$. Suppose the normal bundle $\nu(E)$ is trivial, i.e.

$$\nu(E) \cong E\times \boldsymbol{R}^k.$$

If dt on $\nu(E)$ satisfies:

$$(dt)_p(p, v) = (t(p), v) \text{ for } p \in E, \ v \in \mathbf{R}^k,$$

then we say the action of t on $\nu(E)$ is compatible with the trivialization, or simply t acts on $\nu(E)$ trivially.

Lemma 3.1. *Let E be a smooth submanifold of an N-dimensional euclidean space $\boldsymbol{R}^N$, and T a circle subgroup of $\boldsymbol{SO}(N, \boldsymbol{R})$ acting principally on E. Suppose the normal bundle $\nu(E)$ of E in $\boldsymbol{R}^N$ is trivial and the action of T on $\nu(E)$ is compatible with the trivialization. Then the tangent bundle $\tau(M)$ of the orbit manifold M is stably equivalent to the vector bundle obtained from $E \times \boldsymbol{R}^N$ by the identification $(p, v) = (t(p), t(v))$, for all $t \in T$.*

Proof. At each point p of $\boldsymbol{R}^N$, we have the usual identification of $\tau_p(\boldsymbol{R}^N)$ with $\mathbf{R}^N$, which is denoted by h_p. First we remark that for any element t in $\boldsymbol{GL}(N, \boldsymbol{R})$, the differential dt of t is compatible with the above identifications, i.e.

$$h_{t(p)} \circ (dt)_p = t \circ h_p \qquad \text{at each} \quad p \in \boldsymbol{R}^N. \tag{3.1}$$

Denote by $\lambda(E)$ the restriction of $\tau(\boldsymbol{R}^N)$ over E given by

$$\lambda_p(E) = \tau_p(\boldsymbol{R}^N), \text{ for } p \in E.$$

Consider the equivalence relation on $\lambda(E)$ as follows: $X \sim Y$ if and only if $Y = (dt)(X)$ for some t in T. Now (3.1) shows that the bundle over M obtained from $\lambda(E)$ by the above relation is isomorphic with the vector bundle stated in Lemma 3.1. Let $\gamma_p(E)$ be the kernel of $(d\pi)_p : \tau_p(E) \to \tau_p(M)$, and $\tau'_p(E)$ the orthogonal complement of $\gamma_p(E)$ in $\tau_p(E)$. We have the decomposition:

$$\lambda(E) = \tau'(E) \oplus \gamma(E) \oplus \nu(E),$$

where T acts on each factor. From the assumption in Lemma 3.1, $\nu(E)$ is trivial and T acts trivially on $\nu(E)$. The bundle $\gamma(E)$ is trivial and the action of T on $\gamma(E)$ is compatible with this trivialization since T is abelian. Thus the bundle over M obtained from $\lambda(E)$ by the above equivalence relation is stably equivalent to the bundle over M obtained from $\tau'(E)$ by the same relation. The differential $d\pi$ of the projection $\pi : E \to M$ gives an isomorphism $\tau'_p(E)$ with $\tau_{\pi(p)}(M)$ and $d\pi$ is compatible with the action of T, i.e.

$$d\pi \circ dt = d\pi \qquad \text{for any} \quad t \in T.$$

Now it is easy to see that the bundle obtained from $\tau'(E)$ is isomorphic with the tangent bundle $\tau(M)$. q.e.d.

Theorem 3. *Under the same notations as in section 2, the total Pontrjagin class of the orbit manifold* $M=M(a_0,\cdots,a_m;\ b_0,\cdots,b_n)$ *is*

$$P(M)=\prod_{i=0}^{m}(1+a_i^2c^2)\cdot\prod_{j=0}^{n}(1+b_j^2c^2),$$

where c is the Euler class of the principal S^1-bundle associated with the circle action $\varphi(a_0,\cdots,a_m;\ b_0,\cdots,b_n)$.

Proof. For the unit $(2m+1)$-sphere S^{2m+1} in $\boldsymbol{C}^{m+1}=\boldsymbol{R}^{2m+2}$, we choose a unit normal vector field X on S^{2m+1}. Then each element in $\boldsymbol{SO}(2m+2,\boldsymbol{R})$ fixes X, thus the action of $\boldsymbol{SO}(2m+2,\boldsymbol{R})$ is trivial on the normal bundle of S^{2m+1}. S^{2n+1} in $\boldsymbol{C}^{n+1}$ has the same property. It is easy to see that our action $\varphi(a_0,\cdots,a_m;\ b_0,\cdots,b_n)$ on $E=S^{2m+1}\times S^{2n+1}$ in $\boldsymbol{C}^{m+1}\times\boldsymbol{C}^{n+1}=\boldsymbol{R}^N$ satisfies the required assumption in Lemma 3.1, where $N=2m+2n+2$ and $S^1=T$ is expressed as

$$S^1=\left\{\begin{pmatrix} z^{a_0} & & & & & 0\\ & \ddots & & & & \\ & & z^{a_m} & & & \\ & & & z^{b_0} & & \\ & & & & \ddots & \\ 0 & & & & & z^{b_n}\end{pmatrix}\right\}$$

by the complex coordinates. On the other hand, if we denote by ξ the complex line bundle over M associated with the principal S^1-bundle $\pi: E\to M$, then the bundle constructed in Lemma 3.1. is isomorphic with

$$\zeta=\xi^{a_0}\oplus\cdots\oplus\xi^{a_m}\oplus\xi^{b_0}\oplus\cdots\oplus\xi^{b_n},$$

where ξ^a denotes the a-fold tensor product of ξ. Thus, by Lemma 3.1, the tangent bundle $\tau(M)$ of the orbit manifold is stably equivalent to the real restriction of the complex vector bundle ζ. Now the conclusion of Theorem 3 follows from properties of Pontrjagin classes ([5], Chapter XII). q.e.d.

Corollary. *If two principal circle actions $\varphi_{a,b}$ and $\varphi_{c,d}$ are equivalent, where* $a=(a_0,\cdots,a_m)$, $b=(b_0,\cdots,b_n)$, $c=(c_0,\cdots,c_m)$ *and* $d=(d_0,\cdots,d_n)$, *then*

$$(1)\qquad \sigma_k(a_0^2,\cdots,a_m^2,b_0^2,\cdots,b_n^2)=\sigma_k(c_0^2,\cdots,c_m^2,d_0^2,\cdots,d_n^2)$$

for $2k\leqq m\leqq n$,

$$(2)\qquad |\prod_{i=0}^{m}a_i|=|\prod_{i=0}^{m}c_i|\quad \textit{for}\quad m<n,$$

and

$$(3)\qquad \sigma_k(a_0^2,\cdots,a_m^2,b_0^2,\cdots,b_n^2)=\sigma_k(c_0^2,\cdots,c_m^2,d_0^2,\cdots,d_m^2)$$

mod $|\prod_{i=0}^{m} a_i|$ *for* $m<2k\leqq n$. *Here* σ_k *is the k-th elementary symmetric function on* $(m+n+2)$*-variables.*

4. Gysin homomorphism

Let ξ be an oriented n-plane bundle over a topological space X with a Thom class $U\in H^n(D(\xi),\ S(\xi))$, where $p: D(\xi)\to X$ and $\pi: S(\xi)\to X$ are the associated disk bundle and the associated sphere bundle respectively. Then there is a commutative diagram:

$$\begin{tikzcd}
H^*(S(\xi)) \arrow[r, "\delta"] \arrow[d, equal] & H^*(D(\xi),\ S(\xi)) \arrow[r] & H^*(D(\xi)) \arrow[r, "i^*"] & H^*(S(\xi)) \arrow[d, equal] \\
H^*(S(\xi)) \arrow[r, "\pi_*"] & H^*(X) \arrow[r, "\cdot e(\xi)"] \arrow[u, "\cong", "\phi_\xi"'] & H^*(X) \arrow[r, "\pi^*"] \arrow[u, "\cong", "p^*"'] & H^*(S(\xi))
\end{tikzcd}$$

where the homomorphism ϕ_ξ is a Thom isomorphism defined by $\phi_\xi(x)=p^*(x)U$, $e(\xi)$ is a Euler class of ξ and the homomorphism π_* is a Gysin homomorphism. The lower horizontal line is a Thom-Gysin sequence for the oriented n-plane bundle ξ ([5] p. 60).

Lemma 4.1.

(1) $\pi_*(\pi^*x\cup y)=(-1)^{\deg x}x\cup\pi_*y$ *for* $x\in H^*(X)$ *and* $y\in H^*(S(\xi))$, ([4] p. 71; [7] p. 121)

(2) $\pi_*(Sq^iu)=\sum_{j+k=i} Sq^j\pi_*u\cup W_k(\xi)$ *for* $u\in H^*(S(\xi);\ Z_2)$ *where* $W_k(\xi)$ *is a k-th Stiefel-Whitney class of* ξ, ([5]p. 35; [7] p. 137)

(3) $\pi_*(P^iv)=\sum_{j+k=i} P^j\pi_*v\cup Q_k(\xi)$ *for* $v\in H^*(S(\xi);\ Z_p)$ *where* p *is an odd prime,* P^i *is a reduced power operation and* $Q_k(\xi)\in H^{2k(p-1)}(X;\ Z_p)$ *is a k-th Wu class defined by* $Q_k(\xi)=\phi_\xi^{-1}P^kU$, ([5] p. 120).

Proof.
$$\begin{aligned}
\phi_\xi\pi_*(\pi^*x\cup y) &= \delta(\pi^*x\cup y)\\
&= \delta(i^*p^*x\cup y)\\
&= (-1)^{\deg x}\ p^*x\cup\delta y\\
&= (-1)^{\deg x}\ p^*x\cup(p^*\pi_*y\cup U)\\
&= (-1)^{\deg x}\ \phi_\xi(x\cup\pi_*y).
\end{aligned}$$

This implies (1), since ϕ_ξ is an isomorphism. Next

$$\begin{aligned}
\phi_\xi\pi_*(Sq^iu) &= \delta(Sq^iu)\\
&= Sq^i(\delta u)\\
&= Sq^i(p^*\pi_*u\cup U)\\
&= \sum_{j+k=i} Sq^jp^*\pi_*u\cup Sq^kU \qquad \text{(Cartan formula)}
\end{aligned}$$

$$= \sum_{j+k=i} p^*Sq^j\pi_*u \cup (p^*W_k(\xi) \cup U) \qquad ([5]\text{p. } 35;\ [7]\text{p. } 137)$$
$$= \phi_\xi\Big(\sum_{j+k=i} Sq^j\pi_*u \cup W_k(\xi)\Big).$$

This implies (2). The relation (3) is proved similarly by the Cartan formula of reduced power operations ([6] p. 76) and the definition of Wu classses ([5] p. 120). q.e.d.

5. Miscellaneous principal circle actions

In this section we give some examples of principal circle actions on a closed orientable smooth manifold E which is cohomologically a product $S^{2m}\times S^{2n+1}$.

5.1. Given a sequence $a=(a_1,\cdots, a_m)$ of integers, let ψ_a be a principal smooth circle action on $S^{2m}\times S^{2n+1}$ given by

$$\psi_a(z, ((u_0,\cdots, u_m), (v_0,\cdots, v_n))) = ((u_0, z^{a_1}u_1,\cdots, z^{a_m}u_m), (zv_0,\cdots, zv_n))$$

in complex coordinates, where u_0 is a real number. Denote by M_a the orbit manifold. Then there is a mapping $p: M_a\to P^n(\boldsymbol{C})$ given by the following commutative diagram as in section 2:

$$\begin{array}{ccc} S^{2m}\times S^{2n+1} & \xrightarrow{p_2} & S^{2n+1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_0} \\ M_a & \xrightarrow{p} & P^n(\boldsymbol{C}) \end{array}$$

where p_2 is a projection to the second factor, π and π_0 are natural projections.

The projection $p: M_a\to P^n(\boldsymbol{C})$ is a sphere bundle associated with a real $(2m+1)$-plane bundle

$$\zeta = \theta_R^1\oplus\xi_n^{a_1}\oplus\cdots\oplus\xi_n^{a_m}$$

where ξ_n is the canonical complex line bundle over $P^n(\boldsymbol{C})$ and θ_R^1 is a trivial real line bundle (see Lemma 2.1), and there is a cross-section $s: P^n(\boldsymbol{C})\to M_a$ defined by $s([v_0,\cdots, v_n])=\pi((1, 0,\cdots, 0), (v_0,\cdots, v_n))$, so the Euler class $e(\zeta)=0$. Then, by the Thom-Gysin sequence for ζ, there is a short exact sequence:

$$(5.1.1) \qquad 0\longrightarrow H^k(P^n(\boldsymbol{C}))\xrightarrow{p^*} H^k(M_a)\xrightarrow{p_*} H^{k-2m}(P^n(\boldsymbol{C}))\longrightarrow 0.$$

Proposition 5.1.

(1) *The integral cohomology ring of* M_a *is*

$$H^*(M_a) = \boldsymbol{Z}[c, x]/(c^{n+1}, x^2-(\prod_i a_i)\cdot xc^m)$$

where $c=p^*e(\xi_n)$, $\deg x=2m$, $p_*x=1$ *and* $s^*x=0$.

(2) *The total Pontrjagin class of* M_a *is*

$$P(M_a) = (1+c^2)^{n+1}\cdot\prod_{i=1}^{m}(1+a_i^2c^2).$$

Proof. The module structure of $H^*(M_a)$ and the relation $c^{n+1}=0$ are obtained directly by the exact sequence (5.1.1). And the total Pontrjagin class of M_a is calculated similarly as Theorem 3. Finally the relation $x^2=(\prod_i a_i)\cdot xc^m$ is obtained from Lemma 4.1 (2), (3), a property of the reduced power operations ([6] p. 1, p. 76) and a property of Wu classes ([5] p. 120), so we leave it to the reader. q.e.d.

Corollary. *If the corresponding actions* ψ_a *and* ψ_b *are equivalent for sequences* $a=(a_1,\cdots,a_m)$ *and* $b=(b_1,\cdots,b_m)$ *of integers. Then*

$$\sigma_p(a_1^2,\cdots,a_m^2) = \sigma_p(b_1^2,\cdots,b_m^2)$$

for any positive integer p *with* $2p\leqq n$, *where* σ_p *is the* p*-th elementary symmetric function on* m*-variables.*

5.2. Let ξ_1 be the canonical complex line bundle over $P^1(\boldsymbol{C})$. Given a sequence $a=(a_0,\cdots,a_n)$ of integers, denote by

$$S(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n})$$

the total space of a sphere bundle associated with the complex $(n+1)$-plane bundle $\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n}$ over $P^1(\boldsymbol{C})$. Then there is a natural principal circle action φ on $S(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n})$ whose orbit space is $\boldsymbol{CP}(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n})$, the total space of a projective space bundle.

Proposition 5.2.

(1) $H^*(\boldsymbol{CP}(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n}))\cong\boldsymbol{Z}[c,x]/(x^2, c^{n+1}+(a_0+\cdots+a_n)xc^n)$, *where* deg $c=\deg x=2$, *and* c *is the Euler class of the canonical line bundle over* $\boldsymbol{CP}(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n})$,

(2) $H^*(S(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n}))\cong H^*(S^2\times S^{2n+1})$ if $n>0$,

(3) *If* $a_0+\cdots+a_n=1$ (mod 2), *then the principal circle action* φ *on* $S(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n})$ *does not bord even as unoriented principal smooth circle action.*

Proof. (1), (2) are clear from the cohomology ring structure of the projective space bundle ([2] p. 8, Proposition 3.1, 3.2). Next, assume $a_0+\cdots+a_n=1$ (mod 2), then

$$\langle\bar{c}^{n+1}, [\boldsymbol{CP}(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n})]_2\rangle\neq 0$$

where $\bar{c}$ is a modulo 2 reduction of the Euler class c. Thus the action φ does not bord as unoriented principal smooth circle action ([3] p. 47, Theorem 17.2). q.e.d.

REMARK. If $a_0+\cdots+a_n=1 \pmod 2$, $S(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n})$ is not the same homotopy type as a product $S^{2m}\times S^{2n+1}$, since $Sq^2u\neq 0$ for non-zero element $u\in H^{2n+1}(S(\xi_1^{a_0}\oplus\cdots\oplus\xi_1^{a_n}); \boldsymbol{Z}_2)$ by Lemma 4.1 (2).

5.3. There is a complex $(n+1)$-plane bundle ξ over S^{2m} with $\langle c_m(\xi), [S^{2m}]\rangle =(m-1)!$ for any $n+1\geqq m$ ([1]p. 349, Theorem 26.5 (a)).

Proposition 5.3. *Assume $n\geqq m>0$, then*

(1) $$H^*(\boldsymbol{CP}(\xi)) \cong \boldsymbol{Z}[c, x]/(x^2, c^{n+1}+(m-1)!\, xc^{n-m+1}),$$

where deg $c=2$, deg $x=2m$ *and c is the Euler class of the canonical line bundle over $\boldsymbol{CP}(\xi)$,*

(2) $$H^*(S(\xi)) \cong H^*(S^{2m}\times S^{2n+1}),$$

(3) *the natural principal circle action on $S(\xi)$ does not bord as an orientable principal smooth circle action.*

Proof. (1), (2) are clear. And

$$c^{n+m}=-(m-1)!\, xc^n\neq 0.$$

Hence (3) is obtained (see the proof of Theorem 1). q.e.d.

5.4. Let E be a topological space which is cohomologically a product $S^{2m}\times S^{2n+1}$ with $m>n\geqq 0$. Let $\pi: E\to M$ be a principal S^1-bundle over an orientable closed smooth manifold M. Then,

Proposition 5.4.
(1) *The integral cohomology ring $H^*(M)$ of M is*

$$H^*(M) = \boldsymbol{Z}[c, x]/(c^{n+1}, x^2), \text{ where deg } c = 2, \text{ deg } x = 2m$$

and the element c is the Euler class of the principal S^1-bundle $\pi: E\to M$
(2) *The Stiefel-Whitney classes of M are*

$$W_{2i+1}(M) = 0,\ W_{2i}(M) = a_i\bar{c}^i (a_i = 0, 1)$$

where $\bar{c}$ is a modulo 2 reduction of the Euler class c.

Proof. This is proved similarly as Lemma 1.1, but it makes Lemma 4.1 (1) necessary to determine the ring structure of $H^*(M)$. We leave it to the reader. q.e.d.

Proposition 5.5. *Let E be an orientable closed smooth manifold which is cohomologically a product* $S^{2m}\times S^{2n+1}$ *with* $m>n\geqq 0$. *Then any principal smooth circle action on E bords as unoriented princpal smooth circle action.*

Proof. By Proposition 5.4, all bordism Stiefel-Whitney numbers of an associated principal S^1-bundle vanish (see Theorem 1). Thus the result is obtained ([3] p. 47, Theorem 17.2). q.e.d.

REMARK. There is no principal smooth circle action on a compact smooth manifold whose each odd dimensional integral cohomology group is zero.

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