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EQUIVARIANT K-RING OF G-MANIFOLD (U(n), ad_G) I

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1. Introduction and statement of results

Let G be a compact connected Lie group and H a connected closed subgroup of G. We can consider G a differentiable H-manifold as follows. A differentiable H-action on G ad_H: $H\times G\to G$, called the adjoint operation of H on G, is defined by

$$ad_H(h, g) = hgh^{-1} \qquad h\in H, g\in G.$$

Then by (G, ad_H) we denote the manifold G together with the adjoint operation ad_H.

The purpose of this paper is to calculate K_H-group of (G, ad_H) for (G, ad_H) $=(U(n), ad_H)$ and $(SU(n), ad_H)$ when H is of maximal rank, where $U(n)$ and $SU(n)$ are the n-dimensional unitary group and special unitary group respectively.

Let G denote $U(n)$ or $SU(n)$ henceforth and V the standard n-dimensional G-module over the complex numbers $\boldsymbol{C}$. Moreover, when we regard the G-module V as an H-module, let $\underline{\underline{V}}$ denote a trivial H-vector bundle with a fibre V over G and $\lambda^i(\underline{\underline{V}})$ the i-th exterior power of $\underline{\underline{V}}$ for $i=1, 2, \cdots, n$. Then we can define an H-automorphism θ_i^H of $\lambda^i(\underline{\underline{V}})$ by

$$\theta_i^H(g, z) = (g, \lambda^i(g)(z)) \qquad g\in G, z\in\lambda^i(V)$$

which can be easily check to be compatible with the action of H on $\lambda^i(\underline{\underline{V}})$. Hence θ_i^H determines an element $[\lambda^i(\underline{\underline{V}}), \theta_i^H]$, which we shall also write θ_i^H, in $K_H^1(G, ad_H)$ (See [3]). In particular, $\theta_n^H=0$ in case of $(G, ad_H)=(SU(n), ad_H)$ because $\lambda^n(g)=\det g=1$ for any $g\in SU(n)$ and so the automorphism θ_n^H is the identity map of $\lambda^n(\underline{\underline{V}})$.

In this note we prove the following

Theorem 1. *When* $(G, ad_H)=(U(n), ad_H)$ *or* $(SU(n), ad_H)$ *and* H *is a connected closed subgroup of* G *which is of maximal rank,*

$$K_H^*(G, ad_H)\cong\Lambda_{R(H)}(\theta_1^H, \theta_2^H, \cdots, \theta_n^H)$$

as an algebra over $R(H)$ *where* $\theta_n^H=0$ *in case of* $(G, ad_H)=(SU(n), ad_H)$ *and* $R(H)$

is the complex character ring of H.

L. Hodgkin [2] has stated a more general case of this theorem without proof.

In the following sections we discuss only the case of $(G, ad_H)=(U(n), ad_H)$ as we can compute $K_H^*(G, ad_H)$ analogously in case of $(G, ad_H)=(SU(n), ad_H)$.

2. (T(n), α, U(n−1))-bundle

In this section, we prepare some results, which will be applied in §3.

The standard maximal torus $T(n)$ of $U(n)$ is

$$\left\{\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}\in U(n)\,|\,\lambda_i\in \boldsymbol{C},\, i=1, 2, \cdots, n\right\}.$$

Let ρ_i, $1\leq i\leq n$, be the 1-dimensional complex representations which are given by the i-th projection$T(n)\to U(1)$ defined by

$$\rho_i\begin{pmatrix}\lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n\end{pmatrix}=\lambda_i$$

and let us denote the representation space of $\rho_1^{-1}\rho_2\oplus\rho_1^{-1}\rho_3\oplus\cdots\oplus\rho_1^{-1}\rho_n$ by W.

We identify $U(n-1)$ with a subgroup $1\times U(n-1)$ of $U(n)$. Then $U(n-1)$ is a closed $T(n)$-invariant submanifold of $(U(n), ad_{T(n)})$ and hence the homogeneous space $U(n)/U(n-1)$ becomes a $T(n)$-manifold. When we denote the unit sphere of $\boldsymbol{C}\oplus W$ by $S(\boldsymbol{C}\oplus W)$, we can define a map

$$\pi : U(n) \rightarrow S(\boldsymbol{C}\oplus W)$$

by $\pi(A)=v_A$ for any $A\in U(n)$, where v_A is the 1^{st} column vector of A. Then π is a $T(n)$-equivariant map and furthermore induces a $T(n)$-isomorphism

$$U(n)/U(n-1)\approx S(\boldsymbol{C}\oplus W).$$

Now if we define a homomorphism

$$\alpha : T(n) \rightarrow Aut(U(n-1))$$

by
$$\alpha(t)(u) = tut^{-1} \qquad t\in T(n),\ u\in U(n-1),$$

then we see easily the following

Proposition 1 (*See* [4]). *$\pi : U(n)\rightarrow S(\boldsymbol{C}\oplus W)$ is a $(T(n), \alpha, U(n-1))$-bundle* *in the sense of T. tom Dieck.*

Put

$$S(C\oplus W) = \{(z_1, z_2, \cdots, z_n)\in \boldsymbol{C}^n \mid |z_1|^2+|z_2|^2+\cdots+|z_n|^2 = 1\}$$

and

$$D_1^{\pm} = \{(z_1, z_2, \cdots, z_n)\in S(\boldsymbol{C}\oplus W) \mid (1\pm 1)\pi \leq 2\arg z_1 \leq (3\pm 1)\pi\}$$

respectively. Then $D_1^{\pm}$ are closed $T(n)$-invariant subspaces of $S(C\oplus W)$ such that

$$S(\boldsymbol{C}\oplus W) = D_1^+ \cup D_1^- \tag{2.1}$$

and moreover since $T(n)$ acts on the 1^{st} vectors of n-tuples of $S(\boldsymbol{C}\oplus W)$ trivially, $D_1^{\pm}$ are $T(n)$-contractible to $(1, 0, \cdots, 0)$ respectively by $T(n)$-homotopies $H^{\pm}$: $D_1^{\pm}\times I \to D_1^{\pm}$ defined by

$$H^{\pm}((z_1, \cdots, z_n), t) = \begin{cases}(re^{\theta(t)i}, z_2, \cdots, z_n) & 0\leq t\leq \frac{1}{2}\\ (\sqrt{1-(2-2t)^2(1-r^2)}, (2-2t)z_2, \cdots, (2-2t)z_n) & \frac{1}{2}\leq t\leq 1\end{cases}$$

where $r=|z_1|$ and $\theta(t)=(1-2t)\arg z_1+(2\pi\pm 2\pi)t$. Therefore the restrictions of $\pi : U(n)\to S(\boldsymbol{C}\oplus W)$ onto $D_1^{\pm}$ are trivial $(T(n), \alpha, U(n-1))$-bundles over $D_1^{\pm}$ from the homotopy theorem of [4], §4 and so there exist isomorphisms of $(T(n), \alpha, U(n-1))$-bundles

$$\delta^{\pm} : \pi^{-1}(D_1^{\pm}) \approx D_1^{\pm}\times U(n-1). \tag{2.2}$$

Then we see that $\delta^{\pm}$ induce isomorphisms

$$\begin{aligned} K^*_{T(n)}(\pi^{-1}(D_1^{\pm})) &\cong K^*_{T(n)}(D_1^{\pm}\times U(n-1)) \\ &\cong K^*_{T(n)}(D_1^{\pm}) \underset{R(T(n))}{\otimes} K^*_{T(n)}(U(n-1)) \end{aligned} \tag{2.3}$$

by the $T(n)$-contractibility of $D_1^{\pm}$.

Next we devide $D_1^+\cap D_1^-$ into two closed $T(n)$-contractible subspaces $D_2^{\pm}$ where

$$D_2^{\pm} = \{(r, z_2, \cdots, z_n)\in S(\boldsymbol{C}\oplus W) \mid r\in \boldsymbol{R},\ \pm r\geq 0\}.$$

Then

$$D_1^+\cap D_1^- = D_2^+\cup D_2^- \quad \textit{and} \quad D_2^-\cap D_2^- = S(W) \tag{2.4}$$

as $T(n)$-spaces. The restrictions of δ^+ (or δ^-) onto $D_2^{\pm}$ and $S(W)$ are $T(n)$-isomorphisms

$$\pi^{-1}(D_2^{\pm}) \approx D_2^{\pm}\times U(n-1) \ \text{ and } \ \pi^{-1}(S(W)) \approx S(W)\times U(n-1)$$

and induce isomorphisms

$$(2.5)\qquad K^*_{T(n)}(\pi^{-1}(D^\pm_2)) \cong K^*_{T(n)}(D^\pm_2\times U(n-1)) \cong K^*_{T(n)}(D^\pm_2) \underset{R(T(n))}{\otimes} K^*_{T(n)}(U(n-1))$$

since $D^\pm_2$ are $T(n)$-contractible.

Here we consider the following diagram

$$\begin{array}{ccc} K^*_{T(n)}(P) \underset{R(T(n))}{\otimes} K^*_{T(n)}(U(n-1)) & \xrightarrow{\varphi_{1*}\otimes 1} & K^*_{T(n)}(W) \underset{R(T(n))}{\otimes} K^*_{T(n)}(U(n-1)) \\ \xi_1\downarrow & & \downarrow\xi_2 \\ K^*_{T(n)}(P\times U(n-1)) & \xrightarrow[\varphi_{2*}]{} & K^*_{T(n)}(W\times U(n-1)) \end{array}$$

in which φ_{1*} and φ_{2*} are the Thom isomorphisms for trivial $T(n)$-vector bundles $W\to P(=\text{a point})$ and $W\times U(n-1)\to U(n-1)$ respectively and ξ_i, $i=1, 2$, the homomorphisms induced by the external products. Then, since the diagram is commutative and ξ_1 is an isomorphism we see

$$(2.6)\qquad \xi_2 : K^*_{T(n)}(W) \underset{R(T(n))}{\otimes} K^*_{T(n)}(U(n-1)) \to K^*_{T(n)}(W\times U(n-1))$$

is an isomorphism.

Finally we prove the following

Lemma 1. *$K^*_{T(n)}(S(\boldsymbol{C}\oplus W))$ is an exterior algebra over $R(T(n))$ with one generator g satisfying*

$$\pi^*(g) = \textstyle\sum_{i=1}^n(-1)^i\rho_1^{-i}\theta_i^{T(n)}.$$

Proof. We observe the exact sequence of the pair $(D(\boldsymbol{C}\oplus W), S(\boldsymbol{C}\oplus W))$ where $D(\boldsymbol{C}\oplus W)$ is the unit disk of $\boldsymbol{C}\oplus W$. Then we see that $K^1_{T(n)}(S(\boldsymbol{C}\oplus W))$ is a free $R(T(n))$-module generated by $\delta^{-1}\lambda_{\boldsymbol{C}\oplus W}$ from the exact sequence

$$\begin{array}{ccc} 0 = K^1_{T(n)}(D(\boldsymbol{C}\oplus W)) \to K^1_{T(n)}(S(\boldsymbol{C}\oplus W)) & \xrightarrow{\delta} & K^0_{T(n)}(\boldsymbol{C}\oplus W) \\ & & \uparrow\varphi_* \\ \to \tilde{K}^0_{T(n)}(D(\boldsymbol{C}\oplus W))=0 & & K^0_{T(n)}(P) \end{array}$$

where δ is a coboundary homomorphism, φ_* the Thom isomorphism for the trivial $T(n)$-vector bundle $\boldsymbol{C}\oplus W\to P(=\text{a point})$ and $\lambda_{\boldsymbol{C}\oplus W}=\varphi_*(1)$, and also we get

$$\tilde{K}^0_{T(n)}(S(\boldsymbol{C}\oplus W)) = 0$$

since $\tilde{K}^0_{T(n)}(D(\boldsymbol{C}\oplus W))=K^1_{T(n)}(\boldsymbol{C}\oplus W)=0$. Therefore, (2.7) $$K^*_{T(n)}(S(C\oplus W)) \cong \Lambda_{R(T(n))}(\delta^{-1}\lambda_{C\oplus W})$$

as an algebra over $R(T(n))$.

Now $\sum_{i=0}^{[n/2]}\oplus\lambda^{2i}(C\oplus W)$ and $\sum_{i=0}^{[n/2]}\oplus\lambda^{2i+1}(C\oplus W)$

are isomorphic as $T(n)$-modules where $\lambda^j(C\oplus W)$ denotes the j-th exterior power of $C\oplus W$ for $j=0, 1, \cdots, n$. Because,

$$\begin{aligned}&\textit{the character of } \textstyle\sum_{i=0}^{[n/2]}[\lambda^{2i}(C\oplus W)] - \sum_{i=0}^{[n/2]}[\lambda^{2i+1}(C\oplus W)]\\ &=(1-1)(1-\rho_1^{-1}\rho_2)(1-\rho_1^{-1}\rho_3)\cdots(1-\rho_1^{-1}\rho_n) = 0\end{aligned}$$

where the brackets denote the isomorphism classes of $T(n)$-modules. So we identify the above two $T(n)$-modules and describe it M.

Let $\rho : U(n)\to U(n)$ be the identity homomorphism and $\lambda^i\rho$ the j-th exterior power of ρ for $j=0, 1, \cdots, n$, and let us denote

$$\textstyle\sum_{i=0}^{[n/2]}\oplus\lambda^{2i}\rho \quad \text{and} \quad \sum_{i=0}^{[n/2]}\oplus\lambda^{2i+i}\rho : U(n) \to U(2^{n-1})$$

by α and β respectively. Then we can define a map

$$\gamma : U(n)/U(n-1) \to U(2^{n-1})$$

by $$\gamma(hU(n-1)) = \alpha(h)\beta(h)^{-1} \qquad h\in U(n)$$

because α and β agree on $U(n-1)$ and so a $T(n)$-automorphism $\tilde{\gamma}$ of M by

$$\tilde{\gamma}(hU(n-1), v) = (hU(n-1), \gamma(h)(v)) \qquad h\in U(n),\ v\in M.$$

Therefore $\tilde{\gamma}$ determines an element $[\underline{\underline{M}}, \tilde{\gamma}]$ in $K^1_{T(n)}(S(C\oplus W))$ since $S(C\oplus W)\approx U(n)/U(n-1)$. This element satisfys the condition we require. Because if we denote by V' the representation space of $\rho_1\oplus\rho_2\oplus\cdots\oplus\rho_n$ of $T(n)$, then

$$\begin{aligned}\pi^*[\underline{\underline{M}}, \tilde{\gamma}] &= [\underline{\underline{M}}, \tilde{\alpha}]-[\underline{\underline{M}}, \tilde{\beta}]\\ &= \textstyle\sum_{i=0}^{[n/2]}[\lambda^{2i}(\underline{\underline{C\oplus W}}), \tilde{\lambda}^{2i}(\rho)]-\sum_{i=0}^{[n/2]}[\lambda^{2i+1}(\underline{\underline{C\oplus W}}), \tilde{\lambda}^{2i+1}(\rho)]\\ &= \textstyle\sum_{i=0}^{[n/2]}\rho_1^{-2i}[\lambda^{2i}(\underline{\underline{V'}}), \tilde{\lambda}^{2i}(\rho)]-\sum_{i=0}^{[n/2]}\rho_1^{-2i-1}[\lambda^{2i+1}(\underline{\underline{V'}}), \tilde{\lambda}^{2i+1}(\rho)]\end{aligned}$$

and since $\theta_i^{T(n)}=[\lambda^i(\underline{\underline{V'}}), \tilde{\lambda}^i(\rho)]$ by the definition of $\theta_i^{T(n)}$ for $i=1, 2, \cdots, n$ we have

$$\begin{aligned}\pi^*[\underline{\underline{M}}, \tilde{\gamma}] &= \textstyle\sum_{i=0}^{[n/2]}\rho_1^{-2i}\theta_{2i}^{T(n)}-\sum_{i=0}^{[n/2]}\rho_1^{-2i-1}\theta_{2i+1}^{T(n)}\\ &= \textstyle\sum_{i=0}^{n}(-1)^i\rho_1^{-i}\theta_i^{T(n)}\end{aligned}$$

where the definition of $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\lambda}^i(\rho)$, $0\leq i\leq n$, are similar to that of $\tilde{\gamma}$. Terefore a proof of $[\underline{\underline{M}}, \tilde{\gamma}]=\delta^{-1}\lambda_{C\oplus W}$ concludes Lemma 1.

Let $f : K^1_{T(n)}(S(C\oplus W))\to K^1(S(C\oplus W))$ be the forgetful homomorphism and $j^* : K_{T(n)}(C\oplus W)\to K_{T(n)}(C)$ a homomorphism induced by the natural inclusion map $j : C\to C\oplus W$. When we forget the action of $T(n)$, we have

$$\delta^{-1}\lambda_{C\oplus W} = [\underline{\underline{M}}, \tilde{\gamma}]$$

from [1], p. 115. Namely

$$f(\delta^{-1}\lambda_{C\oplus W}) = f([\underline{\underline{M}}, \tilde{\gamma}]).$$

Hence, since $K^1_{T(n)}(S(\boldsymbol{C}\oplus W))$ is a free $R(T(n))$-module generated by $\delta^{-1}\lambda_{C\oplus W}$ according to (2. 7), there exists an element r of $R(T(n))$ satisfying

(2.8) $$r(\delta^{-1}\lambda_{C\oplus W}) = [\underline{\underline{M}}, \tilde{\gamma}]$$

and $$r = 1 \mod \tilde{R}(T(n))$$

where $\tilde{R}(T(n))$ is the reduced character ring of $T(n)$.

Next we consider the j^*-image of the two elements $\lambda_{C\oplus W}$ and $\delta([\underline{\underline{M}}, \tilde{\gamma}])$. If we compute $j^*\lambda_{C\oplus W}$ and $j^*\delta([\underline{\underline{M}}, \tilde{\gamma}])$ directly by using the technique of the proof of [1], Lemma 2.6.10, then we obtain

(2.9) $$\begin{aligned} j^*\lambda_{C\oplus W} &= j^*\delta([\underline{\underline{M}}, \tilde{\gamma}]) \\ &= -\textstyle\sum_{i=0}^{n}(-1)^i\lambda^i(W)\lambda_C \end{aligned}$$

where λ_C is the Thom element for the trivial $T(n)$-vector bundle $\boldsymbol{C}\to a$ point. Therefore

$$(r-1)\textstyle\sum_{i=0}^{n}(-1)^i\lambda^i(W)\lambda_C = 0$$

follows from (2.8) and (2.9). Now, since $K_{T(n)}(\boldsymbol{C})$ is a free $R(T(n))$-module generated by λ_C and

$$\textstyle\sum_{i=0}^{n}(-1)^i\lambda^i(W) = (1-\rho_1^{-1}\rho_2)\cdots(1-\rho_1^{-1}\rho_n)$$

is non zero element of $R(T(n))$, we get

$$r = 1.$$

This shows

$$\delta^{-1}\lambda_{C\oplus W} = [\underline{\underline{M}}, \tilde{\gamma}].$$

q. e. d.

3. $K^*_{T(n)}(U(n), ad_{T(n)})$

In this section we give a proof of Theorem 1 in case of $H=T(n)$ by induction on n. For convenience we denote $(U(n), ad_{T(n)})$ by $(U(n), ad)$ and $\theta_j^{T(n)}$ by $\theta_j(n)$, $1\leq i\leq n$. Then the theorem is as follows.

Theorem 2. *$K^*_{T(n)}(U(n), ad)\cong\Lambda_{R(T(n))}(\theta_1(n), \theta_2(n), \cdots, \theta_n(n))$ as an algebra over $R(T(n))$.*

Proof. In case of $n=1$, since $(U(1), ad)$ is trivial $T(1)$-space, we have

$$K^*_{T(1)}(U(1), ad) \cong R(T(1))\otimes K^*(U(1))$$

from [3], Proposition 2.2 and since $K^*(U(1))$ is the exterior algebra with one generator $\theta_1(1)$, we get

$$K^*_{T(1)}(U(1), ad) \cong \Lambda_{R(T(1))}(\theta_1(1)).$$

Suppose the assertion is true for $n=k-1$. When we put $T(k)=U(1)\times T(k-1)$, the action of $U(1)$ on $U(k-1)(=1\times U(k-1))$ is trivial. So we have

$$K^*_{T(k)}(U(k-1)) \cong R(U(1)) \otimes K^*_{T(k-1)}(U(k-1))$$

(This is shown by a parallel argument to the proof of [3], Proposition 2. 2). This formula and the inductive hypothesis imply

$$(3.1) \qquad K^*_{T(k)}(U(k-1)) = \Lambda_{R(T(k))}(\theta_1(k-1), \theta_2(k-1),\cdots, \theta_{k-1}(k-1)).$$

As (3.1) shows that $K^*_{T(k)}(U(k-1))$ is a free $R(T(k))$-module, $K^*_{T(k)}(X) \underset{R(T(k))}{\otimes} K^*_{T(k)}(U(k-1))$ becomes an equivariant cohomology theory for $T(k)$-spaces X. We denote this cohomology theory by $h^*_{T(k)}(X)$. $K^*_{T(k)}(X\times U(k-1))$ is another equivariant cohomology theory. So we observe a natural transformation

$$\xi : h^*_{T(k)}(X) \to K^*_{T(k)}(X\times U(k-1))$$

of equivariant cohomology theories induced by the external products.

If we apply the five lemma to the exact sequences for the pair of the unit disk $D(W)$ and the unit sphere $S(W)$ of W in the two cohomology theories $h^*_{T(k)}(X)$ and $K^*_{T(k)}(X\times U(k-1))$, then it follows from (2.6) that

$$(3.2) \qquad \xi : h^*_{T(k)}(S(W)) \to K^*_{T(k)}(S(W)\times U(k-1))$$

is an isomorphism.

Here we consider the following commutative diagram

$$\begin{array}{ccc}
\to h^*_{T(k)}(D_2^+\cup D_2^-) & \longrightarrow & h^*_{T(k)}(D_2^+)\oplus h^*_{T(k)}(D_2^-) \to \\
\downarrow \xi & & \downarrow \xi\oplus\xi \\
\to K^*_{T(k)}((D_2^-\cup D_2^-)\times U(k-1)) & \to & K^*_{T(k)}(D_2^+\times U(k-1))\oplus K^*_{T(k)}(D_2^-\times U(k-1)) \to
\end{array}$$

$$\begin{array}{c}
h^*_{T(k)}(D_2^+\cap D_2^-) \to \\
\downarrow \xi \\
K^*_{T(k)}((D_2^+\cap D_2^-)\times U(k-1)) \to
\end{array}$$

where the rows are the Mayer-Vietoris sequences for the pair (D_2^+, D_2^-). Then

(2.5) and (3.2) shows that the 2^{nd} nd and 3^{rd} homomorphisms $\xi\oplus\xi$ and ξ are isomorphisms respectively since $D_2^+\cap D_2^-=S(W)$ by (2.4). So applying the five lemma, we see that the 1^{st} homomorphism ξ is an isomorphism and so since $D_1^+\cap D_1^-=D_2^+\cup D_2^-$ by (2.4)

(3.3) $\xi: K^*_{T(k)}(D_1^+\cap D_1^-)\underset{R(T(k))}{\otimes} K^*_{T(k)}(U(k-1)) \to K^*_{T(k)}((D_1^+\cap D_1^-)\times U(k-1))$

is an isomorphism.

Let $j: U(k-1)\to U(k)$ be the canonical inclusion of $U(k-1)$ and $j^*: K^*_{T(k)}(U(k))\to K^*_{T(k)}(U(k-1))$ the homomorphism induced by j. Then we get

$$(3.4)\qquad \begin{aligned} j^*\theta_1(k) &= \theta_1(k-1) \\ j^*\theta_i(k) &= \theta_i(k-1)+\rho_1\theta_{i-1}(k-1),\ k-1\geq i\geq 2 \\ j^*\theta_k(k) &= \rho_1\theta_{k-1}(k-1) \end{aligned}$$

easily.

Let $\mathfrak{M}^*$ be the free Z_2-graded module over $R(T(k))$ generated by

$$1 \qquad \text{and} \qquad \theta_{i_1}(k)\theta_{i_2}(k)\cdots\theta_{i_s}(k),\ 1\leq i_1<\cdots<i_s\leq k-1.$$

Then from (3.1) and (3.4) we see

(3.5) $\mathfrak{M}^*$ *is isomorphic to* $K^*_{T(k)}(U(k-1))$ *as an* $R(T(k))$*-module by the correspondence*

$$\theta_1(k)\to\theta_1(k-1) \text{ and } \theta_i(k)\to\theta_i(k-1)+\rho_1\theta_{i-1}(k-1),\ i=2,3,\cdots,k-1.$$

Now we can define a homomorphism

$$\lambda: K^*_{T(k)}(X)\underset{R(T(k))}{\otimes}\mathfrak{M}^*\to K^*_{T(k)}(\pi^{-1}(X))$$

by

$$\lambda(x\otimes v) = \pi^*(x)i^*(v) \qquad x\in K^*_{T(k)}(X),\ v\in\mathfrak{M}^*$$

for any closed $T(k)$-invariant subspace X of $S(C\oplus W)$ where $i:\pi^{-1}(X)\to U(k)$ is the inclusion of $\pi^{-1}(X)$. In particular we see

(3.6) *When* $X=D_1^{\pm}$ *or* $D_1^+\cap D_1^-$, λ *is an isomorphism.*

A proof of (3,6) is as follows: We consider the following diagram for $X=D_1^{\pm}$ or $D_1^+\cap D_1^-$

$$\begin{array}{ccc} K^*_{T(k)}(X)\underset{R(T(k))}{\otimes}\mathfrak{M}^* & \xrightarrow{\lambda} & K^*_{T(k)}(\pi^{-1}(X)) \\ \downarrow{\scriptstyle 1\otimes\mu} & & \uparrow{\scriptstyle\tau} \\ K^*_{T(k)}(X)\underset{R(T(k))}{\otimes}K^*_{T(k)}(U(k-1)) & \xrightarrow{\xi} & K^*_{T(k)}(X\times U(k-1)) \end{array}$$

where μ denotes the isomorphism of (3.5) and τ the isomorphism induced by δ^+ or δ^-, and first we show the commutativity of this diagram. We have

$$\lambda^*(x\otimes 1) = \tau\xi(x\otimes 1) \qquad \text{for any } x\in K^*_{T(k)}(X)$$

since $\delta^{\pm}$ are the bundle homomorphisms and

$$\lambda(1\otimes v) = \tau\xi(1\otimes\mu(v)) \qquad \text{for any } v\in\mathfrak{M}^*$$

in case of $X=D_1^{\pm}$ from the $T(k)$-contractibility of $D_1^{\pm}$ and also when we observe the restriction of this formula to $K^*_{T(k)}(\pi^{-1}(D_1^+\cap D_1^-))$, we get the same formula in case of $X=D_1^+\cap D_1^-$.

Then,
$$\begin{aligned}\lambda(x\otimes v) &= \lambda((x\otimes 1)(1\otimes v))\\ &= \lambda(x\otimes 1)\lambda(1\otimes v)\\ &= \tau\xi(x\otimes 1)\tau\xi(1\otimes\mu(v))\\ &= \tau\xi(x\otimes\mu(v))\\ &= \tau\xi(1\otimes\mu)(x\otimes v) \qquad x\in K^*_{T(k)}(X),\ v\in\mathfrak{M}^*.\end{aligned}$$

This shows that the above diagram is commutative. Therefore we obtain (3.6) from (2.3) and (3.3).

Thus, by applying the five lemma in the following commutative diagram

$$\begin{array}{ccc}
\to K^*_{T(k)}(D_1^+\cup D_1^-)\underset{R(T(k))}{\otimes}\mathfrak{M}^* & \to & K^*_{T(k)}(D_1^+)\underset{R(T(k))}{\otimes}\mathfrak{M}^*\oplus K^*_{T(k)}(D_1^-)\underset{R(T(k))}{\otimes}\mathfrak{M}^* \\
\downarrow\lambda & & \downarrow\lambda\oplus\lambda \\
\to K^*_{T(k)}(\pi^{-1}(D_1^+\cup D_1^-)) & \longrightarrow & K^*_{T(k)}(\pi^{-1}(D_1^+))\oplus K^*_{T(k)}(\pi^{-1}(D_1^-))
\end{array}$$

$$\begin{array}{c}
\to K^*_{T(k)}(D_1^+\cap D_1^-)\underset{R(T(k))}{\otimes}\mathfrak{M}^*\to \\
\downarrow\lambda \\
\to K^*_{T(k)}(\pi^{-1}(D_1^+\cap D_1^-))\to
\end{array}$$

where the rows are the Mayer-Vietoris sequences for the pairs $(\pi^{-1}(D_1^+), \pi^{-1}(D_1^-))$ and (D_1^+, D_1^-) respectively, we see that the 1^{st} homomorphism λ is an isomorphism and since $S(C\oplus W)=D_1^+\cup D_1^-$ by (2.1) we see

$$(3.7) \qquad \lambda : K^*_{T(k)}(S(C\oplus W))\underset{R(T(k))}{\otimes}\mathfrak{M}^*\to K^*_{T(k)}(U(k))$$

is an isomorphism.

From Lemma 1 and (3.7), it follows that $K^*_{T(k)}(U(k), ad)$ is an exterior algebra over $R(T(k))$ generated by $\theta_1(k)$, $\theta_2(k)$, $\cdots$, $\theta_k(k)$ as required. This completes that induction. q.e.d.

4. $K_T^*(X)^{W(H)}$

Let H be a compact connected Lie group and $i : T\to H$ the inclusion of a maximal torus. Then from [3], Proposition 3. 8 we see that $i^* : K_H^*(X)\to K_T^*(X)$ is injective for any compact H-space X.

Here we define an action of the Weyl group $W(H)(=N(T)/T)$ on $K_T^*(X)$ where $N(T)$ is a normalizer of T. Let $\pi : E\to X$ be a T-vector bundle over an H-space X. For each $n\in N(T)$, n^*E admits a T-vector bundle structure if we regard n as a continuous map $n : X\to X$ by its action on X. Namely we can define a T-action on $n^*E : T\times n^*E\to n^*E$ by

$$(t, (x, u))\to(tx, ntn^{-1}(u)) \qquad \text{for } t\in T,\ x\in X \text{ and } u\in E_{nx}.$$

In particular, if $n\in T$, then n^*E and E are isomorphic by a T-isomorphism n^{-1}. So the operation of $W(H)$ on $K_T(X) : W(H)\times K_T(X)\to K_T(X)$ is defined by $(nT, [E])\to[n^*E]$. Further if E is an H-vector bundle, then n^*E admits an H-vector bundle structure, and n^*E and E are siomorphic by an H-isomorphism n^{-1}. Similarly we can define the operation of $W(H)$ on $K_T^1(X)$: $W(H)\times K_T^1(X)\to K_T^1(X)$ by $(nT, [E, \alpha])\to[n^*E, n^*\alpha]$ and if (E, α) is a pair of an H-vector bundle over X and an H-automorphism of it, then $(n^*E, n^*\alpha)$ and (E, α) are isomorphic by an H-isomorphism n^{-1}. Thus we see

Lemma 2. *$i^* : K_H^*(X)\to K_T^*(X)$ is an injection into $K_T^*(X)^{W(H)}$ which is a submodule of $K_T^*(X)$ consisting of invariant elements under the action of $W(H)$.*

Proof of Theorem 1: Let T be a maximal torus of H. Since the rank of H is n, T is conjugate to $T(n)$ in $U(n)$. Therefore we have

$$K_T^*(G, ad_T) \cong K_{T(n)}^*(G, ad_{T(n)})$$

and so

$$K_T^*(G, ad_T) \cong \Lambda_{R(T)}(\theta_1^T, \theta_2^T, \cdots, \theta_n^T)$$

from Theorem 2. Thus

$$\begin{aligned} K_T^*(G, ad_T)^{W(H)} &\cong \Lambda_{R(T)^{W(H)}}(\theta_1^T, \theta_2^T, \cdots, \theta_n^T) \\ &\cong \Lambda_{R(H)}(\theta_1^T, \theta_2^T, \cdots, \theta_n^T). \end{aligned}$$

From this formula and Lemma 2, we see that $K_H^*(G, ad_H)$ and $K_T^*(G, ad_T)^{W(H)}$ are isomorphic because of $i^*\theta_j^H=\theta_j^T$ for $j=1, 2,\cdots, n$. This shows that Theorem 1 is true. q.e.d.

REMARK. In particular, we see that if $G=U(n)$ or $SU(n)$, $H=G$ and $K_T^*(X)$ is torsion free for a compact G-space X, then $K_G^*(X)$ and $K_T^*(X)^{W(G)}$ are isomorphic.

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