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A SPECTRAL SEQUENCE ASSOCIATED WITH A COHOMOLOGY THEORY OF INFINITE CW-COMPLEXES

SHÔRÔ ARAKI AND ZEN-ICHI YOSIMURA

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Introduction. Let h be an additive cohomology theory and X a CW-complex given with a filtration

$$X_0 \subset X_1 \subset \cdots \subset X_i \subset \cdots, \quad \cup X_i = X,$$

by subcomplexes. Milnor [5] established a short exact sequence

$$0 \to \varprojlim{}^1 h^{n-1}(X_i) \to h^n(X) \to \varprojlim h^n(X_i) \to 0$$

for each degree n. In the present paper the authors will give a version of the above exact sequence for the more general situation, *i.e.*, X is given with a direct system of subcomplexes X_α such that $X = \cup X_\alpha$. The result will be given in a form of a spectral sequence (Theorem 2).

In §1 we construct classifying spaces of direct systems of CW-complexes which behave as a generalization of Milnor's telescope constructions (Theorem 1). In §2 we summarize some basic facts needed in the sequel. In §3 we discuss some convergence conditions of certain spectral sequences. In §4 we construct the spectral sequences mentioned above (Theorem 2) and discuss their convergences under some assumptions on h. As a corollary we obtain Anderson's version of Milnor's short exact sequence [2].

All categories in the present work are small categories.

1. Classifying spaces

1.1. Let C be a category. As is customary we associate with C a semi-simplicial complex $C_* = \{C_0, C_1, \cdots, C_n, \cdots\}$ as follows: an n-simplex is a sequence

$$\sigma = \{X_0, \cdots, X_n; f_1, \cdots, f_n\}$$

of $n+1$ objects X_i, $0 \leqq i \leqq n$, and n morphisms f_j, $1 \leqq j \leqq n$, such that $f_j : X_{j-1} \to X_j$; i-th faces $F_i\sigma$, $0 \leqq i \leqq n$, of the n-simplex σ are $(n-1)$-simplexes defined by

$$F_0\sigma = \{X_1, \cdots, X_n; f_2, \cdots, f_n\},$$
$$F_i\sigma = \{X_0, \cdots, X_{i-1}, X_{i+1}, \cdots, X_n; f_1, \cdots, f_{i-1}, f_{i+1}f_i, f_{i+2}, \cdots, f_n\}, 0<i<n,$$
$$F_n\sigma = \{X_0, \cdots, X_{n-1}; f_1, \cdots, f_{n-1}\};$$

i-th degeneracies $D_i\sigma$, $0\leqq i\leqq n$, of the n-simplex σ are $(n+1)$-simplexes defined by

$$D_i\sigma = \{X_0, \cdots, X_i, X_i, \cdots, X_n; f_1, \cdots, f_i, 1, f_{i+1}, \cdots, f_n\}, 0\leqq i\leqq n.$$

$\mathcal{C}_n$ is the set of all n-simplexes of $\mathcal{C}$. Thus $\mathcal{C}_0=\text{obj}\,\mathcal{C}$ and $\mathcal{C}_1=\text{morph}\,\mathcal{C}$.

1.2. As usual we regard a category as a functor defined on an index category. An index category $\mathcal{J}$ is said to be *ordered* when Hom (α, β) consists at most of a single element for any $\{\alpha, \beta\}\subset\text{obj}\,\mathcal{J}$ and $\alpha=\beta$ whenever Hom $(\alpha, \beta)\neq\phi$ and Hom $(\beta, \alpha)\neq\phi$; then the set obj $\mathcal{J}$ is ordered as usual: $\alpha<\beta$ if and only if Hom $(\alpha, \beta)\neq\phi$ and Hom $(\beta, \alpha)=\phi$. A category $\mathcal{C}$ is an ordered system when its index category is ordered.

The *classifying spaces* of categories were discussed by Segal [9]. When a category $\mathcal{C}$ is ordered all faces of a non-degenerate simplex of $\mathcal{C}$ are non-degenerate. Hence, to construct the classifying space $B\mathcal{C}$ of $\mathcal{C}$ it is sufficient to use non-degenerate simplexes and identifications with respect to face operations only.

Let $\mathcal{C}$ be an ordered system of based topological spaces. For each n-simplex σ of $\mathcal{C}$ we associate a space X_σ by

$$X_\sigma = X_0, \text{ the leading vertex of } \sigma.$$

Let $\mathcal{C}'_n$ denote the set of all non-degenerate n-simplexes of $\mathcal{C}$ and put

$$\overline{B\mathcal{C}}_n = \bigvee_{\sigma\in\mathcal{C}'_n} (X_\sigma\wedge\Delta^{n,+})$$

where $\Delta^{n,+}$ is the standard ordered n-simplex (closed) added with a point at infinity (base point). Form one-point union

$$\overline{B\mathcal{C}} = \overline{B\mathcal{C}}_0\vee\overline{B\mathcal{C}}_1\vee\cdots\vee\overline{B\mathcal{C}}_n\vee\cdots.$$

Define continuous maps

$$\varphi_{i,\sigma}: X_\sigma \to X_{F_i\sigma},$$

$0\leqq i\leqq n$, for each n-simplex σ by

$$\varphi_{0,\sigma} = f_1 \text{ and } \varphi_{i,\sigma} = 1 \text{ for } 0<i\leqq n$$

$(f_1: X_0\to X_1)$; define relations

$$(x, F_i u) \sim (\varphi_{i,\sigma}\, x, u) \qquad \text{for } x \in X_\sigma \text{ and } u \in \Delta^{n-1},$$

where $F_i : \Delta^{n-1} \to \Delta^n$, $0 \leqq i \leqq n$, is the standard i-th face map; extend these relations and trivial relations to an equivalence relation $\sim$ in $\overline{BC}$. We define BC as the quotient space

$$BC = \overline{BC}/\sim.$$

1.3. Let τ be a non-degenerate m-simplex of C and σ a face of τ ($\dim \sigma = n$). Remark that the way to embed σ as a face of τ is unique; hence we have a unique face map

$$F_{\sigma,\tau} : \Delta^n \to \Delta^m$$

and its corresponding map

$$f_{\tau,\sigma} : X_\tau \to X_\sigma$$

defined by

$$f_{\tau,\sigma} = f_i^\tau f_{i-1}^\tau \cdots f_1^\tau$$

when $\tau = \{X_0, \cdots, X_m; f_1^\tau, \cdots, f_m^\tau\}$ and $X_\sigma = X_i$.

Let

$$\pi : \overline{BC} \to BC$$

be the projection. For each $\sigma \in C_n'$ put

$$X_\sigma^- = X_\sigma - \{\text{base point}\}.$$

Then we have a decomposition

$$(1.1) \quad BC = \{\text{base point}\} \cup \{\bigcup_{n \geqq 0} \bigcup_{\sigma \in C_n'} \pi(X_\sigma^- \times \operatorname{Int}\Delta^n)\}$$

into a disjoint union, and $\pi | X_\sigma^- \times \operatorname{Int} \Delta^n$ is one-one.

Lemma 1. *BC is a Hausdorff space if all objects X_α of C are Hausdorff.*

Proof. For each point $u \in \Delta^n$ we define its ε-neighborhood in Δ^n by making use of barycentric coordinates as

$$U_\varepsilon(u) = \{v = (v_0, \cdots, v_n) \in \Delta^n; |u_i - v_i| < \varepsilon, 0 \leqq i \leqq n\},$$

where $u = (u_0, \cdots, u_n)$ and $\varepsilon > 0$. ε-neighborhoods of subsets of Δ^n are similarly defined.

We construct certain neighborhoods of points of BC. Suppose $p = \pi(x, u)$,

$(x, u) \in X_\sigma^- \times \text{Int } \Delta^n$; choose an open neighborhood V of x and $\varepsilon > 0$ so small that $U_\varepsilon(u) \subset \text{Int } \Delta^n$ when $n > 0$; then the set

$$\bigcup_\tau f_{\tau,\sigma}^{-1} V \times U_\varepsilon(F_{\sigma,\tau} u)$$

(where the union $\bigcup_\tau$ runs over all non-degenerate simplexes containing σ as a face) is a saturated open set of $\overline{BC}$, hence its π-image is a neighborhood of p. As to neighborhoods of the base point of BC, choose an open neighborhood V_α of the base point of each objects X_α of C, and for each simplex $\sigma = \{X_{\alpha_0}, \cdots, X_{\alpha_n}; \cdots\}$ we put

$$V_\sigma = \bigcap_{i=0}^n f_{\sigma,\{\alpha_i\}}^{-1} V_{\alpha_i} \subset X_\sigma .$$

Choose $\varepsilon > 0$; for each non-degenerate simplex σ the set

$$W(\sigma; \varepsilon) = \bigcup_\tau f_{\sigma,\tau}^{-1} V_\tau \wedge U_\varepsilon(\text{Im } F_{\tau,\sigma})^+$$

(where the union $\bigcup_\tau$ runs over all faces τ of σ) is an open neighborhood of the base point of X_σ. Now the union

$$\bigvee_\sigma W(\sigma; \varepsilon)$$

taken over all non-degenerate simplexes σ is a saturated open set as is easily seen, and its π-image is a neighborhood of the base point of BC.

By suitable choices of neighborhoods of the above types it is now easy to see that BC is Hausdorff under the assumption of the lemma.

By a *k-space* we mean a Hausdorff space with compactly generated topology (cf., [10]).

Proposition 2. *Let C be an ordered system of based k-spaces, then BC is a k-space.*

Obviously $\overline{BC}$ is a k-space and BC is Hausdorff by the above lemma. Thus the proposition follows from [10], 2.6.

Corollary 3. *Let C be an ordered system of based CW-complexes and cellular maps, then BC is a CW-complex.*

1.4. Suppose C is an ordered system of based k-spaces and put

$$BC_n = \pi(\overline{BC}_0 \vee \cdots \vee \overline{BC}_n)$$

for each $n \geqq 0$. As is easily seen $\pi^{-1} BC_n$ is closed in $\overline{BC}$, hence BC_n is a k-space and we have a filtration

$$BC_0 \subset BC_1 \subset \cdots \subset BC_n \subset \cdots, \ \bigcup BC_n = BC, \tag{1.2}$$

of BC by closed subspaces. The topology of BC is the same as the weak topology with respect to this sequence. When C is a system of based CW-complexes and cellular maps, (1.2) is a filtration by subcomplexes.

Let C' be a subsystem of C. The inclusion $C' \subset C$ induces a one-one map

$$BC' \to BC$$

and π-inverse images of closed sets of BC' are closed in $\overline{BC}$ as is easily seen. Hence BC' is a closed subspace of BC.

1.5. Let C be a direct system of based k-spaces, *i.e.*, its index category is directed. Let $\{C_\gamma, \gamma \in \Gamma\}$ be the set of all finite sub direct systems of C. Then it is directed by inclusions and every simplex of C is a simplex of a suitable C_γ. Thus

$$BC = \bigcup_{\gamma \in \Gamma} BC_\gamma$$

and $\{BC_\gamma, \gamma \in \Gamma\}$ is a direct system (by inclusions) of closed subspaces of BC. Remark that each BC_γ contains only finitely many distinct subsets of type $BC_\gamma \cap BC_\delta$, $\delta \in \Gamma$. Thus, by a standard argument we see that every compact set of BC is contained in a suitable BC_γ and that

$$[K, BC]_0 \cong \varinjlim [K, BC_\gamma]_0 \tag{1.3}$$

for any compact based space K, where $[\ ,\ ]_0$ denotes the set of based homotopy classes of maps.

Lemma 4. *Let C' be a finite direct system of based k-spaces and X_ω the final object of C'. Then X_ω is a deformation retract of BC'.*

Proof. Remark that every finite direct system contains a unique final object X_ω and X_ω is a closed subset of BC' by the inclusion

$$X_\omega \subset BC'_0 \subset BC'.$$

Every simplex of C' is a face of a simplex with X_ω as its last vertex; hence, denoting by $\hat{C}'$ the set of all non-degenerate simplexes of C' containing X_ω as the last vertex, we see that

$$BC' = \bigcup_{\sigma \in \hat{C}'} BC'(\sigma),$$

where $C'(\sigma)$ denotes the subsystem of C' consisting of all vertexes and edges of σ. Define a deformation retraction D_σ of $X_\sigma \times \Delta^n (\dim \sigma = n)$ into $X_\sigma \times \{(0, \cdots, 0, 1)\}$ by

$$D_\sigma((x, a), t) = (x, (a_0(1-t), \cdots, a_{n-1}(1-t), t + a_n(1-t))),\ a = (a_0, \cdots, a_n) \in \Delta^n,$$

for each $\sigma \in \hat{C}'$. $\bigvee_\sigma D_\sigma$ is visibly compatible with the equivalence relations in $\overline{BC}'$ and induces the desired deformation retraction of BC' to X_ω.

C is again an arbitrary direct system of based k-spaces. The inclusions

$$X_\alpha \subset BC_0 \subset BC$$

of each object X_α of C induces a morphism

$$\varinjlim\, [K, X_\alpha]_0 \to [K, BC]_0 \tag{1.4}$$

of sets (or of groups when K is a suspension) for any compact based space K in virtue of the structure of BC_1. As a corollary of (1.3) and Lemma 4 we obtain

Theorem 1. *Let $C=\{X_\alpha, f_{\alpha\beta}\}$ be a direct system of based k-spaces and K a compact based space. Then the morphism* (1.4) *is an isomorphism*

$$\varinjlim\, [K, X_\alpha]_0 \cong [K, BC]_0.$$

1.6. Let X be a (connected) based CW-complex and $C=\{X_\alpha, \alpha\in\mathcal{I}\}$ a direct system of based subcomplexes (by inclusions) such that $\cup X_\alpha = X$. As is well known

$$\varinjlim\, [K, X_\alpha]_0 \cong [K, X]_0 \tag{1.5}$$

for any compact based space K. The projections

$$X_\sigma \wedge \Delta^{n,+} \to X_\sigma \subset X$$

for simplexes σ of C are visibly compatible with the equivalence relations in $\overline{BC}$ and induce the canonical projection

$$\varpi : BC \to X.$$

Now the isomorphisms, Theorem 1 and (1.5), are compatible with the projection ϖ and ϖ induces an isomorphism

$$\varpi_* : [K, BC]_0 \cong [K, X]_0$$

for any compact based space K. Hence ϖ is a weak homotopy equivalence. Since BC is a CW-complex by Corollary 3 we obtain

Proposition 5. *$\varpi : BC \to X$ is a homotopy equivalence.*

2. Inverse limit functor

2.1. Let Λ be a ring and $\mathcal{A}=\{A_\alpha, g_\alpha^\beta\}$ an inverse system of Λ-modules and Λ-homomorphisms, *i.e.*, a cofunctor defined on an index category which is directed. For each n-simplex $\sigma=\{A_0,\cdots,A_n; g_1,\cdots,g_n\}$ of $\mathcal{A}$ we associate a

Λ-module A^*_σ by

$$A^*_\sigma = A_n, \text{ the terminal vertex of } \sigma,$$

and define Λ-homomorphisms

$$\varphi^*_{i,\sigma} : A^*_{F_i\sigma} \to A^*_\sigma,\ 0 \leqq i \leqq n,$$

by

$$\varphi^*_{i,\sigma} = 1 \text{ for } 0 \leqq i < n \text{ and } \varphi^*_{n,\sigma} = g_n.$$

Following Nöbeling [6] and Roos [7] we define n-cochain groups $\Pi^n\mathcal{A}$ of the inverse system $\mathcal{A}$ by

$$\Pi^n\mathcal{A} = \Pi_{\sigma\in\mathcal{A}_n'}\, A^*_\sigma \text{ (the direct product)}$$

where $\mathcal{A}_n'$ denotes the set of all non-degenerate n-simplexes of $\mathcal{A}$, and coboundary homomorphisms

$$\delta^{n-1} : \Pi^{n-1}\mathcal{A} \to \Pi^n\mathcal{A}$$

by

$$p_\sigma\delta^{n-1} = \sum_{i=0}^{n} (-1)^i \varphi^*_{i,\sigma}\, p_{F_i\sigma} \tag{2.1}$$

for each n-simplex σ where p_τ is the projection of $\Pi^m\mathcal{A}$ onto the τ-factor A^*_τ for each $\tau\in\mathcal{A}_m'$. Then we obtain a cochain complex of Λ-modules

$$0 \to \Pi^0\mathcal{A} \xrightarrow{\delta^0} \Pi^1\mathcal{A} \xrightarrow{\delta^1} \Pi^2\mathcal{A} \to \cdots.$$

The inverse limit functor $\varprojlim$ and its n-th derived functor $\varprojlim^n$, $1 \leqq n$, are defined respectively by

$$\varprojlim \mathcal{A} = \varprojlim A_\alpha = H^0(\Pi^*\mathcal{A};\ \delta^*) \tag{2.2}$$

and

$$\varprojlim{}^n \mathcal{A} = \varprojlim{}^n A_\alpha = H^n(\Pi^*\mathcal{A};\ \delta^*),\ n \geqq 1. \tag{2.3}$$

In [8] Roos proved the following theorem on the vanishing of $\varprojlim^n$.

Theorem (Roos). *Let Λ be a commutative Noetherian ring of finite global dimension and $\{A_\alpha\}$ an inverse system of finitely generated Λ-modules. Then*

$$\varprojlim{}^p A_\alpha = 0 \quad \textit{for all } p > \dim \Lambda.$$

2.2. Here we shall restrict index sets to the direct set of *non-negative* *integers*. Let $\mathcal{A}=\{A_n, g_n^{n+1}\}_{n\geqq 0}$ be an inverse system of Λ-modules. Then it is well known that

$$\varprojlim{}^p A_n = 0 \quad \text{for all } p>1 \tag{2.4}$$

(see [7], and also [6]).

An inverse system $\{A_n\}$ is said to satisfy the *Mittag-Leffler condition* (ML) [4] if for each n there exists $n_0=n_0(n)\geqq n$ such that

$$\mathrm{Im}\{A_{n_0}\to A_n\} = \varprojlim_i \mathrm{Im}\{A_{n+i}\to A_n\}. \tag{2.5}$$

We say that an element $x_n\in A_n$ is *distinguished* if

$$x_n\in\varprojlim_i \mathrm{Im}\{A_{n+i}\to A_n\}.$$

Lemma 6. *An inverse system $\{A_n, g_n^{n+1}\}_{n\geqq 0}$ satisfies (ML) if and only if for each n there exists $n_0=n_0(n)\geqq n$ such that*

$$\mathrm{Im}\{A_{n_0}\to A_n\} = \mathrm{Im}\{\varprojlim_i A_{n+i}\to A_n\}.$$

Proof. Suppose that $\{A_n\}$ satisfies (ML). Let $x_n\in A_n$ be distinguished, *i.e.*, $x_n=g_n^{n+i}(y_{n+i})$ for some $y_{n+i}\in A_{n+i}$, $i\geqq 0$. By the assumption $g_{n+1}^{m_0+n+1}(y_{m_0+n+1})\in A_{n+1}$ is distinguished for some $m_0=m_0(n+1)$. Thus there exists a distinguished element

$$x_{n+1}\in A_{n+1} \quad \text{with} \quad g_n^{n+1}(x_{n+1}) = x_n.$$

Repeating this construction we obtain a series of distinguised elements

$$\{x_n, x_{n+1}, \cdots\} \quad \text{such that} \quad g_{n+i}^{n+i+1}(x_{n+i+1}) = x_{n+i},\ i\geqq 0.$$

This series gives an element

$$x\in\varprojlim A_{n+i} \quad \text{with} \quad \pi_n(x) = x_n$$

where $\pi_n : \varprojlim A_{n+i}\to A_n$ is the canonical projection. Thus we have

$$\mathrm{Im}\{A_{n_0}\to A_n\} = \varprojlim \mathrm{Im}\{A_{n+i}\to A_n\} = \mathrm{Im}\{\varprojlim A_{n+i}\to A_n\}.$$

The "if" part is evident.

The following result is well known.

(2.6) *If an inverse system $\{A_n, g_n^{n+1}\}_{n\geqq 0}$ satisfies (ML), then*

$$\varprojlim{}^1 A_n = 0.$$

3. Convergence conditions of certain spectral sequences

3.1. Let h be a (reduced general) cohomology theory defined on arbitrary based CW-complexes and X a based CW-complex given with a filtration

$$X_0 \subset X_1 \subset \cdots \subset X_p \subset \cdots, \quad \bigcup X_p = X$$

by subcomplexes. We shall observe the spectral sequence of h associated with the filtration $\{X_p\}$ of X.

Following [3] we put

$$\begin{aligned} Z_r^{p,q} &= \operatorname{Ker}\{h^{p+q}(X_p/X_{p-1}) \to h^{p+q+1}(X_{p+r-1}/X_p)\}, \\ B_r^{p,q} &= \operatorname{Im}\{h^{p+q-1}(X_{p-1}/X_{p-r}) \to h^{p+q}(X_p/X_{p-1})\}, \\ E_r^{p,q} &= Z_r^{p,q}/B_r^{p,q} \text{ for each } 1 \leqq r \leqq \infty \end{aligned}$$

and define a decreasing filtration of $h^n(X)$ by

$$F^{p,n-p} = F^p h^n(X) = \operatorname{Ker}\{h^n(X) \to h^n(X_{p-1})\},$$

where we used the conventions

$$X_\infty = X \quad \text{and} \quad X_{-p} = \{*\}, \text{ the base point of } X,\ 1 \leqq p \leqq \infty.$$

In this case we have

$$B_{p+1}^{p,q} = B_{p+2}^{p,q} = \cdots = B_\infty^{p,q}; \tag{3.1}$$

hence there exists the canonical inclusion

$$E_\infty^{p,q} \to \varprojlim_{r>p} E_r^{p,q}.$$

As is well known we obtain an isomorphism

$$F^{p,q}/F^{p+1,q-1} \xrightarrow{\cong} E_\infty^{p,q}.$$

Combining this with the above inclusion, there exists a natural homomorphism

$$\psi : F^{p,q}/F^{p+1,q-1} \to \varprojlim_{r>p} E_r^{p,q} \tag{3.2}$$

which is a monomorphism.

The projections $u_p : h^n(X) \to h^n(X)/F^p h^n(X)$ induce a natural homomorphism

$$u : h^n(X) \to \varprojlim_{p} h^n(X)/F^p h^n(X). \tag{3.3}$$

The spectral sequence $\{E_r, d_r\}$ is said to be *weakly convergent* if ψ is an isomorphism, and *convergent* or *strongly convergent* if it is weakly convergent and u is a monomorphism or an isomorphism. In addition it is said to be *finitely convergent* if there exists $r_0=r_0(p, q)<\infty$ for each p, q such that $E_{r_0}^{p,q}=E_r^{p,q}$ for all r, $r_0\leqq r<\infty$.

3.2. We define groups $C_s^{p,q}$ by

$$C_s^{p,q} = \operatorname{Im}\{h^{p+q}(X_{p+s-1}) \to h^{p+q}(X_p)\}$$

for each s, $1\leqq s\leqq\infty$. The groups $E_r^{p,q}$ are closely related with the groups $C_s^{p,q}$.

Lemma 7. *Fix an integer n.*

i) *For each p there exists* $s_0=s_0(p,n)<\infty$ *such that*

$$C_{s_0}^{p,n-p} = C_s^{p,n-p} \textit{ for all } s,\ s_0\leqq s<\infty,$$

if and only if there exists $r_0=r_0(p, n)<\infty$ *for each p such that*

$$E_{r_0}^{p,n-p} = E_r^{p,n-p} \textit{ for all } r,\ r_0\leqq r<\infty.$$

ii) *For each p there exists* $s_0=s_0(p, n)<\infty$ *such that*

$$C_\infty^{p,n-p} = C_{s_0}^{p,n-p}$$

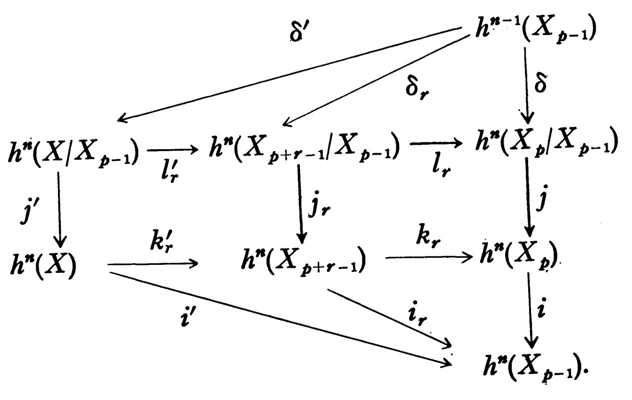
if and only if there exists $r_0=r_0(p, n)<\infty$ *for each p such that*

$$E_\infty^{p,n-p} = E_{r_0}^{p,n-p}.$$

iii) $C_\infty^{p,n-p}=\varprojlim_s C_s^{p,n-p}$ *for each p if and only if* $E_\infty^{p,n-p}=\varprojlim_{r>p} E_r^{p,n-p}$ *for each p.*

Proof. We prove only iii). The proofs of the other parts i) and ii) are more or less parallel to iii) and simpler.
It is sufficient to show that $C_\infty^{p,n-q}=\varprojlim C_s^{p,n-p}$ for each p if and only if $Z_\infty^{p,n-p}=\varprojlim Z_r^{p,n-p}$ for each p by (3.1).

We shall use the following commutative diagram.



We put $l=l_r l_r'$ and $k=k_r k_r'$.

The "only if" part: Take any distinguished element $x\in h^n(X_p/X_{p-1})$, *i.e.*,

$$x\in\varprojlim_r \operatorname{Im}\{h^n(x_{p+r-1}/X_{p-1})\to h^n(X_p/X_{p-1})\}=\varprojlim_r Z_r^{p,q}.$$

Then $j(x)\in h^n(X_p)$ is also distinguished. By the assumption there exists $y\in h^n(X)$ such that $k(y)=i(x)$. Then $i'(y)=0$, whence $y=i'(z)$ for some $z\in h^n(X/X_{p-1})$. So $j(x-l(z))=0$, thus $x=l(z)+\delta(w)$ for some $w\in h^{n-1}(X_{p-1})$. This means that $l(z+\delta'(w))=x$. Hence

$$x\in\operatorname{Im}\{h^n(X/X_{p-1})\to h^n(X_p/X_{p-1})\} = Z_\infty^{p,n-q}.$$

The "if" part: We prove by an induction on p. In case $p=0$ the proof is trivial because $C_r^{0,q}=Z_r^{0,q}$. Take any distinguished element $x\in h^n(X_p)$, $p\geqq 1$, then $i(x)\in h^n(X_{p-1})$ is also distinguished. By the assumption of the induction, $i(x)=i'(y)$ for some $y\in h^n(X)$. Hence $x=k(y)+j(z)$ for some $z\in h^n(X_p/X_{p-1})$. Here we show that z is distinguished. We may put $x=k_r(x_r)$ for some $x_r\in h^n(X_{p+r-1})$, $1\leqq r<\infty$, because x is distinguished. Then $i_r(x_r-k_r'(y))=0$, *i.e.*, $x_r=k_r'(y)+i_r(u_r)$ for some $u_r\in h^n(X_{p+r-1}/X_{p-1})$. Now $j(z-l_r(u_r))=0$, *i.e.*, $z=l_r(u_r)+\delta(v_r)$ for some $v_r\in h^{n-1}(X_{p-1})$. This yields that $z=l_r(u_r+\delta_r(v_r))$ for all r, $1\leqq r<\infty$, thus z is distinguished. By the assumption there exists $w\in h^n(X/X_{p-1})$ such that $l(w)=z$. Then $k(y+i'(w))=x$; hence

$$x\in\operatorname{Im}\{h^n(X)\to h^n(X_p)\} = C_\infty^{p,n-q}.$$

As an immediate corollary of Lemma 7, i), we have

Corollary 8. *The spectral sequence $\{E_r, d_r\}$ of h associated with a filtration $\{X_p\}_{p\geqq 0}$ of X is finitely convergent if and only if the inverse system $\{h^n(X_p)\}_{p\geqq 0}$ satisfies (ML) for all degree n.*

3.3. In this subsection we suppose that a cohomology theory h is *additive*, *i.e.*, h^n (for all degree n) satisfies the wedge axiom (cf., [5], and also [1] for the terminology) for arbitrary collections of CW-complexes.

Milnor [5] established

Theorem (Milnor). *Let h be an additive (reduced) cohomology theory and $\{X_p\}_{p\geqq 0}$ an increasing filtration by subcomplexes of a based CW-complex X. There is an exact sequence*

$$0\to\varprojlim_p{}^1 h^{n-1}(X_p)\to h^n(X)\to\varprojlim_p h^n(X_p)\to 0$$

for all degree n.

Let $i_p: X_p\subset X$ be the inclusions. From the exact sequences

$$0 \to F^{p+1}h^n(X) \to h^n(X) \to \operatorname{Im} i_p^* \to 0$$

and

$$0 \to \operatorname{Im} i_p^* \to h^n(X_p)$$

we obtain the following commutative diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ 0 \to \varprojlim F^{p+1}h^n(X) & \to & h^n(X) & \to & \varprojlim \operatorname{Im} i_p^* & \to & \varprojlim{}^1 F^{p+1}h^n(X) & \to & 0 \\ & & \| & & \downarrow & & & & \\ 0 \to \varprojlim{}^1 h^{n-1}(X_p) & \to & h^n(X) & \to & \varprojlim h^n(X_p) & \to & 0 & & \end{array}$$

in which the rows and column are exact. Therefore

$$\varprojlim_p{}^1 F^{p+1}h^n(X) = 0 \tag{3.4}$$

and

$$\varprojlim_p F^{p+1}h^n(X) \cong \varprojlim_p{}^1 h^{n-1}(X_p). \tag{3.5}$$

And we have an exact sequence

$$0 \to \varprojlim_p{}^1 h^{n-1}(X_p) \to h^n(X) \xrightarrow{u} \varprojlim_p h^n(X)/F^p h^n(X) \to 0. \tag{3.6}$$

This implies that the convergence of the spectral sequence $\{E_r, d_r\}$ of h associated with a filtration $\{X_p\}$ of X is equivalent to the strong convergence of it when h is additive.

Proposition 9. *Suppose that h is additive. If the spectral sequence $\{E_r, d_r\}$ of h associated with a filtration $\{X_p\}_{p\geqq 0}$ of X is finitely convergent, then it is strongly convergent.*

Proof. By Corollary 8 the inverse system $\{h^n(X_p)\}_{p\geqq 0}$ satisfies (ML) for each degree n. From Milnor's Theorem and Lemma 6 it follows that there exists $s_0 = s_0(p, n) < \infty$ such that

$$\begin{aligned} \operatorname{Im}\{h^n(X) \to h^n(X_p)\} &= \operatorname{Im}\{\varprojlim h^n(X_{p+i}) \to h^n(X_p)\} \\ &= \operatorname{Im}\{h^n(X_{p+s_0}) \to h^n(X_p)\}. \end{aligned}$$

Thus by Lemma 7, ii),

$$\psi : E_\infty^{p,n-q} \to \varprojlim E_r^{p,n-q}$$

is an isomorphism. On the other hand, by (2.6) and (3.6)

$$u : h^n(X) \to \varprojlim h^n(X)/F^p h^n(X)$$

is an isomorphism. Thus the spectral sequence $\{E_r, d_r\}$ is strongly convergent.

4. The spectral sequence associated with an inverse system of CW-complexes

4.1. Let h be an additive (reduced general) cohomology theory defined on arbitrary CW-complexes and $C=\{X_\alpha, f_{\alpha\beta}\}$ a direct system of based CW-complexes and cellular maps. We shall observe the spectral sequence of h associated with the filtration (1.2) of BC. Then, by definition

$$\begin{aligned} E_1^{p,q} &= h^{p+q}(BC_p, BC_{p-1}) \\ &\cong h^{p+q}(\bigvee_\sigma S^p X_\sigma) && \text{by the decomposition (1.1)} \\ &\cong \prod_\sigma h^q(X_\sigma) && \text{by the wedge axiom,} \end{aligned}$$

where σ runs over all non-degenerate p-simplexes of C. Thus $E_1^{p,q}$ is isomorphic to the p-cochain group of the inverse system $\{h^q(X_\alpha), f_{\alpha\beta}^*\}$. Now, by the standard argument as in Atiyah-Hirzebruch spectral sequences, we see that d_1 is transformed to the coboundary homomorphism of these cochain groups. Thus

$$E_2^{p,q} \cong \varprojlim{}^p h^q(X_\alpha).$$

E_∞ is the bigraded module associated with $h^*(BC)$ by the filtration induced by (1.2). Thus we obtain

Proposition 10. *Let h be an additive cohomology theory defined on CW-complexes and $C=\{X_\alpha\}$ a direct system of based CW-complexes. There holds a bigraded spectral sequence associated with $h^*(BC)$ such that*

$$E_2^{p,q} = \varprojlim{}^p h^q(X_\alpha).$$

As a corollary of Propositions 5 and 10 we obtain

Theorem 2. *Let h be an additive (reduced) cohomology theory defined on arbitrary CW-complexes, X a based CW-complex and $C=\{X_\alpha\}$ a direct system of based subcomplexes of X such that $X=\bigcup_\alpha X_\alpha$. There holds a bigraded spectral sequence associated with $h^*(X)$ by a suitable filtration such that*

$$E_2^{p,q} = \varprojlim{}^p h^q(X_\alpha).$$

4.2. Let Λ be a commutative Noetherian ring and h a cohomology theory of Λ-modules *of finite type, i.e.*, $h^n(S^0)$ (for all degree n) is a finitely generated

Λ-module, then $h^n(X)$ is a finitely generated Λ-module for any based finite CW-complex X.

Proposition 11. *Let Λ be a commutative Noetherian ring of finite global dimension, h an additive cohomology theory of Λ-modules of finite type, X a based CW-complex and $C=\{X_\alpha\}$ a direct system of finite subcomplexes of X. Then the spectral sequence of Theorem 2 is strongly convergent.*

Proof. By the Theorem of Roos we know that

$$\varprojlim{}^p\, h^q(X_\alpha) = 0 \qquad \text{for all } p > \dim \Lambda.$$

Hence $d_r=0$, $r>\dim\Lambda$, in the spectral sequence of Theorem 2. The conclusion follows immediately from Proposition 9.

Since the global dimension of Z is 1, we obtain

Corollary 12. *Let h be an additive cohomology theory of finite type (as Z-modules), X and $C=\{X_\alpha\}$ be as in the above proposition. Then there is a short exact sequence*

$$0 \to \varprojlim{}^1\, h^{n-1}(X_\alpha) \to h^n(X) \to \varprojlim\, h^n(X_\alpha) \to 0$$

for each degree n.

The above corollary was also obtained by Anderson [2] by an entirely different method.

4.3. Let h be an additive (reduced general) homology theory defined on arbitrary CW-complexes, *i.e.*, satisfying the wedge axiom [5]. Let $C=\{X_\alpha\}$ be a direct system of CW-complexes and cellular maps. Observe the spectral sequence associated with $h_*(BC)$ by the filtration (1.2), then we obtain

$$E^2_{p,q} \cong \varinjlim{}_p\, h_q(X_\alpha),$$

(cf., Nobeling [6] for the definition of $\varinjlim_p$). Since $\varinjlim_p$ are successive derived functors of the right exact functor $\varinjlim$ on ordered systems of abelian groups [6] and it is exact whenever the underlying ordering is directed, we see that

$$\varinjlim{}_p\, h_q(X_\alpha) = 0 \qquad \text{for all } p>0.$$

Thus the spectral sequence collapses,

$$\text{(4.1)} \qquad \operatorname{Im}\,[h_n(BC_0) \to h_n(BC)] = \operatorname{Im}\,[h_n(BC_p) \to h_n(BC)]$$

for $p>0$ and

$$\text{(4.2)} \qquad \operatorname{Im}\,[h_n(BC_0) \to h_n(BC)] \cong \varinjlim h_n(X_\alpha)$$

for all degree n. On the other hand, by [5], Lemma 1, we see that

$$\text{(4.3)} \qquad \varinjlim h_n(BC_p) \cong h_n(BC)$$

for all degree n. Now the isomorphisms (4.1), (4.2) and (4.3) imply

$$\text{(4.4)} \qquad \varinjlim h_n(X_\alpha) \cong h_n(BC).$$

(4.4) and Proposition 5 imply

Theorem 3. *Let h be an additive (reduced) homology theory defined on arbitrary CW-complexes, X a based CW-complex and $C=\{X_\alpha\}$ a direct system of based subcomplexes of X such that $X=\bigcup_\alpha X_\alpha$. There hold the isomorphisms*

$$\varinjlim h_n(X_\alpha) \cong h_n(X)$$

for all degree n.

OSAKA CITY UNIVERSITY

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