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ON POTENT RINGS AND MODULES

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There are two kinds of "quotient ring". One is called a classical quotient ring, that is, an extension ring $Q(R)$ of a ring R is called an classical right quotient ring of R if

(i) $Q(R) \ni 1$,

(ii) every element of $Q(R)$ has the form ac^{-1}, where $a, c \in R$ and c is a regular element of R,

(iii) every regular element of R has an inverse in Q.

In [6], [7], [19], [20] and [21] etc., many authors studied the structure of those rings which have an artinian classical right quotient ring. Such rings have finite dimensions in the sense of Goldie. It seems to the author that there does not exist too many rings with infinite dimensions which have the classical right quotient ring (even when the right singular ideal of such rings vanishes).

The other quotient ring is called a (homological) quotient ring and was defined by R. E. Johnson [10], Y. Utumi [22], G. D. Findlay and J. Lambek [5]. An extension ring S of a ring R is a right quotient ring of R if for each a, $0 \neq b \in S$, there exist $r \in R$ and $n \in Z$ such that $ar+na \in R$ and $br+nb \neq 0$, where Z is the ring of integers. If R is a left faithful ring, then R has a unique maximal right quotient ring $\hat{R}$. In particular, if R has zero right singular ideal, then $\hat{R}$ is a right self-injective von Neumann regular ring. So when we investigate rings with zero right singular ideal, it is useful to consider the (homological) maximal right quotient rings of such rings. But a ring R need not be semi-prime even in the case where $\hat{R}$ is simple and artinian, as the following example shows. Let D be a right Ore domain and let F be the right quotient division ring of D. We put

$$R = \left\{ \begin{bmatrix} a_{11} & 0\cdots 0 \\ a_{21} & 0\cdots 0 \\ \vdots & \vdots \quad \vdots \\ a_{n1} & 0\cdots 0 \end{bmatrix} \middle| \, a_{i1} \in D \right\} \text{ and } \hat{R} = (F)_n \,.$$

Then $\hat{R}$ is the maximal right quotient ring of R. The above example suggests that there are even various those rings which have the simple artinian maximal right quotient ring. So it is important to investigate those rings which have a self-injective von Neumann regular ring as the maximal right quotient ring. In [15] R. E. Johnson defined potent rings and determined those potent rings which have the simple artinian maximal right quotient ring. A ring R is called a potent ring if every non-zero closed right ideal A of R is potent, that is, $A^n \neq 0$ for all $n>0$. The main theme of this paper is to investigate those potent rings which have a right self-injective von Neumann regular ring as the maximal right quotient ring. After several definitions (section 1) we define, in section 2, the concepts of residue-finite and locally residue-finite rings and show that a right locally uniform potent ring with zero right singular ideal which is locally residue-finite is an essentially irredundant subdirect sum of potent irreducible rings with zero right singular ideal and conversely. In section 3, we investigate countably dimensional potent irreducible rings with zero right singular ideal (for short: *CPI*-rings). We define the concept of rings which have matrix representable conditions (m. r. conditions) and give examples of residue-finite *CPI*-rings with m. r. conditions. If R is a residue-finite *CPI*-ring, then the set of closed two-sided ideals is a chain and there are the following two cases:

(A): $R = T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_p \supset \cdots$ and $\bigcap_{p=0}^{\infty} T_p = 0$,

(B): There exists an integer p such that

$$R = T_0 \supset T_1 \supset T_2 \supset \cdots \supset T_p \supset T_{p+1} = 0\,.$$

If R satisfies the condition (A), then we call the ring R of type (A). If R satisfies the condition (B), then we call the ring R of type (B). We give, in Theorem 3. 22, a characterization of *CPI*-rings with m. r. conditions which are of type (A). In section 4, we give a characterization of *CPI*-rings with m. r. conditions which are of type (B). We also show that if the maximal right quotient ring $\hat{R}$ of a ring R is also a left quotient ring of R, then R is of type (B) which has m. r. conditions. This is a generalization of Faith's result [2] on prime rings. In section 5, we give a necessary and sufficient condition that the maximal right quotient ring of a right locally uniform potent ring with zero right singular ideal is a left quotient ring of the same ring. In section 6, we generalize some of Goldie's results on semi-prime Goldie rings to the cases of potent rings or infinite dimensional semi-prime rings. In section 7, applying the methods developed in section 2 to modules, we give a characterization of semi-prime modules over a locally uniform semi-prime ring with zero right singular ideal.

Some of the results in this paper were announced without proofs in [17] and [18].

1. Definitions and notations

Let R be an associative ring and let M be a right R-module. A non-zero

R-submodule U of M is *uniform* if U is an essential extension of every non-zero R-submodule contained in U. An R-module M is said to be *locally uniform* if any non-zero R-submodule of M contains a uniform R-submodule. Clearly, if M is finite dimensional in the sense of Goldie, then M is locally uniform. M is called *countably dimensional* if M contains a direct sum of countable infinite R-submodules but M does not contain a direct sum of non-countable R-submodules. An R-submodule C of M is called *closed* if it has no proper essential extensions in M. Clearly, the concept of closed submodules of M coincides with the one of complemented submodules in the sense of Goldie [7]. A submodule L of M is called *large* if M is an essential extension of L (in symbol: $L\subset' M$).

In the case $M=R$, adapting the terminology of the above, we use the terms *uniform right ideal* and *right locally uniform ring* and so on. We call $Z_R(M)=\{m\in M \mid mE=0 \text{ for some } E\subset' R\}$ the *singular R-submodule* of M. In particular, $Z_R(R)$ is an ideal. We call $Z_R(R)$ the *right singular ideal* of R and denote it by $Z_r(R)$. If $Z_R(M)=0$, then each non-zero submodule N of M has a unique maximal essential extension N^* in M. In this paper, we assume that all rings have zero right singular ideals. If S is a non-empty set of elements of R, then we define $S^r=\{x\in R \mid Sx=0\}$. The set S^r is a right ideal of R and is the *right annihilator* of S. The left ideal S^l is defined in a similar manner and is the *left annihilator* of S. Any right ideal of the form S^r, where S is a non-empty subset of R, is an *annihilator right ideal.* The set $L_r(R)$ $(=L_r)$ of closed right ideals is a complete complemented modular lattice under the inclusion. If $\{C_i \mid i\in I\}$ is any collection of closed right ideals of R, then $\bigcup^*_{i\in I} C_i=(\sum_{i\in I} C_i)^*$. If $(J_r\colon \cap, \cup)$ denotes the lattice of all annihilator right ideals of R, then it is easily seen that $J_r\subseteqq L_r$. For convenience, we put $L_{r2}=L_r\cap L_2$ and $J_{r2}=J_r\cap L_2$, where L_2 is the set of two-sided ideals of R. Corresponding left properties of a ring R are indicated by replacing each "r" by an "l". If R is right locally uniform, then L_r is an atomic lattice and $A\in L_r$ is an atom if and only if A is a closed uniform right ideal. We say that right ideals I and J are *similar* if and only if $E_R(I)\cong E_R(J)$, where $E_R(I)$ is an injective hull of I as a right R-module (in symbol: $I\sim J$). It is clear that if A and B are uniform right ideals of R, then $A\sim B$ if and only if A and B contain mutually isomorphic non-zero right ideals A' and B' respectively. A ring R is said to be *right irreducible* if and only if R is right locally uniform and $A\sim B$ for all uniform right ideals A and B of R. A right locally uniform irreducible ring with zero right singular ideal is called an *I-ring*. We note that a ring R is an I-ring if and only if R is an I-ring in the sense of R.E. Johnson [15]. Following R. E. Johnson, we call a ring R a *right potent ring* (for short: *P-ring*) if every non-zero closed right ideal of R is potent. An I-ring which is also a P-ring will be called a *PI-ring*. A ring R is said to be *residue-finite* if the following conditions is satisfied:

The factor ring R/T is finite dimensional as a right R-module for any non-zero $T\in L_{r2}$.

If R is finite dimensional, then R is residue-finite. If R is a prime ring, then R is residue-finite, because $L_{r2}=\{0, R\}$. A PI-ring which is countably dimensional will be called a *CPI-ring*. Let M be a right R-module. If M is n-dimensional in the sense of Goldie, then we write $n=\dim_R M$. A ring S is called a *right quotient ring* of a subring R if for each a, $0\neq b\in S$, there exist $r\in R$ and $n\in Z$ such that $ar+na\in R$ and $0\neq br+nr$, where Z is the ring of integers (in symbol: $R\leqq S$). A left quotient ring is defined similarly. If S is a left and right quotient ring of R, then we write $R\leqq_l S$. If R has zero right singular ideal, then S is a right quotient ring of R if and only if S is a right quotient ring of R in the sense of R. E. Johnson (see. [2]).

Concerning the terminologies we refer to [7] and [15].

2. Locally residue-finite P-rings

In this section it is shown that it suffices to find the structure of a residue-finite PI-ring in order to determine the structure of an arbitrary locally residue-finite P-ring[1] which is a right locally uniform ring with zero right singular ideal. We start with the proposition which is a generalization of Goldie's result [7] on finite dimensional rings to infinite dimensional modules.

Proposition 2.1. *Let M be a right locally uniform R-module with $Z_R(M)=0$ and let N be an R-submodule of M and let N^* be a unique maximal essential extension of N in M. Then $N^*=\{m\in M\mid mE\subseteqq N$ for some $E\subset' R\}$.*

Proof. We put $N'=\{m\in M\mid mE\subseteqq N$ for some $E\subset' R\}$. Clearly, N' is an R-submodule which contains N. If $m\in N'$, then $0\neq mE\subseteqq N$, where E is a large right ideal and thus $0\neq mE\subseteqq mR\cap N$. Hence $N\subset' N'$ as right R-modules and thus $N^*\supseteqq N'$. Conversely, let $x\in N^*$ and let $E=\{r\in R\mid xr\in N\}$. Then we have $E\subset' R$ and $xE\subseteqq N$. Hence $N^*\subseteqq N'$ and we obtain $N^*=N'$, as desired.

Let R be a right locally uniform ring with $Z_r(R)=0$ and let $\hat{R}$ be the maximal right quotient ring of R. Then $\hat{R}$ is a right self-injective (von Neumann) regular ring and the mappings

$$A\rightarrow E_R(A),\ A\in L_r(R);\ \hat{A}\rightarrow \hat{A}\cap R,\ \hat{A}\in L_r(\hat{R})$$

are mutually inverse isomorphisms between $L_r(R)$ and $L_r(\hat{R})$, where $E_R(A)$ is a right R-injective hull of A in $\hat{R}$ (see [2]). Let A be a right ideal of R. Then we write the R-injective hull of A in $\hat{R}$ by $\hat{A}$. Clearly, $\hat{A}$ is a right ideal of $\hat{R}$

1) The term "locally residue-finite rings" will be defined in this section.

$\hat{R}$ and is right $\hat{R}$-injective. Now the set of all uniform right ideals of R can be classified by the similartity. $\{A_\alpha\}$ will denote the class containing the uniform right ideal A_α. We set $R_\alpha=(\sum_{A\in\{A_\alpha\}} A)^*$ and call R_α an *irreducible component* of R. Then we obtain

Proposition 2. 2. *Let R be a right locally uniform ring with $Z_r(R)=0$. Then*

(1) *$\sum_{A\in\{A_\alpha\}} A$ is a two-sided ideal.*

(2) *R_α is a two-sided ideal.*

(3) *If B is a uniform right ideal of R and if $B\subseteqq R_\alpha$, then $B\sim A_\alpha$.*

(4) *The sum $\sum R_\alpha$ is a direct sum.*

Proof. (1) Let A be a uniform right ideal and let A^* be a unique maximal essential extension of A in R. Then A^* is an atom of L_r. Hence if x is an element of R, then we obtain $x^r\supseteqq A^*$ or $x^r\cap A^*=0$. From these (1) follows immediately.

(2) We put $R_\alpha'=\sum_{A\in\{A_\alpha\}} A$ and let a be an element of R_α and let r be an element of R. Then, by Proposition 2. 1, $aE\subseteqq R_\alpha'$ for some $E\subset' R$ and hence $(ra)E=r(aE)\subseteqq R_\alpha'$ by (1). Again, by Proposition 2. 1, $ra\in R_\alpha$. Hence R_α is an ideal.

(3) Let B be a uniform right ideal of R, and $B\subseteqq R_\alpha$ for some α. Then there exists an independent set $\{B_i\}$ of uniform right ideals which satisfies $A_\alpha\sim B_i$ and $\sum_i \oplus B_i\subset' R_\alpha'$, because R is right locally uniform. Then $B\cap(\sum \oplus B_i)\neq 0$ and the mapping

$$\theta_i\colon b\rightarrow b_i\,,\quad \text{where}\quad b=\sum_i b_i\in B\cap(\textstyle\sum_i \oplus B_i)\,,$$

is a monomorphism or zero by Lemma 5. 4 of [8]. Hence $B\sim B_i$ for each i such that $\theta_i\neq 0$ and thus $B\sim A_\alpha$.

(4) We assume that $R_\alpha\cap(\sum_{\beta\neq\alpha} R_\beta)\neq 0$. Then, applying the method of proof of (3) for a uniform right ideal B contained in $R_\alpha\cap(\sum_{\beta\neq\alpha} R_\beta)$, we obtain $B\sim A_\alpha$ and $B\sim A_\beta$ for some $\beta\neq\alpha$. This is a contradiction and hence the sum $\sum_\alpha R_\alpha$ is a direct sum.

Proposition 2. 3. *Let R be a right locally uniform ring with $Z_r(R)=0$, let $\{R_\alpha\,|\,\alpha\in\Lambda\}$ be the irreducible components of R and let $\hat{R}$ be the maximal right quotient ring of R. Then*

(1) *$\hat{R}_\alpha$ is a right self-injective, regular and prime ring with a minimal right ideal.*

(2) *$\hat{R}_\alpha$ is the maximal right quotient ring of R_α.*

(3) *$L_r(R_\alpha)=\{I\in L_r(R)\,|\,I\subseteqq R_\alpha\}$.*

(4) *If R is a potent ring, then R_α is a PI-ring.*

Proof. (1) If A and B are uniform right ideals such that $A\sim B$, then

$\hat{A}\simeq\hat{B}$ and $\hat{A}$ is a minimal right ideal of $\hat{R}$. Hence $\hat{R}_\alpha$ is an $\hat{R}$-injective hull of the sum of all minimal right ideals which are isomorphic to $\hat{A}_\alpha$ and thus $\hat{R}_\alpha$ is a direct summand of $\hat{R}$ and is an two-sided ideal of $\hat{R}$ by the same argument as in (2) of Proposition 2. 2. From these (1) follows immediately.

(2) Since $\hat{R}_\alpha$ is a regular ring and is a right self-injective ring by (1), it is enough to prove that $\hat{R}_\alpha\text{'}\supset R_\alpha$ as a right R_α-module. Let q be a non-zero element of $\hat{R}_\alpha$. Then there exists $r\in R$ such that $0\neq qr\in R\cap\hat{R}_\alpha=R_\alpha$. Since $R_\alpha R_\beta=0$ $(\alpha\neq\beta)$, $\sum_\alpha\oplus R_\alpha\subset' R$ and $Z_r(R)=0$, we obtain $qrR_\alpha\neq 0$. Hence there exists $r'\in R_\alpha$ such that $0\neq q(rr')=(qr)r'\in R_\alpha$ and $rr'\in R_\alpha$, as desired.

(3) Let I be a closed right ideal of R such that $I\subseteq R_\alpha$. Then $\hat{I}$ is a direct summand of $\hat{R}_\alpha$ and hence $\hat{I}\in L_r(\hat{R}_\alpha)$. Since $I=\hat{I}\cap\hat{R}=(\hat{I}\cap\hat{R}_\alpha)\cap R=\hat{I}\cap(\hat{R}_\alpha\cap R)=\hat{I}\cap R_\alpha$, we obtain $I\in L_r(R_\alpha)$. Conversely, let I be a closed right ideal of R_α and let $\bar{I}=E_{R_\alpha}(I)$. Then clearly $\bar{I}$ is a right ideal of $\hat{R}$ and is a direct summand of $\hat{R}$. Hence $\bar{I}\in L_r(\hat{R})$. Since $\bar{I}\cap R=(\hat{I}\cap\hat{R}_\alpha)\cap R=\bar{I}\cap(\hat{R}_\alpha\cap R)=\bar{I}\cap R_\alpha=I$, we obtain $I\in L_r(R)$ and $I\subseteq R_\alpha$.

(4) follows from (1) and (3).

Let R be a right locally uniform potent ring with $Z_r(R)=0$. Then R is saidt to be *locally residue-finite* if and only if the irreducible components of R are residue-finite as a ring. By Proposition 2. 3, if R is locally residue-finite, then R_α is a residue-finite PI-ring for each α.

Now we set

(2. 4) $P_\alpha=(\sum_{\beta\neq\alpha}R_\beta)^*$ and $\bar{R}_\alpha=R/P_\alpha$ for each α. Then the following lemma holds.

Lemma 2. 5. (1) *P_α is a two-sided ideal of R.*

(2) *$\cap_\alpha P_\alpha=0$ and $\cap_{\beta\neq\alpha}P_\beta\neq 0$.*

(3) *$\bar{R}_\alpha\text{'}\supset R_\alpha$ as right R_α-modules.*

(4) *If R_α is a residue-finite PI-ring, then so is $\bar{R}_\alpha$.*

Proof. (1) and (2) are trivial.

(3) The mapping

$$x\rightarrow\bar{x}=x+P_\alpha$$

is a ring monomorphism from R_α to $\bar{R}_\alpha$, where $x\in R_\alpha$. Hence we may assume that $\bar{R}_\alpha\supset R_\alpha$. Let $\bar{x}$ be a non-zero element of $\bar{R}_\alpha$, where $x\notin P_\alpha$, $x\in R$. By Proposition 2. 1, $xE\subseteq R_\alpha\oplus P_\alpha$ for some $E\subset' R$. Clearly $(E\cap R_\alpha)\oplus(E\cap P_\alpha)\subset' R$. If $x(E\cap R_\alpha)=0$. then $x[(E\cap P_\alpha)\oplus(E\cap R_\alpha)]=x(E\cap P_\alpha)\subseteq P_\alpha$, because P_α is an ideal and hence $x\in P_\alpha^*=P_\alpha$ by Proposition 2. 1. This is a contradiction and hence $0\neq x(E\cap R_\alpha)\subseteq R_\alpha$, i.e., $xR_\alpha\cap R_\alpha\supseteq x(E\cap R_\alpha)\neq 0$. Hence $\bar{R}_\alpha\text{'}\supset R_\alpha$ as right R_α-modules.

(4) By (3), we may assume that $\hat{R}_\alpha\supseteq\bar{R}_\alpha\supseteq R_\alpha$. By Theorem 4 of [2, p. 70],

$L_r(\bar{R}_\alpha)$ is isomorphic to $L_r(R_\alpha)$ under the contraction. Hence if R_α is a residue-finite PI-ring, then $\bar{R}_\alpha$ is a residue-finite PI-ring.

REMARK. If R is a right locally uniform ring with $Z_r(R)=0$, then (1)~(3) hold and $\bar{R}_\alpha$ is an I-ring.

Let S be a subdirect sum of a family $\{S_\alpha\}$ of rings (that is, $S\subset\Pi_\alpha S_\alpha$ and the projection $S\to S_\alpha$ is onto for each α). The subdirect sum will be called *essentially irredundant* if and only if $\Pi_\alpha S_\alpha \supset' \sum\oplus(S\cap S_\alpha)$ as right S-modules (see [2]).

Let $\bar{x}=(\bar{x}_\alpha)$ be a non-zero element of $\Pi_\alpha \bar{R}_\alpha$ and let $\bar{x}_\alpha\neq 0$ for some α. We put $E_\alpha=\{r\in R_\alpha \mid \bar{x}_\alpha r\in R_\alpha\}$. Then, since $R_\alpha\subset'\bar{R}_\alpha$, we obtain $R_\alpha\supset' E_\alpha$ as right R_α-modules. Since $Z_{R_\alpha}(R_\alpha)=0$, there exists an element r of E_α such that $0\neq\bar{x}_\alpha r\in R_\alpha$. Hence $0\neq\bar{x}r=\bar{x}_\alpha r\subseteq R_\alpha\subseteq\sum\oplus(R\cap\bar{R}_\alpha)$.

Now, we can summarize the above-mentioned results as follows:

Theorem 2. 6. *Let R be a right locally uniform (potent) ring with $Z_r(R)=0$ and let $\{\bar{R}_\alpha\}$ be as in (2. 4). Then R is an essentially irredundant subdirect sum of $\{\bar{R}_\alpha\}$ and $\bar{R}_\alpha$ is a (potent) I-ring for each α. Furthermor, if R is locally residue-finite, then $\bar{R}_\alpha$ is residue-finite.*

We now give a converse of Theorem 2. 6.

Theorem 2. 7. *Let $\{\bar{R}_\alpha\}$ be a family of PI-rings and R be an essentially irredundant subdirect sum of $\{\bar{R}_\alpha\}$. Then*

(1) *R is a right locally unifrom potent ring with $Z_r(R)=0$.*

(2) *If $\bar{R}_\alpha$ is residue-finite for each α, then R is locally residue-finite.*

Before proving this, we establish the following proposition, which is of interest in itself.

Proposition 2. 8. *Let S be a ring. Then S is a right locally uniform ring with $Z_r(S)=0$ if and only if S is an essentially irredundant subdirect sum of $\{\bar{S}_\alpha\}$, where $\bar{S}_\alpha$ is an I-ring for each α. Furthermore $\{S_\alpha\}$ are the irreducible components of S, where $S_\alpha=\bar{S}_\alpha\cap S$.*

Proof. The "only if" part was proved by Theorem 2. 6. The "if" part: we first prove that S is a right locally uniform ring with $Z_r(S)=0$. Let $\hat{\bar{S}}_\alpha$ be the maximal right quotient ring of $\bar{S}_\alpha$ for each α. Then $\hat{\bar{S}}_\alpha$ is a full left linear ring over a division ring. We set $K=\Pi_\alpha \bar{S}_\alpha$. Then, by Proposition of [16, p. 72], $\hat{K}=\Pi_\alpha \hat{\bar{S}}_\alpha$ is the maximal right quotient ring of S. By Theorem 3. 9 of [2, p. 117], $\hat{K}$ is right self-injective, right locally uniform and regular as a ring. Hence S is a right locally uniform ring with $Z_r(S)=0$.

Before proving that $\{S_\alpha\}$ are the irreducible components of S, where $S_\alpha=\bar{S}_\alpha\cap S$, we need the following two lemmas.

Lemma 2. 9. *$\bar{S}_\alpha$ is a right quotient ring of S_α.*

Proof. Let $\bar{s}_\alpha$ be a non-zero element of $\bar{S}_\alpha$. Then $0 \neq \bar{s}_\alpha s \in \sum_\alpha \oplus S_\alpha$ for some $s \in S$ and hence $\bar{s}_\alpha s \in S_\alpha$. Since $Z_r(S)=0$, $\sum_\alpha \oplus S_\alpha \subset' S$ and $S_\alpha S_\beta = 0$ $(\alpha \neq \beta)$, we obtain $\bar{s}_\alpha s S_\alpha \neq 0$. Hence $0 \neq \bar{s}_\alpha s s' \in \bar{s}_\alpha S_\alpha \cap S_\alpha$ for some $s' \in S_\alpha$. Since $Z_r(\bar{S}_\alpha)=0$, $\bar{S}_\alpha$ is a right quotient ring of S_α.

Lemma 2. 10. (1) *$S_\alpha \in L_{r2}(S)$ and S_α is an I-ring as a ring for each α.*

(2) *If A is a uniform right ideal of S contained in S_α, then A is a uniform right ideal of the ring S_α.*

(3) *If A_α is a fixed uniform right ideal of S contained in S_α and if A is an arbitrary uniform right ideal of S, then $A \sim A_\alpha$ if and only if $A \subseteq S_\alpha$.*

Proof. (1) Clearly S_α is an ideal and $S_\alpha \cap (\sum_{\beta \neq \alpha} S_\beta)=0$. Let L be a right ideal of S such that $L \supsetneqq S_\alpha$ and let $a=(a_\alpha) \in L$, $a \notin S_\alpha$. Then $a_\beta \neq 0$ for some $\beta \neq \alpha$. Since, by lemma 2. 9, $\bar{S}_\alpha$ is a right quotient ring of S_α, there exists an element x_β of S_β such that $0 \neq a_\beta x_\beta \in S_\beta$ and $0 \neq a_\beta x_\beta = a x_\beta \in L \cap S_\beta$. Hence $L \cap (\sum_{\beta \neq \alpha} S_\beta) \neq 0$ and thus $S_\alpha \in L_{r2}(S)$. Since $\hat{\bar{S}}_\alpha = \hat{S}_\alpha$ is the full ring of linear transformations in a right vector space over a division ring, S_α is an I-ring as a ring.

(2) We may assume that A is closed. Assume that A is not a uniform right ideal of S_α. Then there exist right ideals A_i $(i=1, 2)$ of S_α such that $A \supseteqq A_1 \oplus A_2$. Since $\hat{S} = \Pi_\alpha \hat{S}_\alpha$, we obtain $E_S(A) = E_{S_\alpha}(A) \supseteqq E_{S_\alpha}(A_1) \oplus E_{S_\alpha}(A_2)$ in $\hat{S}$ and $E_{S_\alpha}(A_j)$ is a right ideal of $\hat{S}$ $(j=1, 2)$. Hence $E_\alpha(A_j) \cap S \neq 0$ and $A = E_S(A) \cap S \supseteqq (E_{S_\alpha}(A_1) \cap S) \oplus (E_{S_\alpha}(A_2) \cap S)$. This is a contradiction and hence A is a uniform right ideal of S_α.

(3) First suppose that $S_\alpha \supseteqq A$. By (1) and (2), A_α and A contain non-zero right ideals A_α' and A' of S_α, respectively, such that $A_\alpha' \cong A'$ as an S_α-module. Then $E_S(A) = E_{S_\alpha}(A) = E_{S_\alpha}(A') \cong E_{S_\alpha}(A_\alpha') = E_{S_\alpha}(A_\alpha) = E_S(A_\alpha)$ and thus $A \sim A_\alpha$. Conversely, suppose that $A \sim A_\alpha$ and $A \nsubseteq S_\alpha$. If $A \nsubseteq S_\beta$ for each β, then $A \cap S_\beta = 0$ and hence $A^r \supseteqq S_\beta$, because S_β is an ideal of S. This contadicts $Z_r(S)=0$ and $S \supset' \sum_\alpha \oplus S_\alpha$. Hence $A \subseteq S_\beta$ for some $\beta \neq \alpha$ and thus $\hat{A} \subseteq \hat{S}_\beta$. On the other hand, since $A \sim A_\alpha$ we obtain $\hat{A} \cong \hat{A}_\alpha$ and hence $0 \neq \hat{A}\hat{A}_\alpha \subseteq \hat{S}_\beta \hat{S}_\alpha = 0$, which is a contradiction. Hence if $A \sim A_\alpha$, then $A \subseteq S_\alpha$. This completes the proof of Lemma 2. 10.

Clearly $S_\alpha = (\sum_{A \sim A_\alpha} A)^*$ by Lemma 2. 10 and $\sum_\alpha \oplus S_\alpha \subset' S$. Hence $\{S_\alpha\}$ are the irreducible components of S. This completes the proof of Proposition 2. 8.

The proof of Theorem 2. 7: By Proposition 2. 8, R is a right locally uniform ring with $Z_r(R)=0$ and $\{R_\alpha\}$ are the irreducible components of R, where $R_\alpha = \bar{R}_\alpha \cap R$. For the sake of the completion of the proof of Theorem 2.7, we need several lemmas.

Lemma 2. 11. *Let I be a closed right ideal of R and let $I_\alpha=\{x_\alpha\in\bar{R}_\alpha \mid a=(x_\alpha)\in I$ for some $a\in I\}$. Then I_α is a closed right ideal of $\bar{R}_\alpha$.*

Proof. Let K be a relative complement of I in the sense of Goldie and let $K_\alpha=\{x_\alpha\in\bar{R}_\alpha \mid a=(x_\alpha)\in K$ for some $a\in K\}$. We shall prove that $I_\alpha\cap K_\alpha=0$. Suppose that $I_\alpha\cap K_\alpha\neq 0$ and $0\neq x_\alpha\in I_\alpha\cap K_\alpha$. Then there exist $a=(\cdots x_\alpha, \cdots)\in I$ and $b=(\cdots, x_\alpha, \cdots)\in K$. Since $\bar{R}_\alpha$ is a right quotient ring of R_α by Lemma 2 .9, $0\neq x_\alpha r_\alpha\in R_\alpha$ for some $r_\alpha\in R_\alpha$. Then $0\neq ar_\alpha=br_\alpha\in I\cap K$, which is a contradiction. Hence $I_\alpha\cap K_\alpha=0$. Suppose that I_γ is not a closed right ideal of $\bar{R}_\gamma$ for some γ. Then there exists a right ideal L_γ of $\bar{R}_\gamma$ such that $L_\gamma\supsetneqq I_\gamma$ and $L_\gamma\cap K_\gamma=0$. Now we set $L_\alpha=I_\alpha$ for $\alpha\neq\gamma$ and put $L=\{r=(r_\alpha) \mid r\in R$ and $r_\alpha\in L_\alpha$ for each $\alpha\}$. Then L is a right ideal of R which contains I. If $L=I$, then $L_\gamma=I_\gamma$, which is a contradiction. Hence $L\supsetneqq I$ and thus $L\cap K\neq 0$. Let $a=(a_\alpha)$ be a non-zero element of $L\cap K$. Then $0\neq a_\beta\in L_\beta\cap K_\beta$ for some β. This is a contradiction. Hence I_α is a closed right ideal for each α.

Lemma 2. 12. *Let T be a non-zero element of $L_{r2}(R_\alpha)$. Then*

(1) *$T\in L_{r2}(R)$.*

(2) *$\bar{T}\in L_{r2}(\bar{R}_\alpha)$, where $\bar{T}=\hat{T}\cap\bar{R}_\alpha$.*

Proof. (1) we put $T^*=\cap\{A^r \mid A^r\supseteqq T$ and A is an atom of $L_r(R)\}$. Clearly $T^*\in L_{r2}(R)$ and $T^*\supseteqq T$. Suppose that $T^*\supsetneqq T$. Then since $T\in L_r(R)$, by (3) of Proposition 2. 3, there exists an atom B of $L_r(R)$ such that $T^*\supseteqq B$ and $B\cap T=0$. If $B\nsubseteq R_\alpha$, then $B\subseteqq R_\beta$ for some $\beta\neq\alpha$ and $B^r\supseteqq R_\alpha\supseteqq T$. Hence $B^r\supseteqq T^*$ and thus $B^2=0$. This is a contradiction. If $B\subseteqq R_\alpha$, then since $T\in L_{r2}(R_\alpha)$ and $T\cap B=0$, we obtain $B^r\supseteqq T$. Hence $B^r\supseteqq T^*$ by the definition of T^* and thus $B^2=0$. This is a contradiction and hence $T=T^*\in L_{r2}(R)$.

(2) Let K be a relative complement of T in R and let $\bar{K}=\hat{K}\cap\bar{R}$. Then clearly $\bar{T}\cap\bar{K}=0$. Suppose that $\bar{T}$ is not an ideal of $\bar{R}$. Then $\bar{x}_\alpha\bar{t}\notin\bar{T}$ for some $\bar{x}_\alpha\in\bar{R}_\alpha$ and $\bar{t}\in\bar{T}$. Hence $(\bar{x}_\alpha\bar{t}\bar{R}_\alpha^{\ 1}+\bar{T})$[2] $\cap\bar{K}\neq 0$. Let $\bar{k}$ be a non-zero element of $\bar{K}\cap(\bar{x}_\alpha\bar{t}\bar{R}_\alpha^{\ 1}+\bar{T})$ and let $\bar{k}=\bar{t}_1+\sum_{j=1}^m\bar{x}_\alpha\bar{t}\bar{r}_j+n\bar{x}_\alpha\bar{t}$, where $\bar{t}_1\in\bar{T}$ and $\bar{r}_j\in\bar{R}_\alpha$. Since $Z_r(\bar{R}_\alpha)=0$ and $\bar{R}_\alpha$ is a right quotient ring of R_α, there exists an element of $r\in R_\alpha$ such that $0\neq\bar{k}r\in K$ and $\bar{t}_1r$, $\bar{t}\bar{r}_jr$, $\bar{t}r\in T$. Since R is a subdirect sum of $\{\bar{R}_\alpha\}$, there exists $s\in R$ such that $s=(\cdots\bar{x}_\alpha, \cdots)$. Since $T\in L_{r2}(R)$ and $\bar{t}\bar{r}_jr$, $\bar{t}r\in T$, we obtain $\bar{x}_\alpha\bar{t}\bar{r}_jr=s\bar{t}\bar{r}_jr$, $\bar{x}_\alpha\bar{t}r=s\bar{t}r\in T$. Hence $0\neq\bar{k}r\in T\cap K=0$. This is a contradiction and hence $\bar{T}\in L_{r2}(\bar{R}_\alpha)$. This completes the proof of Lemma 2. 12.

By Lemma 2. 11, R is a potent ring. Since $L_r(R_\alpha)\cong L_r(\bar{R}_\alpha)$ under the contraction, R_α is residue-finite by Lemma 2. 12 if $\bar{R}_\alpha$ is residue-finite. Hence

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2) The principal right ideal of a ring R, generated by a, is denoted by aR^1.

if $\bar{R}_\alpha$ is residue-finite for each α, then R is locally residue-finite. This completes the proof of Theorem 2. 7.

3. Residue-finite PI-rings which are of type (A)

Theorem 3. 1. *Let R be a residue-finite CPI-ring. Then*

(1) $L_{r2}=J_{r2}=\{A^r \mid A\in L_r\text{: atom}\}\cup\{0, R\}$.

(2) *L_{r2} is a chain and there are the following two cases:*

(A): *L_{r2} is an infinite chain $R=T_0\supset T_1\supset T_2\supset\cdots$ such that $\cap_{p=0}^{\infty} T_p=0$.*

(B): *L_{r2} is a finite chain $R=T_0\supset T_1\supset T_2\supset\cdots\supset T_p\supset T_{p+1}=0$.*

(3) *For each non-zero $T_p\in L_{r2}$, there exists an independent set $\{A_1, \cdots, A_n\}$ of atoms of L_r such that $A_1\cup^*\cdots\cup^* A_n\cup^* T_p=T_{p-1}$ and $(A_1\cup^*\cdots\cup^* A_n)\cap T_p=0$.*

(4) *If A is an atom of L_r, then $A\subseteqq T_p$ and $A\nsubseteqq T_{p+1}$ if and only if $A^r=T_{p+1}$.*

Proof. (1) By Proposition 5 of [2, p. 71], $L_{r2}\supseteqq\{A^r\mid A\in L_r\text{: atom}\}$. Conversely, if $T\in L_{r2}$ such that $T\neq R$, $T\neq 0$, then the set $S=\{A^r\mid A^r\supseteqq T, A\in L_r\text{: atom}\}$ is non-empty, becauce there exists an atom $A\in L_r$ such that $A\cap T=0$ and hence $A^r\supseteqq T$. Since $\dim_R R/T<\infty$, there exists a minimal element A^r in S by Lemma 3. 6 of [9]. If $A^r\supsetneqq T$, then there exists an atom $C\in L_r$ such that $A^r\supseteqq C$ and $C\cap T=0$. Hence $C^r\supseteqq T$, i.e., $C\in S$. By Theorem 1. 4 of [15], $A^r\supsetneqq C^r$ or $C^r\supseteqq A^r$. If $C^r\supseteqq A^r$, then $C^r\supseteqq A^r\supseteqq C$ and $C^2=0$. This is a contradiction. If $A^r\supsetneqq C^r$, then this contradicts the choice of A^r. Hence we obtain $T=A^r$, as desired.

(2) It is clear that L_{r2} is a chain by (1) and Theorem 1. 4 of [15]. We shall show that the condition (B) holds if and only if there exists an atom A of L_r such that $A^r=0$. At first, suppose that $T_p\neq 0$ and $T_{p+1}=0$ for some p. Then there exists an atom A of L_r such that $T_p\supseteqq A$. By (1), $A^r=T_k$ for some k. If $k\leqq p$, then $A^r=T_k\supseteqq T_p\supseteqq A$ and thus $A^2=0$. This is a contradiction and hence $A^r=T_{p+1}=0$. Conversely, suppose that $A^r=0$ for some A of L_r and that L_{r2} is an infinite chain, i.e.,

$$L_{r2}\colon R=T_0\supset T_1\supset\cdots\supset T_p\supset\cdots .$$

Let $T=\cap_{p=0}^{\infty} T_p$. Then $T=0$, because R is residue-finite. Hence we may assume that $T_{p-1}\supseteqq A$ and $T_p\not\supseteqq A$ for some p. Then $A\cap T_p=0$, because A is an atom. Thus $A^r\supseteqq T_p$ and hence $T_p=0$, which is a contradiction. Hence L_{r2} is a finite chain. If L_{r2} is an infinite chain, then it is clear that $\cap_{p=0}^{\infty} T_p=0$, because R is residue-finite.

Since R is a right locally uniform residue-finite ring, (3) follows from the definition of Goldie's dimension.

(4) First we suppose that $A\subseteqq T_p$ and $A\nsubseteqq T_{p+1}$. By (1), $A^r=T_k$ for some k. If $k\leqq p$, then $A^r=T_k\supseteqq T_p\supseteqq A$ and thus $A^2=0$. This is a contradiction. Hence we obtain $k\gneqq p$ and thus $A^r\subseteqq T_{p+1}$. Since $A\nsubseteqq T_{p+1}$, it is clear that

$A^r \supseteqq T_{p+1}$ and hence $A^r = T_{p+1}$. Conversely, suppose that $A^r = T_{p+1}$. Then if $A \subsetneqq T_p$, then $A^r \supseteqq T_p$, which is a contradiction. Hence $A \subseteqq T_p$. It is clear that $T_{p+1} \not\supseteqq A$, because A is potent.

The lattices J_r and J_l are dual isomorphic under the correspondence $A \rightarrow A^l$, $A \in J_r$. Hence if J_{r2} consists of $\{T_p\}_{p=0}^{\infty}$ such that $R = T_0 \supset T_1 \supset \cdots$, $\cap_{p=0}^{\infty} T_p = 0$, then J_{l2} consists of $\{T_p^l\}_{p=0}^{\infty}$ such that

$$(3.2) \qquad 0 = T_0^l \subset T_1^l \subset \cdots \subset T_p^l \cdots, \ \cup_{p=0}^{\infty} T_p^l = R.$$

If J_{r2} consists of $\{T_i\}_{i=0}^{p+1}$ such that $R = T_0 \supset T_1 \supset \cdots \supset T_p \supset T_{p+1} = 0$, then J_{l2} consists of $\{T_i^l\}_{i=0}^{p+1}$ such that

$$(3.3) \qquad 0 = T_0^l \subset T_1^l \subset \cdots \subset T_p^l \subset T_{p+1}^l = R.$$

Lemma 3.4. *Let R be a residue-finite CPI-ring and $J_{l2} = \{T_0^l, T_1^l, \cdots\}$ be given by* (3.2) *or by* (3.3). *Then*

(1) *For each $T_p^l \neq R$, there exists a potent atom $B \in J_l$ such that $B \subseteqq T_{p+1}^l$ and $B \cap T_p^l = 0$.*

(2) *If B is a potent atom of J_l, then $B \subseteqq T_{p+1}^l$ and $B \subsetneqq T_p^l$ if and only if $B^l = T_p^l$.*

Proof. (1) By Theorem 3.1, there exists an atom A of L_r such that $A^r = T_{p+1}$ and $T_p \supseteqq A$. Since A is potent, $aA \neq 0$ for some $a \in A$ and thus $a^r \cap A = 0$, because A is atomic. By Theorem 6.9 of [12], a^r is maximally closed and thus a^r is a maximal annihilator. Hence $B = a^{rl}$ is an atom of J_l. Furthermore, since $a^r \cap A = 0$ and $a \in A$, we obtain that B is potent and $B \subseteqq T_{p+1}^l$. If $B \cap T_p^l \neq 0$, then $B \subseteqq T_p^l$ and $B^r = a^r \supseteqq T_p \supseteqq A$. This contradicts the choice of a. Hence $B \cap T_p^l = 0$.

(2) First we assume that $B \subseteqq T_{p+1}^l$ $B \subsetneqq T_p^l$ and B is potent. Then it is clear that $B \cap T_p^l = 0$ and hence $B^l \supseteqq T_p^l$. If $B^l \supsetneqq T_p^l$, then $B^l \supseteqq T_{p+1}^l$ and thus $B^2 = 0$. This is a contradiction and hence $B^l = T_p^l$. Conversely, suppose that $B^l = T_p^l$ and B is potent. Then clearly $T_p^l \not\supseteqq B$. If $T_{p+1}^l \not\supseteqq B$, then $B \cap T_{p+1}^l = 0$ and thus $B^l \supseteqq T_{p+1}^l$. This is a contradiction and hence $T_{p+1}^l \supseteqq B$.

By Theorem 2.3 of [14], the lattice J_l is upper semi-modular. Now let $B \in J_l$. If there exists a finite chain in J_l $0 = B_0 < B_1 < \cdots < B_d = B$ such that B_i is a cover of B_{i-1} $(1 \leqq i \leqq d)$, then, by Theorem 14 of [1], we can define the *dimention of B* as such an integer d and write $d = \dim B$.

Following R. E. Johnson [13], R is said to be a *right stable ring* if R is a right locally uniform ring with $Z_r(R) = 0$ and $(\sum A_\alpha)^r = 0$, where A_α runs all over uniform right ideals. Clearly, if R is a PI-ring, then R is a right stable ring.

Lemma 3. 5. *Let R be a right stable ring and let B be an atom of J_l. Then B^r is maximally closed.*

Proof. Since R is a right stable ring, there exists an atom A of L_r such that $AB \neq 0$. Then b^r is maximally closed for $0 \neq b \in B \cap A$. Hence b^{rl} is a minimal annihilator and $b^{rl} \cap B \neq 0$. Thus $B = b^{rl}$ and hence $B^r = b^r$ is maximally closed.

Lemma 3. 6. *Let R be a residue-finite CPI-ring. Then*

(1) $\dim_R(R/T_p) = d_p$ *if and only if* $\dim T_p^l = d_p$

(2) *For each non-zero T_p, there exists an independent set $\{B_i\}_{i=1}^n$ of potent atoms of J_l such that $T_p^l = T_{p-1}^l \cup (B_1 \cup \cdots \cup B_n)$, $(B_1 \cup \cdots \cup B_n) \cap T_{p-1}^l = 0$, where $n = \dim_R T_{p-1}/T_p$.*

Proof. Since $\dim_R(R/T_p)$ is finite, (1) immediately follows from Lemma 2. 2 of [14].

(2) By Lemma 3. 4, there exists a potent atom B_1 of J_l such that $T_p^l \supseteqq B_1$ and $T_{p-1}^l \cap B_1 = 0$. Assume that we have selected an independent set $\{B_1, \cdots, B_k\}$ of potent atoms of J_l such that $C \subseteqq T_p^l$ and $C \cap T_{p-1}^l = 0$, where $C = B_1 \cup \cdots \cup B_k$. If $C \cup T_{p-1}^l \subsetneqq T_p^l$, then $C^r \cap T_{p-1} \supsetneqq T_p$. Hence there exists an atom $A \in L_r$ such that $C^r \cap T_{p-1} \supseteqq A$ and $A \cap T_p = 0$. By (4) of Theorem 3. 1, $A^r = T_p$. By the same way as in (1) of Lemma 3. 4, there exists an atom B of J_l such that $B \subseteqq T_p^l$, $B \cap T_{p-1}^l = 0$ and $B = a^{rl}$ with $a \in A$, $a^r \cap A = 0$. Assume that $B \cap (C \cup T_{p-1}^l) \neq 0$. Then $B \subseteqq (C \cup T_{p-1}^l)$ and so $B^r = a^r \supseteqq C^r \cap T_{p-1} \supseteqq A$, which is a contradiction. Hence we obtain that $B \cap (C \cup T_{p-1}^l) = 0$. Then, by the same way as in Corollary 2. 4 of [14], we obtain that $(B \cup C) \cap T_{p-1}^l = 0$ and thus, by (1), the assertion of (2) now follows by induction.

Let $\dim_R(R/T_p) = d_p$ for each non-zero $T_p \in L_{r2}$. Then evidently $\dim_R(T_{p-1}/T_p) = d_p - d_{p-1}$. If R satisfies (A) in Theorem 3. 1, then we shall call the *ring R of type (A)* and $(d_1, d_2 - d_1, \cdots, d_p - d_{p-1}, \cdots)$ *the set of block numbers of R.*

If R satisfies (B) in Theorem 3. 1, then we shall call the *ring R of type* (B) and $(d_1, d_2 - d_1, \cdots, d_p - d_{p-1}, \infty)$ the *set of block numbers* of R.

Let L be an atomic lattice with 1. A set $\{a_i\}$ of atoms of L is *independent* if $a_i \cap (\cup_{j \neq i} a_j) = 0$ for each i. An independent set $\{a_i\}$ of atoms of L is called a *basis* of L if $\cup_i a_i = 1$.

In order to make further progress we need the following definitions:

Let R be a residue-finite *CPI*-ring which is of type (A), let $L_{r2} = \{T_0, T_1, T_2, \cdots\}$ and let $\dim_R R/T_p = d_p$ for each p. Then we say that R has *matrix representable conditions* (for short: *m.r.* conditions), if there exists a set $\{B_i\}_{i=1}^\infty$ of potent atoms of J_l such that

(a) $T^l_p = T^l_{p-1} \cup (B_{d_{p-1}+1} \cup \cdots \cup B_{d_p})$, $T^l_{p-1} \cap (B_{d_{p-1}+1} \cup \cdots \cup B_{d_p}) = 0$

for each p,

(b) $T_p \cup^* T^c_p = R$ and $T_p \cap T^c_p = 0$ for each p, where $T^c_p = (\cup_{j>d_p} B_j)^r$,

(c) $\cup^{*\infty}_{p=0} T^c_p = R$.

Let R be a residue-finite *CPI*-ring which is of type (B) and let $L_{r2} = \{T_0, T_1, \cdots, T_p, T_{p+1}\}$, where $T_{p+1} = 0$ and let $\dim_R R/T_k = d_k$ for each $k \leqq p$. Then we say that R has *m.r. conditions* if there exists a basis $\{B_i\}^\infty_{i=1}$ of potent atoms of J_l such that

(d) $T^l_k = T^l_{k-1} \cup (B_{d_{k-1}+1} \cup \cdots \cup B_{d_k})$, $T^l_{k-1} \cap (B_{d_{k-1}+1} \cup \cdots \cup B_{d_k}) = 0$ for each $k \leqq p$,

(e) $\cup^{*\infty}_{i=1} A_i = R$, where $A_i = (\cup_{j \neq i} B_j)^r$ for each i.

Now, for the sake of giving examples of residue-finite *CPI*-rings with m.r. conditions, we shall generalize the concept of T-rings which was defined on finite dimensional rings in [15] to the case when the ring considered is infinite dimensional. Let F be a division ring and let ω be a countable ordinal number. We denote by $(F)_\omega$ the ring of all column-finite $\omega \times \omega$ matrices over F. Let F_{ij} be additive subgroups of F such that

$$F_{ij} F_{jk} \subseteq F_{ik} \quad (i, j, k = 1, 2, \cdots). \tag{3.7}$$

Let

$$S = \{a \in (F)_\omega \mid a = (a_{ij}),\ a_{ij} \in F_{ij}\}. \tag{3.8}$$

Clearly S is the subring of $(F)_\omega$. The ring S will be called a *T-ring* (*triangular-block matrix ring*) *with type* (A) *in* $(F)_\omega$ if there exist integers d_n such that $0 = d_0 < d_1 < \cdots < d_n < \cdots$ and

$$F_{ij} \neq 0 \Leftrightarrow i > d_p \text{ and } d_p < j \leqq d_{p+1} \quad (p = 0, 1, \cdots). \tag{3.9}$$

The ring S will be called a *T-ring with type* (B) *in* $(F)_\omega$ if there exist integers d_n such that $0 = d_0 < d_1 < \cdots < d_p$ and

$$F_{ij} \neq 0 \Leftrightarrow \text{(i) if } j \leqq d_p \text{ and if } d_k < j \leqq d_{k+1} \tag{3.10}$$

for some k ($0 \leqq k < p$), then $i > d_k$, (ii) if $j > d_p$, then $i > d_p$.

In both cases, we let

$$M = \{a \in (F)_\omega \mid a = (a_{ij}),\ a_{ij} \in F'_{ij}\}, \quad \text{where } F'_{ij} = F \text{ whenever } F_{ij} \neq 0 \text{ and } F'_{ij} = 0 \text{ otherwise.} \tag{3.11}$$

Following R. E. Johnson, we shall call M the *full cover of* S. Let A and B be subsets of a division ring F. The set $\{ab^{-1} \mid a \in A,\ 0 \neq b \in B\}$ will be denoted by AB^{-1}.

Since $(F)_\omega$ is column-finite, we obtain the following two propositions by the same arguments as in Theorems 3. 5 and 3. 7 of [15].

Proposition 3. 12. *Let S be a T-ring in $(F)_\omega$ given by* (3. 9) *or by* (3. 10). *Then $S \leqq (F)_\omega$ if and only if $F_{11}F_{11}^{-1}=F$.*

Proposition 3. 13. *Let S be a T-ring in $(F)_\omega$ given by* (3. 9) *or by* (3. 10) *such that $S \leqq (F)_\omega$. Then S is potent if and only if $F_{jj}F_{kj}^{-1}=F$ for $j<k$ $(j, k=2, 3, \cdots)$.*

Proposition 3. 14. *Let S be a T-ring with type (A) in $(F)_\omega$ whose blocks are defined by the numbers $d_0, d_1, \cdots, d_n, \cdots$ with $0=d_0<d_1<\cdots d_n<\cdots$ in* (3. 9). *If $S \leqq (F)_\omega$ and if S is potent, then*

(1) *S is a residue- finite PI-ring with m.r. conditions which is of type (A).*

(2) *$L_{r2}=\{T_0, T_1, \cdots, T_n, \cdots\}$, where $T_0=S$ and $T_n=\{a \in S \mid a=(a_{ij}),$ $a_{ij}=0$ if $i \leqq d_n\}$ for each n.*

Proof. (2) follows from the same argument as in Theorem 3. 9 of [15].

(1) Let $B_i=\{a \in S \mid a=(a_{ij}),\ a_{ij} \in F_{ij}$ and $a_{ki}=0$ if $k \neq i\}$ for each positive integer i and let

$$b_i = \begin{pmatrix} 0 & & & & & \\ & \ddots & & & & 0 \\ & & 0 & & & \\ & & & f_i & & \\ & & & & 0 & \\ & 0 & & & & \ddots \end{pmatrix}$$

where $0 \neq f_i \in F_{ii}$ and other positions are all zero. Then it is clear that $b_i^{r_l}=B_i$ and that $\{B_i\}_{i=1}^{\infty}$ is a set of potent atoms of J_l. Further, it is easily checked that the set $\{B_i\}_{i=1}^{\infty}$ satisfies the conditions (a), (b) and (c). The other assertions are evident.

Corollary 3. 15. *If M is the full cover of S which is a T-ring with type (A) in $(F)_\omega$, then M is a residue-finite PI-ring with m.r. conditions which is of type (A).*

Proposition 3. 16. *Let S be a T-ring with type (B) in $(F)_\omega$ whose blocks are defined by the numbers $d_0, d_1, \cdots, d_p$ with $0=d_0<d_1<\cdots<d_p$ as in* (3. 10). *If $S \leqq (F)_\omega$ and if S is potent, then*

(1) *S is a residue-finite PI-ring with m.r. conditions of type (B).*

(2) *$L_{r2}=\{T_0, T_1, \cdots, T_p, T_{p+1}\}$, where $T_0=S$, $T_{p+1}=0$ and $T_k=\{a \in S \mid$ $a=(a_{ij}),\ a_{ij}=0$ if $i \leqq d_k\}$ for $1 \leqq k \leqq p$.*

Proof. (2) follows from the same argument as in Theorem 3. 9 of [15].

(1) Let $\{B_i\}_{i=1}^{\infty}$ be as in the proof of Proposition 3. 14. Then it is easily checked that $\{B_i\}$ is a basis of potent atoms of J_l and that it satisfies the conditions (d) and (e).

Corollary 3. 17. *If M is the full cover of S which is a T-ring with type* (B) *in $(F)_\omega$, then M is a residue-finite PI-ring with m.r. conditions of type* (B).

Let R be a residue-finite PI-ring of type (A) with m.r. conditions, and let $\{B_i\}$ be a set of potent atoms of J_l which satisfies the conditions (a), (b) and (c). Now we set $A_i=(\cup_{j\neq i} B_j)^r$. Then the following lemma holds:

Lemma 3. 18. (1) *$\{A_i\}$ and $\{B_i\}$ are bases of potent atoms of L_r and J_l respectively.*

(2) *For each p, $T_{p-1}=T_p\cup^*(A_{d_{p-1}+1}\cup^*\cdots\cup^*A_{d_p})$ and $T_p\cap(A_{d_{p-1}+1}\cup^*\cdots\cup^*A_{d_p})=0$.*

(3) *$B_i=(\cup^*_{j\neq i} A_j)^l$.*

Proof. (1) We first prove that $\{B_i\}$ is an independent set of atoms of J_l. If $B_i\cap A_i^l\neq 0$ for some i $(d_{p-1}<i\leqq d_p)$, then $B_i\subseteqq C\cup T_p^{cl}$ and $B_i^r\supseteqq C^r\cap T_p^c$, where $C=B_1\cup\cdots\cup B_{i-1}\cup B_{i+1}\cup\cdots\cup B_{d_p}$ and $T_p^c=(\cup_{j>d_p}B_j)^r$. Since $T_p^l=B_1\cup\cdots\cup B_{d_p}$, we obtain that $T_p=B_i^r\cap C^r$. By the assumption, $T_p\cup^*T_p^c=R$. Hence $C^r=C^r\cap(T_p\cup^*T_p^c)=C^r\cap[(B_i^r\cap C^r)\cup^*T_p^c]=(C^r\cap T_p^c)\cup^*(B_i^r\cap C^r)\subseteqq B_i^r$ by the modular law and we obtain $C\supseteqq B_i$. This is a contradiction, because $\{B_1,\cdots,B_{d_p}\}$ is an independent set of atoms of J_l. Hence $B_i\cap(B_1\cup\cdots\cup B_{i-1}\cup B_{i+1}\cup\cdots)=0$, i.e., $\{B_i\}$ is independent. Since $\cup_{p=0}^{\infty}T_p^l=R$, $\cup_i B_i=R$ and hence $\{B_i\}$ is a basis of J_l. Clearly $B_i^r\cap A_i=0$, $B_i^r\cup A_i=R$ and B_i^r is a maximal closed right ideal by Lemma 3. 5. Hence $B_i^r\cup^*A_i=R$ and thus A_i is an atom of L_r. If $A_i\cap(A_1\cup^*\cdots\cup^*A_{i-1}\cup^*A_{i+1}\cup^*\cdots)\neq 0$, then $R\neq A_i^l\cup(A_1^l\cap\cdots\cap A_{i-1}^l\cap A_{i+1}^l\cap\cdots)\supseteqq\cup_i B_i=R$, which is contradiction. Hence $\{A_i\}$ is an idependent set of atoms of L_r. Since $T_p^c\supseteqq A_1\oplus\cdots\oplus A_{d_p}$ and $\dim_R R/T_p=d_p$, we obtain $T_p^c=\cup^{*d_p}_{i=1}A_i$. Since $\cup_p^* T_p^c=R$ by the assumption, we obtain $R=\cup_i^* A_i$, as desired.

(2) follows from the same way as in the proof of (1).

(3) Clearly $B_i\subseteqq(\cup^*_{j\neq i}A_j)^l$ and $(\cup^*_{j\neq i}A_j)^l$ is an atom of J_l. Hence $B_i=(\cup^*_{j\neq i}A_j)^l$.

Theorem 3. 19. *If R is a residue-finite PI-ring with m.r. conditions of type (A) and if $(d_1,\cdots,d_p,\cdots)$ is the set of block numbers of R, where d_i is a positive integer, then there exist potent atomic bases $\{B_i\}$ for J_l and $\{A_i\}$ for L_r such* that:

(1) *$A_i=(\cup_{j\neq i}B_j)^r$ and $B_i=(\cup^*_{j\neq i}A_j)^l$ $(i=1, 2, \cdots)$.*

(2) *$J_{r2}=L_{r2}=\{A_i^r\,|\,i=1, 2, \cdots\}$, $J_{l2}=\{B_i^l\,|\,i=1, 2, \cdots\}$.*

(3) *$A_1^r\supseteqq A_2^r\supseteqq\cdots\supseteqq A_p^r\supseteqq\cdots$, $\cap_{p=1}^{\infty}A_p^r=0$ and $0=B_1^l\subseteqq B_2^l\subseteqq\cdots\subseteqq B_p^l\subseteqq\cdots$, $\cup_{p=1}^{\infty}B_p^l=R$.*

(4) $A_i^r=A_j^r$ and $B_i^l=B_j^l$ *if and only if* $d_0+d_1+\cdots+d_p<i$ *and* $j\leqq d_0+d_1+\cdots+d_{p+1}$ *for some* p, *where* $d_0=0$.

(5) $A_iB_j\neq 0$ *if and only if* $i>d_0+\cdots+d_p$ *and* $d_0+\cdots+d_p<j\leqq d_0+\cdots+d_{p+1}$ *for some* p.

Proof. Let $\{B_i\}$ be potent atoms of J_l which satisfies the conditions (a), (b) and (c). And let $A_i=(\cup_{j\neq i}B_j)^r$ for each i. Then, by Theorem 3. 1, Lemmas 3. 4 and 3. 18, (1)~(4) are evident.

(5) For any B_j, there exists an integer p such that $d_0+\cdots+d_p<j\leqq d_0+\cdots+d_{p+1}$. Then $B_j^l=T_p^l$ by Lemma 3. 4. Suppose that $A_iB_j=0$. Then the following implications hold:

$A_iB_j=0\Leftrightarrow T_p^l=B_j^l\supseteqq A_i\Leftrightarrow T_p\subseteqq A_i^r=T_k$ for some $k\Leftrightarrow p\geqq k\Leftrightarrow i\geqq d_0+\cdots+d_p$.
Hence $A_iB_j\neq 0$ if and only if $i>d_0+\cdots+d_p$.

Let R be a residue-finite PI-ring with m.r. conditions of type (A) and let $\{A_i\}$ and $\{B_i\}$ be atomic bases given by Theorem 3. 19. Then $\{\hat{A}_i\}$ is an atomic basis of $L_r(\hat{R})$ which corresponds to the atomic basis $\{A_i\}$ of $L_r(R)$. By Theorem 1. 11 of [2, p. 108], there exist matrix units $\{e_{ij}|i,j=1,2,\cdots\}$ in $\hat{R}$ such that $\hat{A}_i=e_{ii}\hat{R}$ and $\hat{R}=(F)_\omega$, where F is a division ring. Clearly $A_i=e_{ii}\hat{R}\cap R$ and $B_i=(\cup_{j\neq i}A_j)^l=\hat{R}e_{ii}\cap R$. Let

$$A_i\cap B_j=F_{ij}e_{ij}\quad (i,j=1,2,\cdots).$$

Then F_{ij} are additive subgroups of F satisfying (3. 7).

If we put

$$S=\{a\in R\,|\,a=(a_{ij}),\ a_{ij}\in F_{ij}\}\,, \tag{3. 20}$$

then S is a subring of R. By Theorem 3. 19,

$$F_{ij}\neq 0\Leftrightarrow i>d_0+\cdots+d_p \text{ and } d_0+\cdots+d_p<j\leqq d_0+\cdots+d_{p+1} \text{ for some } p\,.$$

Thus, S is a T-ring in $(F)_\omega$ with the same block numbers as in R. Let M be the full cover of S. Then we have

Lemma 3. 21. *If R is a residue-finite PI-ring with m.r. conditions of type (A), if S is a T-ring given by* (3. 20) *and if M is the full cover of S in* $(F)_\omega$, *then*

(1) $S\leqq R\leqq M$.

(2) *S is a potent ring.*

Proof. Since $B_1^l=0$, it is clear that $B_1\leqq R$. Since $\{A_i\cap B_1\}_{i=1}^\infty$ is an atomic basis of the ring B_1 and $Z_r(B_1)=0$, we obtain $\sum_{i=1}^\infty(A_i\cap B_1)\leqq B_1$ Hence $\sum_{i=1}^\infty(A_i\cap B_1)\leqq R$ by Lemma 2 of [2, p. 88]. Since $\sum_{i=1}^\infty(A_i\cap \mathrm{B}_1)\subseteqq S$, we obtain $S\leqq R$. Let b *be* a non-zero element of R, then $b\in\hat{R}$ and $b=(b_{ij})$ for some $b_{ij}\in F$. If $b_{rs}\neq 0$, then $c=(e_{rr}f)b(e_{ss}g)\in R$ for any non-zero $f\in F_{rr}$ and

$g\in F_{ss}$ and thus $c=fb_{rs}ge_{rs}\in A_r\cap B_s$. Hence $fb_{rs}g\in F_{rs}$, i.e., $F_{rs}\neq 0$. Thus $b\in M$.

By the same argument as in Theorem 4. 3 of [15], (2) follows immediately.

By Lemma 3. 21, we have

Theorem 3. 22. *Let R be a left faithful ring and let $\hat{R}$ be the maximal right quotient ring of R. Then R is a residue-finite PI-ring with m.r. conditions of type (A) if and only if it satisfies the following two conditions:*

(1) $\hat{R}=(F)_\omega$, *where F is a division ring,*

(2) $S\leqq R\leqq M$, *where S is a potent T-ring with type (A) in $(F)_\omega$ and M is the full cover of S in $(F)_\omega$.*

4. Residue-finite PI-rings which are of type (B)

Throughout this section, let R be a residue-finite CPI-ring. Let R be a ring of type (B) with m.r. conditions, let $L_{r2}=\{T, T_0, \cdots_1 T_p, T_{p+1}\}$ and let $\dim_R R/T_k=d_k$ for each $k\leqq p$. And let $\{B_i\}$ be a basis of J_l which satisfies the conditions (e) and (d). Now we put $A_i=(\cup_{j\neq i} B_j)^r$ for each i. Then, by the same arguement as in Lemma 3. 18, the following lemma holds:

Lemma 4. 1. (1) *$\{A_i\}$ and $\{B_i\}$ are bases of L_r and J_l respectively,*

(2) $T_{k-1}=T_k\cup^*(A_{d_{k-1}+1}\cup^*\cdots\cup^*A_{d_k})$, $T_k\cap(A_{d_{k-1}+1}\cup^*\cdots\cup^*A_{d_k})=0$ *for each $k\leqq p$ and* $T_p=\cup^*_{j>d_p}A_j$.

(3) $B_i=(\cup^*_{j\neq i} A_j)^l$.

By the validity of Lemma 4. 1, the proof of the following theorem proceeds just like that of Theorem 3. 19 did.

Theorem 4. 2. *Let R be a residue-finite PI-ring with m.r. conditions which is of type (B) and let $(d_1, d_2, \cdots, d_p, \infty)$ be the set of block numbers of R, where d_i is a positive integer. Then there exist potent atomic bases $\{B_i\}$ for J_l and $\{A_i\}$ for L_r such that*

(1) $A_i=(\cup_{j\neq i} B_j)^r$ *and* $B_i=(\cup^*_{j\neq i}A_j)^l$, $(i=1, 2, \cdots)$.

(2) $J_{r2}=L_{r2}=\{A_i^r\,|\,i=1, 2, \cdots\}$, $J_{l2}=\{B_i^l\,|\,i=1, 2, \cdots\}$.

(3) $A_1^r\supseteqq A_2^r\supseteqq\cdots\supseteqq A_n^r\neq 0$, $A_j^r=0\ (j>n)$ *and* $0=B_1^l\subseteqq B_2^l\subseteqq\cdots\subseteqq B_n^l\subsetneqq B_{n+2}^l=B_{n+2}^l=\cdots$, *where* $n=d_1+\cdots+d_p$.

(4) *For* $1\leqq i, j\leqq n$, $A_i^r=A_j^r$ *and* $B_i^l=B_j^l$ *if and only if* $d_0+d_1+\cdots+d_k<i$ *and* $j\leqq d_0+d_1+\cdots+d_{k+1}$ *for some* $0\leqq k<p$, *where* $n=d_1+\cdots+d_p$ *and* $d_0=0$.

(5) $A_iB_j\neq 0 \Leftrightarrow$ (*i*) *If* $j\leqq d_0+\cdots+d_p$ *and if* $d_0+\cdots+d_k<j\leqq d_0+\cdots+d_{k+1}$ *for some k* $(0\leqq k<p)$, *then* $i>d_0+\cdots+d_k$, (*ii*) *if* $j>d_0+\cdots+d_p$, *then* $i>d_0+\cdots+d_p$, *where* $d_0=0$.

Let R be a residue-finite PI-ring with m.r. conditions which is of type (B) and let $\{A_i\}$ and $\{B_i\}$ be given as in Theorem 4. 2. Then we obtain $\hat{R}=(F)_\omega$ and $\hat{A}_i=e_{ii}\hat{R}$, where F is a division ring and $\{e_{ij}\}$ are matrix units for $(F)_\omega$. Clearly $A_i=e_{ii}\hat{R}\cap R$ and $B_i=(\cup^*_{j\neq i} A_j)^l=\hat{R}e_{ii}\cap R$. Let

$$A_i\cap B_j=F_{ij}e_{ij} \quad (i, j=1, 2, \dots).$$

Then F_{ij} are additive subgroups of F satisfying (3. 7).

If we put

$$S=\{a\in R\,|\,a=(a_{ij}),\ a_{ij}\in F_{ij}\}\ , \tag{4. 3}$$

then S is a subring of R. By Theorem 4. 2, we obtain

$F_{ij}\neq 0 \Leftrightarrow$ (i) If $j\leqq d_0+\cdots+d_p$ and if $d_0+\cdots+d_k<j\leqq d_0+\cdots+d_{k+1}$ for some k $(1\leqq k<p)$, then $i>d_0+\cdots+d_k$, (ii) if $j>d_0+\cdots+d_p$ then $i>d_1+\cdots+d_p$.

Thus, S is a T-ring in $(F)_\omega$ with the same blook numbers as in R. Let M be the full cover of S. Then, by the same argument as in Lemma 3. 21, we obtain $S\leqq R\leqq M$ and S is a potent ring. Hence we obtain the following:

Theorem 4. 4. *Let R be a left faithful ring and let $\hat{R}$ be the maximal right quotient ring of R. Then R is a residue-finite PI-ring with m.r. conditions of type (B) if and only if it satisfies the following two conditions:*

(1) *$\hat{R}=(F)_\omega$, were F is a division ring,*

(2) *$S\leqq R\leqq M$, where S is a potent T-ring with type (B) in $(F)_\omega$ and M is the full cover of S in $(F)_\omega$.*

Proposition 4. 5. *Let R be a residue-finite CPI-ring and let $\hat{R}$ be the maximal right quotient ring of R. If $\hat{R}$ is a left quotient ring of R, then R is of type (B).*

Proof. Assume that R is of type (A) and let $L_{r2}=\{T_0, T_1, \cdots\}$. By Theorem 3. 1, there exists an independent set $\{A_i\}$ of atoms of L_r such that $T_{p-1}=T_p\cup^*(A_{d_{p-1}+1}\cup^*\cdots\cup^*A_{d_p})$ and $T_p\cap(A_{d_{k-1}+1}\cup^*\cdots\cup^*A_{d_p})=0$ for each p. Now we put $T_p^c=A_1\cup^*\cdots\cup^*A_{d_p}$. Then we obtain

$$T_p\cup T_p^c=R \quad\text{and}\quad T_p^l\cap T_p^{cl}=0 \quad\text{for each } p\ , \tag{$*$}$$

because $L_r=J_r$ by Theorem 2. 2 of [23]. If $\cup_p T_p^c\neq R$, then $I=\cap_p T_p^{cl}$ contains an atom B of J_l. Since $B^l\in J_{l2}$, $B^l=T_p^l\neq R$ for some p. If $B^2=0$, then $B\subseteqq B^l=T_p^l\subseteqq T_{p+1}^l$ If $B^2\neq 0$, then $B\subseteqq T_{p+1}^l$ by Lemma 3. 4. In either case we have $B\subseteqq T_{p+1}^l\cap I\subseteqq T_{p+1}^l\cap T_{p+1}^{cl}=0$ by $(*)$, a contradiction. Thus we obtain $R=\cup_p T_p^c=\cup_i A_i=\cup_i^* A_i$. Hence there exists a set $\{e_{ij}\,|\,i, j=1, 2, \cdots\}$ of matrix units in $\hat{R}$ such that $\hat{A}_i=e_{ii}\hat{R}$ and $\hat{R}=(F)_\omega$, where F is a division ring.

Hence $A_i=(e_{ii}\hat{R})\cap R$. We put $B_i=(\cup^*_{j\neq i}A_j)^l$. Then the following properties hold:

(1) $\{B_i\}$ is an independent set of atoms of J_l and $\hat{B}_i=\hat{R}e_{ii}\supseteqq B_i$ for each i.

(2) $T^l_p=B_1\cup\cdots\cup B_{d_p}$ for each p.

(1) Since $L_r=J_r$ is a dual-isomorphism to J_l and B^r_i is a maximal right annihilator, it is clear that B_i is an atom of J_l. Furthermore, we obtain

$$B_i=(\cup^*_{j\neq i}A_j)^l=(\cup^*_{j\neq i}\hat{A}_j)^l\cap R=\hat{R}e_{ii}\cap R\,.$$

If $B_i\cap(B_1\cup\cdots\cup B_{i-1}\cup B_{i+1}\cup\cdots)\neq 0$, then we have $R=(\cup^*_{j\neq i}A_j)\cup{}^*A_i\subseteqq B^r_i\cup{}^*(B^r_1\cap\cdots\cap B^r_{i-1}\cap B^r_{i+1}\cap\cdots)\subsetneqq R$, which is a contradiction. Hence $\{B_i\}$ is an independent set.

(2) By the construction of $\{A_i\}$, it is clear that $T_p=\cup_{j>d_p}A_j$. Hence $T^l_p\supseteqq B_i$ $(1\leqq i\leqq d_p)$. Since dim $T^l_p=d_p$, we obtain $T^l_p=B_1\cup\cdots\cup B_{d_p}$.

Now, let q be the element of $\hat{R}$ such that $q=(q_{ij})$, $q_{1j}=1$ for each j and $q_{ik}=0$ otherwise. Since $\hat{R}$ is a left quotient ring of R, there exists an element r of R such that $0\neq rq\in R$. Hence there exists an integer i such that

$$(**)\qquad rq=\begin{bmatrix} & * & \\ r_{i1}\,, & r_{i1}\,, & \cdots \\ & * & \end{bmatrix} 0\neq r_{i1}\in F\,.$$

Since $q\in e_{11}\hat{R}$, $r(\hat{R},q)=\{a\in\hat{R}\,|\,qa=0\}$ is maximally closed in $\hat{R}$. Hence $q^r=r(\hat{R},q)\cap R$ is maximally closed in R and hence $(rq)^r=q^r$. By Theorem 6. 9 of [12], rqR^1 is a uniform right ideal of R. Since $\cup^*_i A_i=R$, there exists an integer p such that $rq\in A_1\cup{}^*\cdots\cup{}^*A_{d_p}$. Clearly $A_1\cup{}^*\cdots\cup{}^*A_{d_p}\subseteqq T^l_p\subseteqq\hat{B}_1\oplus\cdots\oplus\hat{B}_{d_p}$, where $\hat{B}_i=\hat{R}e_{ii}$ for each i. This contradicts (**). Hence R is of type (B).

Theorem 4. 6. *Let R be a left faithful ring and let $\hat{R}$ be the maximal right quotient ring of R. Then R is a residue-finite CPI-ring and $\hat{R}$ is a left quotient ring of R if and only if the following two conditions are satisfied:*

(1) $R\leqq_l\hat{R}$ and $\hat{R}=(F)_\omega$, where F is a division ring,

(2) $S\leqq_l R\leqq_l M$, where S is a potent T-ring with type (B) in $(F)_\omega$ and M is the full cover of S in $(F)_\omega$.

Proof. The "if" part is clear. "Only if" part: By Proposition 4. 5, R is of type (B). Hence $L_{r2}=\{T_0,T_1,\cdots,T_p,T_{p+1}\}$ for some integer p, where $T_0=R$, $T_{p+1}=0$. We put $\dim_R R/T_k=d_k$ for $1\leqq k\leqq p$. By Lemma 3. 6, there exists an independent set $\{B_i'\}$ $(1\leqq i\leqq d_p)$, each of which is a potent atom of J_l such that

$$T^l_k=B_1'\cup B_2'\cup\cdots\cup B'_{d_k}\ (k=1,2,\cdots,p)\,.$$

Since $J_r=L_r$, there exists $T^c_p\in J_r$ such that $T_p\cup{}^*T^c_p=R$ and $T_p\cap T^c_p=0$. For each i $(1\leqq i\leqq d_p)$, we put

$$A_i=(B_1'\cup\cdots\cup B_{i-1}'\cup B_{i+1}'\cup\cdots\cup B_{d_p}')^r\cap T_p^c.$$

Then the following properties hold:

(1) $\{A_i\}$ $(1\leqq i\leqq d_p)$ are independent atoms of L_r.

(2) $T_{k-1}=T_k\cup^*(A_{d_{k-1}+1}\cup^*\cdots\cup^*A_{d_k})$ and $T_k\cap(A_{d_{k-1}+1}\cup^*\cdots\cup^*A_{d_k})=0$ for $1\leqq k\leqq p$.

(3) $T_p^c=A_1\cup^*\cdots\cup^*A_{d_p}$.

To prove (1), we put $B=B_1'\cup\cdots\cup B_{i-1}'\cup B_{i+1}'\cup B_{d_p}'$. Then $T_p\subsetneqq B^r$ and hence $A_i=B^r\cap T_p^c\neq 0$. Suppose that $B_i'^r\cap A_i\neq 0$. Then $B_i'\cup A_i^l\neq R$. On the other hand, by the definition of A_i, we have $B_i'\cup A_i^l=R$. This is a contradiction and hence $B_i'^r\cap A_i=0$. Since $B_i'^r$ is maximally closed, A_i is an atom of L_r. It is clear that $\{A_i\}_{i=1}^{d_p}$ are independent by the definition of A_i.

To prove (2), we suppose that $T_k\cap(A_{d_{k-1}+1}\cup^*\cdots\cup^*A_{d_p})\neq 0$. Then $T_k^l\cup(A_{d_{k-1}+1}^l\cap\cdots\cap A_{d_k}^l)\neq R$. On the other hand $T_k^l\cup(A_{d_{k-1}+1}^l\cap\cdots\cap A_{d_k}^l)\supseteqq T_p^l\cup T_p^{cl}=R$. This is a contradiction and thus $T_k\cap(A_{d_{k-1}+1}\cup^*\cdots\cup^*A_{d_k})=0$. If $T_{k-1}\not\supseteqq A_i$ for some i $(d_{k-1}<i\leqq d_k)$, then $T_{k-1}\cap A_i=0$ and $R=T_{k-1}^l\cup A_i^l=B_1'\cup\cdots\cup B_{i-1}'\cup B_{i+1}'\cup\cdots\cup B_{d_p}'\cup T_p^{cl}$, which is a contradiction. Hence $T_{k-1}\supseteqq A_i$ for $d_{k-1}<i\leqq d_k$. Since $\dim_R T_{k-1}/T_k=d_k-d_{k-1}$, the assertion of (2) is clear.

(3) Clearly $T_p^c\supseteqq A_1\cup^*\cdots\cup^*A_{d_p}$. Since $\dim_R R/T_p=d_p$, we have $T_p^c=A_1\cup^*\cdots\cup^*A_{d_p}$.

Since $L_r=J_r$ and T_p is countably dimensional as an R-module, by Zorn's lemma, there exist independent atoms $\{A_i'\}_{i=1}^\infty$ of L_r such that $T_p=\cup_{i=1}^\infty A_i'$. For a convinience, we put $A_i'=A_{d_p+i}$ for each i. Now we put

$$B_i=(\cup^*_{j\neq i}A_j)^l\quad(i=1,2,\cdots).$$

Then we shall prove that $\{B_i\}$ is a potent atomic basis for J_l which satisfies the conditions (d) and (e). It is clear that $\{B_i\}$ is a basis of J_l. For $1\leqq i\leqq d_p$, $B_i=(\cup^*_{j\neq i}A_j)^l=(A_1\cup^*\cdots\cup^*A_{i-1}\cup^*A_{i+1}\cup^*\cdots\cup^*A_{d_p})^l\cap T_p^l\supseteqq B_i'$. Thus $B_i=B_i'$ and hence B_i is potent for $1\leqq i\leqq d_p$ and the $\{B_i\}$ satisfies the condition (d). For $j>d_p$, since $A_j^r=0$, we obtain $A_jB_j\neq 0$. It is clear that $B_jA_j\neq 0$. Hence $b_jA_j\neq 0$ for $0\neq b_j\in A_j\cap B_j$ and thus $b_j^r\cap A_j=0$. Hence $B_j=b_j^{rl}$ and so B_j is potent. It is clear that $(\cup_{j\neq i}B_j)^r=A_i$ and $\cup_{i=1}^\infty A_i=R$. Hence $\{B_i\}$ satisfies the condition (e). Thus R is a residue-finite PI-ring with m.r. conditions which is of type (B). Hence, by Theorem 4. 4., $S\leqq R\leqq M\leqq\hat{R}=(F)_\omega$, where F is a division ring, S is a potent T-ring with type (B) in $(F)_\omega$ and M is the full cover of S. To prove that $S\leqq_l R$, we shall prove that R is a left stable ring. Since $\hat{R}$ is a left quoient ring of R, R is a left I-ring. For each non-zero $x\in A_i\cap B_1$, $x^l=(e_{ii}\hat{R})^l\cap R=\hat{R}(1-e_{ii})\cap R$ is a maximal closed left ideal of R. Hence R^1x is a uniform left ideal of R by Theorem 6. 9 of [12], where R^1x is the principal left ideal generated by x. Since $(\sum_{i=1}^\infty\oplus(A_i\cap B_1))^l=0$, R is a left stable ring. Hence $\{B_i\}$ is a basis of $L_l(R)$, because B_i is an atom of $L_l(R)$ by

Corollary 2. 3 of [13] and $\cap_{i=1}^{\infty} B_j^r = 0$. On the other hand, since $A_i^r = 0$ for $i > d_p$, R is a left quotient ring of the ring A_i. Hence $\{A_i \cap B_j\}_{j=1}^{\infty}$ is a basis of $L_l(A_i)$ and thus A_i is a left quotient ring of $\sum_j \oplus (A_i \cap B_j)$, $Z_l(A_i) = 0$. Hence $S \leqq_l R \leqq_l M \leqq_l \hat{R}$ by Lemma 2 of [2, p. 88]. This completes the proof of Theorem 4. 6.

5. Left quotient rings of right locally uniform potent rings with zero right singular ideal

In this section, let R be a right locally uniform potent ring with $Z_r(R) = 0$ and let $\hat{R}$ be the maximal right quotient ring of R. We study the conditions under which $\hat{R}$ is a left quotient ring of R.

Proposition 5. 1. *Let R be a right locally uniform potent ring with $Z_r(R) = 0$ and let $\{R_\alpha\}$ be the irreducible components of R. Then $\hat{R}$ is a left quotient ring of R if and only if $\hat{R}_\alpha$ is a left quotient ring of R_α for each α.*

Proof. Suppose that $\hat{R}$ is a left quotient ring of R and let $0 \neq q \in \hat{R}_\alpha$. Then $0 \neq rq \in R$ for some $r \in R$. Since $\hat{R}_\alpha$ is an ideal of $\hat{R}$, $0 \neq rq \in \hat{R}_\alpha \cap R = R_\alpha$. Since R is a right stable ring and $R_\beta R_\alpha = 0$ $(\beta \neq \alpha)$, it is clear that $R_\alpha rq \neq 0$. Hence $0 \neq r_\alpha(rq) = (r_\alpha r)q \in R_\alpha q \cap R_\alpha$. Since $Z_l(R_\alpha) = 0$ by Lemma 2. 1 of [14], $\hat{R}_\alpha$ is a left quotient ring of R_α. Conversely, suppose that $\hat{R}_\alpha$ is a left quotient ring of R_α for each α and let $0 \neq q \in \hat{R}$. Then $q\hat{R}_\alpha \neq 0$ for some α. Since $\hat{R}_\alpha$ is an ideal of $\hat{R}$ and is direct summand, we have $\hat{R}_\alpha = e_\alpha \hat{R}$ for some central idempotent e_α. And thus $0 \neq e_\alpha q = qe_\alpha \in \hat{R}_\alpha$. There exists $r \in R_\alpha$ such that $0 \neq r(qe_\alpha) \in R_\alpha$. Again, for $0 \neq rqe_\alpha \in R_\alpha$, $re_\alpha \in \hat{R}_\alpha$, there exists $r' \in R_\alpha$ such that $0 \neq r're_\alpha \in R_\alpha$, $0 \neq r'rqe_\alpha$. Thus $0 \neq (r're_\alpha)q = r'(rqe_\alpha) \in R_\alpha q \cap R_\alpha$. Since $Rq \cap R \supseteqq R_\alpha q \cap R_\alpha$, $\hat{R}$ is a left quotient ring of R.

Theorem 5. 2. *Let R be a residue-finite CPI-ring and let $\hat{R}$ be the maximal right quotient ring of R. Then $\hat{R}$ is a left quotient ring of R if and only if the following two conditions are satisfied*:

(1) *There exists an atom A of L_r such that $A^r = 0$.*

(2) *Let A be an atom satisfying $A^r = 0$. Put $\Gamma = \mathrm{Hom}_R(A, A)$ and $\Delta = \mathrm{Hom}_R(\hat{A}, \hat{A})$. Then Δ is a left quotient ring of Γ and $\Delta A = \hat{A}$.*

Proof. First, assume that $\hat{R}$ is also a left quotient ring of R. Then, by Proposition 4. 5, R is of type (B) with m.r. conditions and R is a left stable ring. There exists an atom A of L_r such that $A^r = 0$. Let θ and ϕ be non-zero elements of Γ and let u be a non-zero element of A. Then $\theta(u) \neq 0$, $\phi(u) \neq 0$, because every non-zero element of Γ is a non-singular mapping by Lemma 5. 4 of [8]. Since $\theta(u)^r = u^r$, we obtain $(\theta u)^r = (\phi u)^r$ and $(\theta u)^{rl} = (\phi u)^{rl}$. Since $(\theta u)^r$ is a maximal closed right ideal, $(\theta u)^{rl}$ is a minimal annihilator left ideal and hence $(\theta u)^{rl} = (\phi u)^{rl}$ is an atom of L_l by Corollary 2. 3 of [13]. Hence there exist $a, b\in R$ such that $a\theta(u)=b\phi(u)\neq 0$. Since $A^r=0$, $Aa\theta(u)\neq 0$ and hence there exists $v\in A$ such that $va\theta\ (u)=vb\phi(u)\neq 0$. This means that $(\lambda_{va}\theta)(u)=(\lambda_{vb}\phi)(u)$, where $\lambda_{va}(x)=vax$ for $x\in A$. From which we obtain $\lambda_{va}\theta=\lambda_{vb}\phi$, because the elements of Γ, other than zero, are non-singular mappings. Evidently $\lambda_{va}, \lambda_{vb}\in\Gamma$ and $\Gamma\theta\cap\Gamma\phi\neq 0$; thus Γ is a left Ore domain. Let δ be any non-zero element of Δ. Since $\hat{A}$ is $\hat{R}$-right injective, there exists $e=e^2\in\hat{R}$ such that $\hat{A}=e\hat{R}$. For $0\neq\delta(e)$, there exists $r\in R$ such that $0\neq r\delta(e)\in R$. Since $A^r=0$, there exists $a\in A$ such that $0\neq ar\delta(e)\in A$ and $0\neq ar\in A$. Clearly $\lambda_{av}\delta\in\Gamma$, $\lambda_{ar}\in\Gamma$ and $\lambda_{ar}\delta\neq 0$, because $0\neq\lambda_{ar}\delta(e)$. This means that Δ is a left quotient ring of Γ. Evidently $\Delta A\subseteq\hat{A}$. Assume that q is a non-zero element of $\hat{A}$. Then there exists $r\in R$ such that $0\neq rq\in R$. Since $A^r=0$, $Arq\neq 0$ and there exists $u\in A$ such that $0\neq urq$. Since q^r is a maximal closed right ideal, $(urq)^r=(rq)^r=q^r$. Now define $\phi\colon urq\hat{R}\to\hat{A}$ by $\phi(urqy)=qy$ for each $y\in\hat{R}$. Then since $\hat{A}$ is right $\hat{R}$-injective, ϕ can be extended to $\hat{\phi}\in\Delta$ and $\hat{\phi}(urq)=\phi(urq)=q$, $urq\in A$. This means that $\Delta A\supseteq\hat{A}$. Hence we have $\Delta A=\hat{A}$, as desired.

Conversely, assume that (1) and (2) hold. If $0\neq q\in\hat{R}$, then $A^r=0$ implies $Aq\neq 0$. There exists $a\in A$ such that $w=aq\neq 0$. Since $w\in\hat{A}=\Delta A$, there exist $\delta_1, \cdots, \delta_n\in\Delta$ and $a_1, \cdots, a_n\in A$ such that $w=\sum_{i=1}^n\delta_i a_i$. Now Δ is a left quotient ring of Γ. Hence there exists $0\neq\gamma\in\Gamma$ such that $0\neq\gamma\delta_i=\gamma_i\in\Gamma$, $i=1, \cdots, n$. Since $\Gamma A\subseteq A$, we obtain that $0\neq\gamma w=(\gamma a)q=\sum\gamma_i a_i\in Aq\cap A$. Thus we have $Rq\cap R\neq 0$. This means that $\hat{R}$ is a left quotient ring of R.

6. On closed right ideals and annihilator right ideals of right locally uniform rings with zero right singular ideal

In this section, we generalize Goldie's results on closed right ideals and annihilator right ideals of (semi-) prime right Goldie rings to right stable rings or to infinite dimensional semi-prime rings with zero right singular ideal.

Proposition 6. 1. *Let M be a faithful locally uniform right R-module and let K be a closed submodule of M. Then K is an intersection of maximal closed submodules of M.*

Proof. Let K be a relative complement of a submodule L (see. [7]). Then there exists an independent set $\{A_i\}$ of uniform submodules such that $L'\supset\sum_i\oplus A_i$. We set $N_i=K\oplus\sum_{j\neq i}\oplus A_j$ for each i, then $N_i\cap A_i=0$. Choose a maximal closed submodule N_i^* such that $N_i^*\supseteq N_i$ and $N_i^*\cap A_i=0$ for each i. If $(\cap_i N_i^*)\cap(\sum_i\oplus A_i)\neq 0$, then there exist $\{A_i\}_{i=1}^n$ such that $(N_i^*\cap\cdots\cap N_n^*)\cap(A_1\oplus\cdots\oplus A_n)\neq 0$. On the other hand $(N_1^*\cap\cdots\cap N_n^*)\cap(A_1\oplus\cdots\oplus A_n)=0$, as may be seen by repeated application of the modular law. Hence $(\cap_i N_i^*)\cap(\sum_i\oplus A_i)=0$ and $K=\cap_i N_i^*$, as desired.

Following Goldie [7], an element u of R is said to be *right uniform* if uR^1 is a uniform right ideal.

Proposition 6. 2. *If R is a right stable ring, then a right ideal M is a maximal right annihilator ideal if and only if $M=u^r$ for some right uniform element u of R. In particular, u^r is maximally closed.*

Proof. The "if" part is immediately by Theorem 6. 9 of [12]. Suppose that M is a maximal annihilator. Then there exists a uniform right ideal A such that $AM^l \neq 0$, because R is a right stable ring. For $0 \neq u \in A \cap M^l$ we have $u^r \supseteq M$. Hence $u^r=M$, as desired.

Corollary 6. 3. *If R is a right locally uniform potent ring with $Z_r(R)=0$, then a right ideal M is a maximal right annihilator ideal if and only if $M=u^r$ for some right uniform element u of R. In particular, u^r is maximally closed.*

Theorem 6. 4. *Let R be a right stable ring and let $\hat{R}$ be the maximal right quotient ring of R. If $\hat{R}$ is a left quotient ring of R, then every closed right ideal of R is of the form $\cap_\alpha (u_\alpha)^r$, where $\{u_\alpha\}$ are right uniform elements of R.*

Proof. By Theorem 2. 2 of [23], $L_r=J_r$. Hence the assertion follows immediately from Propositions 6. 1 and 6. 2.

Theorem 6. 5. *Let R be a finite dimensional right stable ring. Then every proper right annihilator of R is of the form $u_1^r \cap \cdots \cap u_k^r$, where $\{u_i\}$ are right uniform elements of R.*

Proof. Let I be a non-zero right annihilator ideal of R and let K be a relative complement of I. Choose a uniform right ideal $A_1 \subseteq K$. If $I^l A_1=0$, then $I \supseteq A_1$. This is a contradiction. Hence $I^l A_1 \neq 0$. There exists a uniform right ideal C_1 such that $C_1 I^l A_1 \neq 0$, because R is a right stable ring. Hence there exists an element u_1 of $I^l \cap C_1$ such that $u_1 A_1 \neq 0$ and therefore $u_1^r \cap A_1=0$, $u_1^r \supseteq I$. If $u_1^r \cap K=0$, then clearly $I=u_1^r$. Otherwise we choose a uniform right ideal A_2 in $u_1^r \cap K$. By the same argument as above, there exists a uniform element u_2 of R such that $u_2^r \cap A_2=0$ and $u_2^r \supseteq I$. Since $u_1^r \supseteq A_2$ and $u_2^r \cap A_2=0$, we have $u_1^r \supsetneqq u_1^r \cap u_2^r$. If $u_1^r \cap u_2^r \cap K=0$, then we obtain $I=u_1^r \cap u_2^r$. Otherwise we choose a uniform right ideal A_3 in $u_1^r \cap u_2^r \cap K$ and a uniform element u_3 of R such that $u_3^r \supseteq I$ and $u_3^r \cap A_3=0$. Clearly $u_1^r \cap u_2^r \supsetneqq u_1^r \cap u_2^r \cap u_3^r$. The process is continued until it terminates, which must occur after not more than $\dim_R R$ terms, because the chain $u_1^r \supsetneqq u_1^r \cap u_2^r \supsetneqq u_1^r \cap u_2^r \cap u_3^r \supsetneqq \cdots$ can not have more than $\dim_R R$ terms. Hence there is an integer $k \neq 0$ such that $(u_1^r \cap \cdots \cap u_k^r) \cap K=0$ and $(u_1^r \cap \cdots \cap u_k^r) \supseteq I$. Hence we obtain $I=u_1^r \cap u_2^r \cap \cdots \cap u_k^r$.

Corollary 6. 7. *Let R be a finite dimensional potent ring with $Z_r(R)=0$. Then every proper right annihilator of R is of the form $u_1^r \cap \cdots \cap u_k^r$, where $\{u_i\}$ are* *right uniform elements of R.*

In the remaining of this section, let R be a right locally uniform semi-prime ring with $Z_r(R)=0$ and let $\{R_\alpha \mid \alpha\in\Lambda\}$ be the irreducible components of R, where Λ is an index set. Then we have

Lemma 6. 8. (1) *If A and B are uniform right ideals, then $A\sim B$ if and only if $A^r=B^r$.*

(2) *R_α is a prime ring.*

Proof. (1) Suppose that $A\sim B$. Then A and B contain mutually isomorphic non-zero right ideals A' and B' respectively. Clearly $A'^r=B'^r$ and $B'^2\neq 0$. Hence $0\neq A'B$ and $0\neq aB\cong B$ for some $a\in A$. Therefore we obtain $A^r\subseteq(aB)^r=B^r$. Similarly, $A^r\supseteq B^r$ and hence $A^r=B^r$. Conversely, suppose that $A^r=B^r$. Then $0\neq AB$ and $0\neq aB\cong B$ for some $a\in A$. Hence $A\sim B$.

(2) Let I be a non-zero ideal of R_α. Then clearly $0\neq IR_\alpha$ and IR_α is a right ideal of R. Since R is semi-prime, we have $0\neq(IR_\alpha)^n\subseteq I^n$ for each n. Hence R_α is a semi-prime ring. Since $\hat{R}_\alpha$ is a prime ring, R_α is a prime ring by Theorem 3. 2 of [2, p. 114].

Following Goldie [7], an ideal I of R is an *annihilator ideal* if $I=K^r$ for some right ideal K of R. Since $K^{rlr}=K^r$, we may assume that K is an ideal..

Theorem 6. 9. *Let R be a right locally uniform semi-prime ring with $Z_r(R)=0$ and let $\{R_\alpha \mid \alpha\in\Lambda\}$ be the irreducible components of R. Then*

(1) *$R_\alpha=\bigcap_{\beta\neq\alpha}A_\beta^r$, where A_β is a uniform right ideal contained in R_β.*

(2) *$\{R_\alpha \mid \alpha\in\Lambda\}$ is the set of minimal annihilator ideals of R.*

(3) *R_α is a prime I-ring.*

Proof. (1) Since $R_\beta R_\alpha=0(\alpha\neq\beta)$, we have $R_\alpha\subseteq\bigcap_{\beta\neq\alpha}A_\beta^r$. If $R_\alpha\subsetneqq\bigcap_{\beta\neq\alpha}A_\beta^r$, then there exists a uniform right ideal A such that $A\not\subseteq R_\alpha$ and $A\subseteq\bigcap_{\beta\neq\alpha}A_\beta^r$. Hence $A\sim A_\gamma$ for some $\gamma\in\Lambda$ with $\gamma\neq\alpha$, and $A_\gamma A=0$. But by Lemma 6. 8, $0\neq A_\gamma A$, which is a contradiction. Hence we have $R_\alpha=\bigcap_{\beta\neq\alpha}A_\beta^r$.

(2) If $R_\alpha\supseteq K^r\neq 0$, where K is an ideal, then K^r contains a uniform right ideal B such that $B\sim A_\alpha$, where A_α is a fixed uniform right ideal contained in R_α. Since R is semi-prime, $KB=0$ implies that $BK=0$, i.e., $B^r\supseteq K$. Let C be any uniform right ideal such that $C\sim A_\alpha$. Then, since $B^r=C^r$ by Lemma 6. 8, $C^r\supseteq K$. Again, since R is semi-prime, $K^r\supseteq C$ and thus $K^r\supseteq R_\alpha$. Hence $K^r=R_\alpha$ and thus R_α is a minimal annihilator ideal of R. Conversely, let I be a minimal annihilator ideal of R. Then $IR_\alpha\neq 0$ for some $\alpha\in\Lambda$ and thus $IR_\alpha\subseteq I\cap R_\alpha$. Hence $I=R_\alpha$.

(3) follows from the remark of Lemma 2. 5 and Lemma 6. 8.

Following Goldie, right ideals I and J are said to be *related* $(I\sim_1 J)$ provided that $I \cap X=0$ holds if and only if $J \cap X=0$, where X is a right ideal of R.

Lemma 6.10. *Let R be a right locally uniform semi-prime ring with $Z_r(R)=0$. Then*

(1) *If I is a right ideal of R and if J is an ideal such that $I \sim_1 J$, then $I^*=J^{lr}$.*

(2) *If $\bar{I}$ is a closed ideal of $\hat{R}$, then $\bar{I} \cap R$ is an annihilator ideal of R.*

(3) *If I is a right ideal of R, then there exists an ideal $J \sim_1 I$ if and only if $\hat{I}$ is an ideal of $\hat{R}$.*

Proof. (1) It is clear that $J^l \cap J^{lr}=0$, J^l is a relative complement of J in the sense of Goldie and $J^{lr} \supseteq J$. Hence we obtain $I^*=J^*=J^{lr}$.

(2) Clearly $\bar{I} \cap R$ is a closed ideal of R. Hence $\bar{I} \cap R=(\bar{I} \cap R)^{lr}$ is an annihilator ideal by (1).

(3) The "if" part follows from (2). The "only if" part: suppose that $J \sim_1 I$, where J is an ideal of R. Then $J^{lr} \supseteq R_\alpha$ or $J^{lr} \cap R_\alpha=0$ for each $\alpha \in \Lambda$ by Theorem 6.9. Now we put $\Lambda_0=\{\alpha \in \Lambda \mid J^{lr} \supseteq R_\alpha\}$. If J^{lr} is not an essential extension of $\sum_{\alpha \in \Lambda_0} \oplus R_\alpha$, then there exists a uniform right ideal A such that $J^{lr} \supseteq A$ and $R_\alpha \cap A=0$ for each $\alpha \in \Lambda_0$. Thus $A \subseteq R_\beta \cap J^{lr}$ for some $\beta \notin \Lambda_0$ and hence $R_\beta \subseteq J^{lr}$. This is a contradiction and hence $J^{lr} \supset' \sum_{\alpha \in \Lambda_0} \oplus R_\alpha$. Since $J^{lr} \supset' J$, we have $\sum_{\alpha \in \Lambda_0} \oplus \hat{R}_\alpha \subset' \hat{J}^{lr}=\hat{J}$ as right $\hat{R}$-modules. Hence, by Lemma 1.2 of [24], $\hat{I}=\hat{J}$ is an ideal of $\hat{R}$.

Theorem 6.11. *Let R be a right locally uniform semi-prime ring with $Z_r(R)=0$, let $\{R_\alpha \mid \alpha \in \Lambda\}$ be the irreducible components of R and let $\hat{R}$ be the maximal right quotient ring of R. Then every closed ideal of $\hat{R}$ is of the form $\hat{I}_{\Lambda_0}$, where $I_{\Lambda_0}=\sum_{\alpha \in \Lambda_0} \oplus R_\alpha$ and Λ_0 is a subset of Λ.*

Proof. It is clear that $\hat{I}_{\Lambda_0} \in L_{r2}(\hat{R})$ by Lemma 6.10. Conversely, suppose that $\bar{I} \in L_{r2}(\hat{R})$. Then, by Lemma 6.10, $\bar{I} \cap R$ is an annihilator ideal of R. Now we put $\Lambda_1=\{\alpha \in \Lambda \mid \bar{I} \cap R \supseteq R_\alpha\}$ and assume that $\bar{I} \cap R$ is not an essential extension of K, where $K=\sum_{\alpha \in \Lambda_1} \oplus R_\alpha$. Then there exists an atom A of $L_r(R)$ such that $A \subseteq \bar{I} \cap R$ and $A \cap K=0$. Hence $A \subseteq R_\beta$ for some $\beta \notin \Lambda_1$ and thus $(\bar{I} \cap R) \cap R_\beta \neq 0$. Hence we obtain $\bar{I} \cap R \supseteq R_\beta$, because R_β is a minimal annihilator ideal. This is a contradiction. Hence $\bar{I} \cap R \supset' K$ and thus $\bar{I}=\hat{K}$.

Corollary 6.12. *$\{\hat{R}_\alpha \mid \alpha \in \Lambda\}$ is the set of minimal closed ideals of $\hat{R}$.*

7. Semi-prime modules

In this section, let R be a right locally uniform semi-prime ring with $Z_r(R)=0$, let $\{R_\alpha \mid \alpha \in \Lambda\}$ be the irreducible components of R, let A_α be a fixed uniform right ideal contained in R_α and let $P_\alpha=(\sum_{\beta \neq \alpha, \beta \in \Lambda} R_\beta)^*$ as in (2.4).

Applying the methods developed in section 2 to modules, we shall give, in this section, more detailed results on semi-prime modules, which investigated in [4]. Let M be a right R-module such that $Z_R(M)=0$. Then it is clear that M is locally uniform. Let U be a uniform R-submodule of M. If A_α and U contain mutually isomorphic non-zero R-submodules A_α' and U' respectively, then A_α and U are said to be *similar* ($A_\alpha \sim U$). If M is faithful, then $MA_\alpha \neq 0$ and thus $0 \neq mA_\alpha$ for some $m \in M$. By Theorem 2. 4 of [3], $mA_\alpha \cong A_\alpha$ and thus $mA_\alpha \sim A_\alpha$. Conversely, let U be a uniform R-submodule. Then there exists a uniform right ideal A such that $0 \neq UA$, because $Z_R(M)=0$. Hence $uA \cong A$ for some $u \in U$ and thus $U \sim A_\alpha$ for some $\alpha \in \Lambda$. Now we put $M_\alpha = (\sum_{U \sim A_\alpha} U)^*$, where U runs over uniform R-submodules of M which are similar to A_α. We call M_α an *irreducible component* of M. By the same methods as in Proposition 2. 2 we can easily prove that the sum $\sum_{\alpha \in \Lambda} M_\alpha$ is a direct sum and that if U is a uniform R-submodule of M, then $U \sim A_\alpha$ if and only if $U \subseteq M_\alpha$. We assemble these results below.

Proposition 7. 1. *Let M be a faithful R-module such that* $Z_R(M)=0$. *Then*

(1) *There is one-to-one correspondence between the irreducible components* $\{R_\alpha \mid \alpha \in \Lambda\}$ *of R and the irreducible components* $\{M_\alpha \mid \alpha \in \Lambda\}$ *of M, in the sense of similarity.*

(2) *Let* $\{M_\alpha \mid \alpha \in \Lambda\}$ *be the irreducible components of M. Then the sum* $\sum_{\alpha \in \Lambda} M_\alpha$ *is direct.*

(3) *Let U be a uniform R-submodule. Then* $U \sim A_\alpha$ *if and only if* $U \subseteq M_\alpha$.

In the remainder of this section, M_α will denote an irreducible component of M which corresponds to R_α, in the sense of similarity and we put $Q_\alpha = (\sum_{\beta \neq \alpha, \beta \in \Lambda} M_\beta)^*$. If N is a submodule of M and if I is a right ideal of R, then we denote $(N: I) = \{m \in M \mid mI \subseteq N\}$. Similarly, for submodules K and L, we denote $(K: L) = \{r \in R \mid Lr \subseteq K\}$.

Following [4], a submodule N of an R-module M is said to be *closed-prime* if

(i) $LI \subseteq N \Rightarrow L \subseteq N$ or $I \subseteq (N: M)$, where L is a submodule of M and I is a right ideal of R.

(ii) N is a closed submodule of M.

Proposition 7. 2. *Let M be a faithful R-module such that* $Z_R(M)=0$. *Then*

(1) $Q_\alpha = (0: R_\alpha)$.

(2) Q_α *is closed-prime and* $(Q_\alpha: M) = P_\alpha$.

(3) $\cap_{\alpha \in \Lambda} Q_\alpha = 0$ *and* $\cap_{\beta \neq \alpha, \beta \in \Lambda} Q_\beta \neq 0$.

Proof. (1) Suppose that $mR_\alpha \neq 0$ for some $m \in Q_\alpha$. Then $0 \neq mr \in Q_\alpha$ for some $r \in R_\alpha$. Then there exists a right ideal $E \subseteq' R$ such that $rE \subseteq \sum_{A \sim A_\alpha} A$. Hence $0 \neq mrE \subseteq M_\alpha \cap Q_\alpha = 0$, which is a contradiction and thus $Q_\alpha R_\alpha = 0$. Hence we obtain $Q_\alpha \subseteq (0: R_\alpha)$. Suppose that $(0: R_\alpha) \supsetneqq Q_\alpha$. Then there exists a uniform R-submodule U of M such that $U \nsubseteq Q_\alpha$ and $UR_\alpha = 0$. Since $U \nsubseteq Q_\alpha$, we have $U \subseteq M_\alpha$ and hence $U \sim A_\alpha$. Then clearly $U^r \subseteq A_\alpha^r$, where $U^r = \{x \in R \mid Ux = 0\}$ and thus $UR_\alpha \neq 0$. This is a contradiction and thus $Q_\alpha = (0: R_\alpha)$, as desired.

(2) First we shall prove that $P_\alpha = (Q_\alpha: M)$. Let m be a non-zero element of M and let r be a non-zero element of R_β. Then $rE \subseteq \sum_{A \sim A_\beta} A$ for some $E \subset' R$. Hence $mrE \subseteq M_\beta \subseteq Q_\alpha$. Hence $mr \in Q_\alpha^* = Q_\alpha$, i.e., $R_\beta \subseteq (Q_\alpha: M)$. Now let x be a non-zero element of P_α. Then $xL \subseteq \sum_{\beta \neq \alpha} R_\beta$ for some $L \subset' R$ and $MxL \subseteq Q_\alpha$. Hence $Mx \subseteq Q_\alpha^* = Q_\alpha$ and thus $x \in (Q_\alpha: M)$. Hence $P_\alpha \subseteq (Q_\alpha: M)$. If $(Q_\alpha: M) \supsetneqq P_\alpha$, then there exists a uniform right ideal B such that $B \nsubseteq P_\alpha$ and $B \subseteq (Q_\alpha: M)$. Hence $B \subseteq R_\alpha$ and $MB \subseteq Q_\alpha$. By Proposition 2. 2, $B \sim A_\alpha$ and $0 \neq mB \cong B$ for some $m \in M$. Thus $0 \neq mB \subseteq Q_\alpha \cap M_\alpha = 0$, which is a contradiction. Hence $(Q_\alpha: M) = P_\alpha$. To prove that Q_α is a closed-prime R-submodule, we assume that $NI \subseteq Q_\alpha$, $I \nsubseteq (Q_\alpha: M)$ and $N \nsubseteq Q_\alpha$, where N is an R-submodule and I is a right ideal of R. Then there exists a uniform right ideal B such that $B \subseteq I$ and $B \nsubseteq (Q_\alpha: M)$. Since $(Q_\alpha: M) = P_\alpha$, we have $B \sim A_\alpha$ and thus $B^r = A_\alpha^r$ by Lemma 6. 8. Since $N \supsetneqq Q_\alpha$, there exists a uniform R-submodule U such that $U \subseteq N$ and $U \nsubseteq Q_\alpha$, i.e., $U \sim A_\alpha$. Hence $U^r \subseteq A_\alpha^r$ and thus we have $0 \neq UB \subseteq NI \subseteq Q_\alpha$. On the other hand, since $U \sim A_\alpha$, $UB \subseteq M_\alpha$ by Proposition 7. 1. This is a contradiction. Hence Q_α is a closed-prime R-submodule of M.

(3) is obvious.

Following [4], we shall denote the intersection of all closed-prime R-submodules by $P(M)$ and called $P(M)$ the *prime radical of* M. In [4], Feller and Swokowski showed that $P(M) \supseteq Z_R(M)$. By Proposition 7. 2, in our case, we have

Corollary 7. 3. *Let R be a right locally uniform semi-prime ring with $Z_r(R) = 0$ and let M be a faithful R-module. Then $Z_R(M) = 0$ if and only if $P(M) = 0$.*

Let an R-module M be a subdirect sum of R-modules $\{M_\alpha \mid \alpha \in \Lambda\}$ and let η_α be a canonical epimorphism from M to M_α: $\eta_\alpha(m) = m_\alpha$, where $m = (m_\alpha) \in M \subseteq \Pi_\alpha M_\alpha$. The subdirect sum M is *irredundant* if for each $\alpha \in \Lambda$, the kernel of the map: $m \to \{\eta_\beta(m) \mid \beta \neq \alpha, \beta \in \Lambda\}$ of M into $\Pi_{\beta \neq \alpha} M_\beta$ is non-zero.

Let a ring R be an irredundant subdirect sum of rings $\{R_\alpha \mid \alpha \in \Lambda\}$ and let θ_α be a canonical epimorphism from R to R_α. We say that an R-module M is a *canonical R_α-module* if $M(\ker \theta_\alpha) = 0$. This condition satisfies if and only if M becomes an R_α-module when multiplication is defined by $mr_\alpha = mr$, where $m \in M$ and $r = (r_\alpha) \in R \subseteq \Pi_\alpha R_\alpha$.

Following Feller and Swokowski ([3], [4]), an R-module M is called *annihilator-prime* if (0) is a closed-prime submodule of M. M is called a *prime*

R-module if the following two conditions are satisfied:

(i) $N^r=0$ for every non-zero submodule N of M.

(ii) $Z_R(M)=0$.

An R-module M is said to be *semi-prime* if $P(M)=0$. Now we have

Theorem 7.4. *Let R be a right locally uniform semi-prime ring with $Z_r(R)=0$, let $\{R_\alpha \mid \alpha\in\Lambda\}$ be the irreducible components of R and let $\bar{R}_\alpha=R/P_\alpha$, where $P_\alpha=(\sum_{\beta\neq\alpha} R_\beta)^*$. Let M be a faithful semi-prime R-module and let $\{M_\alpha \mid \alpha\in\Lambda\}$ be the irreducible components of M and let $\bar{M}_\alpha=M/Q_\alpha$, where $Q_\alpha=(\sum_{\beta\neq\alpha} M_\beta)^*$. Then M is an irredundant subdirect sum of $\{\bar{M}_\alpha \mid \alpha\in\Lambda\}$, where $\bar{M}_\alpha$ is an annihilator-prime R-module as well as $\bar{M}_\alpha$ is a canonical prime $\bar{R}_\alpha$-module.*

Proof. By Proposition 7.2, it is clear that M is an irredundant subdirect sum of $\{\bar{M}_\alpha \mid \alpha\in\Lambda\}$. Since Q_α is closed-prime by Proposition 7.2, we have $\bar{M}_\alpha=M/Q_\alpha$ is an annihilator-prime R-module by Proposition 2.3 of [4]. Since $MP_\alpha\subseteqq Q_\alpha$ by Proposition 7.2, $\bar{M}_\alpha$ is an canonical $\bar{R}_\alpha$-module. To prove that $\bar{M}_\alpha$ is a prime $\bar{R}_\alpha$-module, we assume that $Z_{\bar{R}_\alpha}(\bar{M}_\alpha)\neq 0$. Then $Z_{\bar{R}_\alpha}(\bar{M}_\alpha)$ contains a non-zero $\bar{R}_\alpha$-submodule $\bar{N}$, where N is an R-submodule of M. Hence there exists a uniform R-submodule U of M such that $U\subseteqq N$ and $U\nsubseteqq Q_\alpha$. Thus $U\sim A_\alpha$, i.e., $U\subseteqq M_\alpha$. Let u be a non-zero element of U. Then $\bar{u}\bar{E}=0$ for some $\bar{E}\subset'\bar{R}_\alpha$, because $\bar{U}\subseteqq Z_{\bar{R}_\alpha}(\bar{M}_\alpha)$. Let E be the inverse image of $\bar{E}$ in R. Then clearly $E\subset' R$ and we have $uE\subseteqq M_\alpha\cap Q_\alpha=0$, which is a contradiction and thus $Z_{\bar{R}_\alpha}(\bar{M}_\alpha)=0$. Since $\bar{R}_\alpha$ is a prime ring, $\bar{M}_\alpha$ is a prime $\bar{R}_\alpha$-module by Proposition 1.3 of [3]. This completes the proof of Theorem 7.4.

We now prove the converse of Theorem 7.4.

Theorem 7.5. *Let R, $\{R_\alpha \mid \alpha\in\Lambda\}$ and $\{\bar{R}_\alpha\}$ be as in Theorem 7.4. Let M be an irredundant subdirect sum of $\{\bar{M}_\alpha \mid \alpha\in\Lambda\}$, where $\bar{M}_\alpha$ is an annihilator-prime R-module and is a canonical prime $\bar{R}_\alpha$-module. Then M is a faithful semi-prime R-module.*

Proof. First we shall prove that M is faithful. If $Mr=0$, where $r=(\bar{r}_\alpha)\in(\Pi_\alpha \bar{R}_\alpha\cap R)$, then $\bar{M}_\alpha\bar{r}_\alpha=0$ and thus $\bar{r}_\alpha=0$ for all $\alpha\in\Lambda$. Hence $r=0$. To prove that M is semi-prime, we let $m=(\bar{m}_\alpha)\in Z_R(M)$ and $\bar{m}_\alpha\in\bar{M}_\alpha$. Then $mE=0$ for some $E\subset' R$. It follows that $E\cap R_\alpha\subset' R_\alpha$ as right R_α-modules and that $m(E\cap R_\alpha)=\bar{m}_\alpha(E\cap R_\alpha)=0$. Since $\bar{R}_\alpha$ is a right quotient ring of R_α, $Z_{\bar{R}_\alpha}(\bar{M}_\alpha)=Z_{R_\alpha}(\bar{M}_\alpha)$. Hence $\bar{m}_\alpha=0$ and thus $m=0$. Hence $Z_R(M)=0$ and thus M is a semi-prime R-module by Corollary 7.3.

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