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ON PRIME IDEALS OF A WITT RING OVER A LOCAL RING

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In this note, we consider commutative local rings with invertible element 2, and give a relation between an ordered local ring and a prime ideal of Witt ring over it which is a generalization of the results of Lorenz and Leicht [3] related to prime ideals of Witt ring over a field. By [5], any non-degenerate and finitely generated projective quadratic module (V, q) over a local ring R can be written as a form $(V, q)=\langle a_1\rangle\perp\langle a_2\rangle\perp\cdots\perp\langle a_r\rangle$, where a_i is in the unit group $U(R)$ of R and $\langle a_i\rangle$ denotes a rank one free quadratic submodule $(Rv_i, q|Rv_i)$ such that $q(v_i)=\frac{a_i}{2}$. If, for any element a in $U(R)$, the element having the representative $\langle a\rangle$ in the Witt ring $W(R)$ is denoted by a, then any element of $W(R)$ can be written as a sum of elements of $U(R)$. We use $\perp$, $\top$ and $\otimes$ for the notations of sum, difference and product in $W(R)$. In §1, we have essentially same argument for Witt ring over a local ring as one in [3]. In §2, we study about an ordered local ring R which is an ordered ring such that every unit in R is either >0 or <0, and give a generalization of Sylvester's theorem. In §3, we give an one to one correspondence between such orderings on R and prime ideals $\mathfrak{P}$ of $W(R)$ such that $W(R)/\mathfrak{P}\approx Z$. Throughout this paper, we assume that the ring R is commutative local ring with invertible element 2, and every R-module is unitary.

1. Let R be a local ring with the maximal ideal $\mathfrak{m}$ and the unit group $U(R)$. Since $\langle a\rangle\otimes_R\langle b\rangle\approx\langle ab\rangle$ for $a, b\in U(R)$, we have $(a\perp 1)\otimes(a\top 1)=a^2\top 1=1\perp(-1)=0$ in $W(R)$ for any a in $U(R)$. Therefore, we have the following analogous argument on local ring R to [3]. If $\mathfrak{P}$ is any prime ideal of $W(R)$, then any element a in $U(R)$ is either $a\equiv 1 \pmod{\mathfrak{P}}$ or $a\equiv -1 \pmod{\mathfrak{P}}$. We denote $\varepsilon_{\mathfrak{P}}(a)=1$ or -1, if $a\equiv 1 \pmod{\mathfrak{P}}$ or $a\equiv -1 \pmod{\mathfrak{P}}$, respectively. Then for any element $\alpha\in W(R)$, say $\alpha=a_1\perp a_2\perp\cdots\perp a_n$ for $a_i\in U(R)$, we have $\alpha\equiv\varepsilon_{\mathfrak{P}}(a_1)\perp\varepsilon_{\mathfrak{P}}(a_2)\perp\cdots\perp\varepsilon_{\mathfrak{P}}(a_n) \pmod{\mathfrak{P}}$, therefore there exists an epimorphism $Z\rightarrow W(R)/\mathfrak{P}$, and so $W(R)/\mathfrak{P}\approx Z$ or $\approx Z/(p)$ for some prime number p in the integers Z. Accordingly, we have

(1. 1) *$W(R)/\mathfrak{P}\approx Z$ if and only if $\mathfrak{P}$ is a minimal prime ideal of $W(R)$ which is not maximal.*

(1. 2) *$W(R)/\mathfrak{P}\approx Z/(p)$ for some prime number p if and only if $\mathfrak{P}$ is a maximal ideal of $W(R)$.*

(1. 3) *$W(R)$ is a Jacobson ring, i.e. every prime ideal is an intersection of maximal ideals.*

There is an epimorphism $W(R)\to Z/(2)$ such that if $\alpha=a_1\perp a_2\perp\cdots\perp a_n$ is in $W(R)$ for $a_i\in U(R)$ then α corresponds to n (mod 2). Then we denote ker $(W(B)\to Z/(2))$ by $\mathfrak{M}$.

(1. 4) *A prime ideal $\mathfrak{P}$ is $\mathfrak{P}\neq\mathfrak{M}$ if and only if $1\not\equiv -1$ (mod $\mathfrak{P}$).*

(1. 5) *Any minimal prime ideal $\mathfrak{P}$ of $W(R)$ is contained in $\mathfrak{M}$.*

2. We call that local ring R is an *ordered local ring* if R is an ordered ring such that every unit is either positive element or negative element (R is not necessarily total ordered). For ordered local ring R, we call that the set of positive units in $U(R)$ is the *positive units part* of R.

(2. 1) Proposition. *A local ring R is an ordered local ring if and only if there exists a subset P satisfying the following conditions*

(1) $P\cup -P=U(R)$

(2) $P\cap -P=\phi$

(3) $P\cdot P\subset P$

(4) $(P+P)\cap U(R)\subset P$.

Proof. Let P be a subset of $U(R)$ satisfying the conditions. We set $\mathfrak{m}_+=\{x\in\mathfrak{m}$; there exists $a\in P$ such that $x-a\in P\}$, and $Q=P\cup\mathfrak{m}_+$. Then we have the following properties:

1) $\mathfrak{m}_+\cap -\mathfrak{m}_+=\phi$. Because, if there exists an element x in $\mathfrak{m}_+\cap -\mathfrak{m}_+$, then there exists a, b in P such that $x-a$ and $-x-b$ are in P, and so $-(a+b)=(x-a)+(-x-b)\in P+P$. If $a+b$ is in $U(R)$, it is impossible by 4) and 2). Therefore, $a+b\in\mathfrak{m}$ and $a-b=(a+b)-2b\in U(R)$. If $a-b\in P$, then $x-b=(x-a)+(a-b)\in P$ and so $-2b=(x-b)+(-x-b)\in P$, it is a contradiction to (2). If $b-a$ is in P, then similarly we have contradiction $-2a=(x-a)+(-x-a)\in P$.
Analogously, we have easily

2) $(P+P)\cap\mathfrak{m}\subset\mathfrak{m}_+$.

3) $(P+\mathfrak{m}_+)\subset P$.

4) $\mathfrak{m}_++\mathfrak{m}_+\subset\mathfrak{m}_+$.

5) $P\cdot\mathfrak{m}_+\subset\mathfrak{m}_+$.

6) $\mathfrak{m}_+\cdot\mathfrak{m}_+\subset\mathfrak{m}_+$.

Therefore Q has the properties (I) $Q\cap -Q=\phi$, (II) $Q\cdot Q\subset Q$ and (III) $Q+Q\subset Q$. By the set Q, we can make R an ordered ring which has positive part Q. The converse is clear.

We denote by $k=R/\mathfrak{m}$ the residue field of R and φ: $R\to k$ the canonical homomorphism.

(2. 2) **Proposition.** *Let R be an ordered local ring with positive units part P. Then it satisfies $P+P\subset P$ if and only if k is a total ordered field such that $\varphi(P)$ is the positive part, i.e. k is a formal real field.*

Proof. If $k=R/\mathfrak{m}$ is a total ordered field such that $\varphi(P)$ is the positive part, then $\varphi(P)+\varphi(P)\subset\varphi(P)$ and $0\notin\varphi(P)$, therefore we have $P+P\subset P$. Conversely, if $P+P\subset P$, then we have $\varphi(P)\cap -\varphi(P)=\phi$. Therefore, we obtain easily that k is total ordered field with positive part φ(P).

Let P be any subset of local ring R satisfying the condtions in (2. 1) and $Q=P\cup\mathfrak{m}_+$, where $\mathfrak{m}_+=\{ x\in\mathfrak{m};\ \exists a\in P;\ x-a\in P\}$.

(2. 3) *For any x, y in R, $x+y\in Q$ implies $x\in Q$ or $y\in Q$.*

Proof. Let $x+y\in Q$. If $x+y\in$ P, then $x\in U(R)$ or $y\in U(R)$. If x and y are in $U(R)$, then $x\in P$ or $y\in P$. If $x\in U(R)$ and y$\in\mathfrak{m}$, then $x\in P$ or y$\in\mathfrak{m}_+$. If $x+y$ is in $\mathfrak{m}_+$, there exists $a\in P$ such that $x+y-a\in P$. Since $x+y-a=\left(x-\frac{a}{2}\right)+\left(y-\frac{a}{2}\right)$, we have $x-\frac{a}{2}\in P$ or $y-\frac{a}{2}\in P$, accordingly $x\in\mathfrak{m}_+$ or $y\in\mathfrak{m}_+$.

(2. 4) **Proposition.** *Let P and Q be as above. Then $\mathfrak{p}=\{x\in R;\ x\notin Q\cup -Q\}$ is a prime ideal of R.*

Proof. From (2. 3), we have $\mathfrak{p}+\mathfrak{p}\subset\mathfrak{p}$. We shall show that for any $r\in R$ and $x\in\mathfrak{p}$ we have $rx\in\mathfrak{p}$. We assume $rx\notin\mathfrak{p}$. Then we may assume $rx\in Q$. It is considered in the three cases; 1) If $r\in U(R)$, then it is impossible that $rx\in Q\cup -Q$. 2) If $r\in\mathfrak{m}_+$, then there exists $a\in$ P such that $r-a=c\in P$, and from (2. 3) $xa+xc=xr\in Q$ implies $xa\in Q$ or $xc\in Q$, it is impossible from the first case. 3) If r$\in\mathfrak{p}$, then $xr\in\mathfrak{m}_+$ and so there exists $a\in$ P such that $xr-a\in$ P. Since $r(x-a)+a(r-1)=xr-a\in Q$, we have $r(x-a)\in Q$ or $a(r-1)\in Q$. But $a(r-1)\in Q$ is impossible. Therefore, it must be $r(x-a)\in Q$. But, it is also impossible from the first case. Accordingly, $rx\in\mathfrak{p}$, and $\mathfrak{p}$ is an ideal of R. Since $(Q\cup -Q)\ (Q\cup -Q)\subset Q\cup -Q$, $\mathfrak{p}$ is a prime ideal.

(2. 5) **Theorem.** *Let R be an ordered local ring with the positive units part P, and let Q and $\mathfrak{p}$ be as* (2.4). *Then the localization $R_{\mathfrak{p}}=Q^{-1}R$ of R by prime ideal*

$\mathfrak{p}$ is also an ordered local ring such that $\hat{Q}=Q^{-1}Q$ is the positive units part and $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a formal real field. Let $\mathfrak{R}$ be the real closure of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Then there exists a ring homomorphism $f: R\rightarrow\mathfrak{R}$ such that f induces the epimorphism $\bar{f}: W(R)\rightarrow W(\mathfrak{R})\approx Z$, and $f(P)$ is contained in the positive part of $\mathfrak{R}$, furthermore ker $\bar{f}$ is generated by $\{x\top 1; x\in P\}$.

Proof. It is obvious that $\hat{Q}\cup-\hat{Q}=U(R_{\mathfrak{p}})$, $\hat{Q}\cap-\hat{Q}=\phi$, $\hat{Q}\hat{Q}\subset\hat{Q}$ and $\hat{Q}+\hat{Q}\subset\hat{Q}$. Therefore, by (2. 2) the canonical homomorphism φ': $R_{\mathfrak{p}}\rightarrow R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ induces a total ordering on $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Therefore, $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$ is a formal real field. Let $\mathfrak{R}$ be the real closure of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. Then $W(\mathfrak{R})\approx Z$. Let $f: R\rightarrow\mathfrak{R}$ be the composition of ring homomorphisms $R\rightarrow R_{\mathfrak{p}}\overset{\varphi'}{\rightarrow}R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}\rightarrow\mathfrak{R}$. The positive units part of R is sent to the positive part of $\mathfrak{R}$. Therefore, f induces the ring epimorphism $\bar{f}$: $W(R)\rightarrow W(\mathfrak{R})$, and ker $\bar{f}$ is generated by $\{x\top 1;\ x\in P\}$. Because, if α is any element in ker $\bar{f}$ and $\alpha=a_1\perp a_2\perp\cdots\perp a_n$, then we have $\varepsilon_p(a_1)\perp\varepsilon_p(a_2)\perp\cdots\perp\varepsilon_p(a_n)=0$, in $W(\mathfrak{R})$, also in $W(R)$, where $\varepsilon_p a()=\begin{cases}1: a\in P\\ -1: a\in -P\end{cases}$. Since $\varepsilon_p(a_i)a_i$ is in P for i=1, 2, ...n, we have $\alpha=a_1\perp a_2\perp\cdots\perp a_n\perp(\varepsilon_p(a_1)\perp\cdots\perp\varepsilon_p(a_n))=\varepsilon_p(a_1)\otimes(\varepsilon_p(a_1)a_1\top 1)\perp\cdots\perp\varepsilon_p(a_n)\otimes(\varepsilon_p(a_n)a_n\top 1)$ in $W(R)$. Therefore we have ker $\bar{f}\subset(\{x\top 1;\ x\in P\})$. ker $\bar{f}\supset\{x\top 1;\ x\in P\}$ is clear.

We have the following Sylvester's theorem for ordered local ring.

(2. 6) Corollary. *Let R be an ordered local ring, and (V, q) a non-degenerate and finitely generated projective quadratic R-module. If $(V, q)\approx\langle a_1\rangle\perp\langle a_2\rangle\perp\cdots\perp\langle a_r\rangle\perp\langle -b_1\rangle\perp\langle -b_2\rangle\perp\cdots\perp\langle -b_s\rangle$ for positive units $a_1, a_2, \cdots a_r$ and $b_1, b_2, \cdots b_s$ in R, then the integer $r-s$ is uniquely determined by (V, q).*

Proof. From (2. 5), there exists a real closed field $\mathfrak{R}$ and a ring homomorphism f: R$\rightarrow\mathfrak{R}$ such that the positive units part of R is sent to the positive part of $\mathfrak{R}$. If $(V, \mathrm{q})\approx\langle a_1\rangle\perp\cdots\perp\langle a_r\rangle\perp\langle\perp b_1\rangle\perp\cdots\perp\langle -b_s\rangle\approx\langle a_1'\rangle\perp\cdots\perp\langle a_{r'}'\rangle\perp\langle -b_1'\rangle\perp\cdots\perp\langle -b_{s'}'\rangle$, then $(\sum_{i=1}^{r}\perp a_i)\top(\sum_{i=1}^{s}\perp b_i)=(\sum_{i=1}^{r'}\perp a_i')\top(\sum_{i=1}^{s'}\perp b_i')$ in $W(R)$, and by the ring homomorphism $\bar{f}$: $W(R)\rightarrow W(\mathfrak{R})\approx Z$ induced by f, it is sent to $r-s=r'-s'$.

3. We shall show the following main theorem.

(3. 1) Theorem. *For any local ring R with invertible 2, there exists an one to one correspondence between the set of minimal prime ideals $\mathfrak{P}$ of $W(R)$ such that $\mathfrak{P}\neq\mathfrak{M}$ and the set of subsets P of $U(R)$ satisfying the conditions* (**1**), (**2**), (**3**) *and* (**4**) *in* (**2. 1**), *i.e. the set of minimal orderings on R such that R makes ordered local ring.*

This theorem is obtained from the following arguments.

(3. 2) *Let* $\mathfrak{P}$ *be a prime ideal of* $W(R)$ *such that* $\mathfrak{P}\neq\mathfrak{M}$, *and put* $P(\mathfrak{P})=\{x\in U(R): x\equiv 1 \ (mod\ \mathfrak{P})\}$. *Then* $P(\mathfrak{P})$ *satisfies the conditions* (1), (2), (3) *and* (4) *in* (2. 1). *Therefore R is an ordered local ring with positive part* $Q(\mathfrak{P})=\mathfrak{P}(\mathfrak{P})\cup\{x\in\mathfrak{m}: \exists a\in P(\mathfrak{P}); x-a\in P(\mathfrak{P})\}$. *If* $\mathfrak{P}(P(\mathfrak{P}))$ *denotes the ideal of* $W(R)$ *generated by* $\{x\top 1: x\in P(\mathfrak{P})\}$, *then* $\mathfrak{P}(P(\mathfrak{P}))$ *is a minimal prime ideal of* $W(R)$ *such that* $\mathfrak{P}\supset\mathfrak{P}(P(\mathfrak{P}))$. *Therefore, if* $\mathfrak{P}$ *is a minimal prime ideal of* $W(R)$, *then* $\mathfrak{P}=\mathfrak{P}(P(\mathfrak{P}))$.

Proof. The proof of conditions (1), (2), (3), and (4) is obtained similarly to the case over field (cf. [4]). The other part is obvious.

(3. 3) *Let P be a subset of R satisfying the conditions* (1), (2), (3) *and* (4) *in* (2. 1). *Then we have* $P(\mathfrak{P}(P))=P$.

Proof. Since $\mathfrak{P}(P)=\{x\top 1; x\in P\}$, $P(\mathfrak{P}(P))\supset P$ is obvious. If there exists an element x in $P(\mathfrak{P}(P))$ such that $x\notin P$, then $x\in -P$, and so $-x\top 1\in\mathfrak{P}(P)$. Therefore, we have $1\equiv x\equiv -1(\text{mod}\ \mathfrak{P}(P))$, it is contradiction to $\mathfrak{P}(P)\neq\mathfrak{M}$. Accordingly, we have $P(\mathfrak{P}(P))=P$.

(3. 4) **Corollary.** *For any local ring R with invertible* 2, *the Witt ring* $W(R)$ *is either a local ring with the maximal ideal* $\mathfrak{M}$ *such that* $\mathfrak{M}$ *is nil ideal and* $W(R)\mathfrak{M}\approx Z/(2)$, *or a Jacobson ring such that every maximal ideal has hight* 1 *and every minimal prime ideal has a residue ring isomorphic to Z.*

(3. 5) **Corollary.** *If R is a local domain with altitude* 1 *and an ordered local ring, then R is a total ordered ring, or the residue field is a formal real field.*

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