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ON ENDOMORPHISM RINGS OF NOETHERIAN QUASI-INJECTIVE MODULES

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Let R be a ring with identity element. One of the authors studied the endomorphism ring of projective right R-module P with chain conditions in [6] and showed that the ring is right artinian (resp. noetherian) if so is P as an R-module.

We shall consider its dual in this short note. Unfortunately, we could not give the complete dual of them.

Recently, many authors have studied structures of injective module Q and given many interesting results between ideals in R and S-submodules in Q, where $S=\mathrm{Hom}_R(Q, Q)$. However, we shall study mainly, in this note, some properties between R-submodules and left ideals in S.

In the first section, we shall consider the above problem in an abelian C_3-category $\boldsymbol{A}$ (see [10], Chap. III), and show that if A is a quasi-injective object in $\boldsymbol{A}$ and A is noetherian (resp. artinian), then the endomorphism ring $[A, A]$ of A is semi-primary (resp. left noetherian).

In the second section, we shall study conditions under which $S=\mathrm{Hom}_R(M, M)$ is left artinian, when M is a right R-quasi-injective noetherian module and shall give a condition that M gives us a Morita duality on categories of finitely generated right R-(resp. left S)-modules.

In this paper, we always assume that R-modules M are unitary and the ring of endomorphism of M operates from the left side.

After having completely settled this note, we have found J.W. Fisher's results in [5]. His Theorem 2 is contained in [6], Theorem 2. 8 and Theorem 3 coincides with our Theorem 1. Further, K. Motose obtained similar results in [12].

1. In cases of C_3-abelian categories

Let $\boldsymbol{A}$ be an abelian C_3-category (see [10], Chap. III). For any object A in $\boldsymbol{A}$, by S_A we denote the ring of morphisms of A to itself. Let B be a sub-object in A. By $l(B)$ we denote the left ideal in S_A whose elements consists of all s in

S such that Ker $s \supseteq B$. We call $l(B)$ the *left annihilator ideal* of B. Conversely, let T be a sub-set in S_A. By $r(T)$ we denote $\bigcap_{t \in T}$ Ker t. We call it an *annihilator sub-object* in A. We define the dual of idempotent sub-object in A, (cf. [6]). If $r(\mathfrak{l})=r(\mathfrak{l}^2)$ for a left ideal $\mathfrak{l}$ in S_A then $r(\mathfrak{l})$ is called a *coidempotent sub-object* in A. If the sub-objects in A satisfy the descending (resp. ascending) chain conditions, we say A is *artinian* (resp. *noetherian*). A is called a *quasi-injective*, if $[A, A] \xrightarrow{[i, A]} [B, A]$ is surjective for any sub-object B and $i: B \to A$ inclusion.

Theorem 1. *Let A be a quasi-injective object in the abelian C_3-category* $\mathbf{A}$. *If A is noetherian with respect to annihilator sub-objects, then $S=[A, A]$ is a semi-primary ring.* (Dual of [6], Proposition 2. 4).

In order to prove it we need some lemmas.

Lemma 1. *Let A be a quasi-injective object in* $\mathbf{A}$ *and $\mathfrak{l}$ a left ideal in S_A such that $lr(\mathfrak{l})=\mathfrak{l}$. Then $lr(\mathfrak{l}+S_A x)=\mathfrak{l}+S_A x$ for any x in S_A.* (Dual of [6], Proposition 2. 3, cf. [1], Lemma 1 in §5 and [9], Theorem 2. 1).

Proof. The proof is analogous to [9], Theorem 2. 1. It is clear that $lr(\mathfrak{l}+Sx) \supseteq \mathfrak{l}+Sx$, where $S=S_A$. Let y be in $lr(\mathfrak{l}+Sx)=l(r(l) \cap r(x))$. Then $r(y) \supseteq r(\mathfrak{l}) \cap r(x)$ and hence, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & r(\mathfrak{l}) \cap r(x) & \to & r(\mathfrak{l}) & \xrightarrow{x \mid r(\mathfrak{l})} & xr(\mathfrak{l}) & \to 0 \\ & \downarrow & & \downarrow i & & \downarrow \theta & \\ 0 \to & r(y) & \to & A & \xrightarrow{y} & yA & \to 0 \end{array}$$

where $yA=\operatorname{Im} y$ and $xr(\mathfrak{l})=\operatorname{Im}(x \mid r(\mathfrak{l}))$. Hence, we have a morphism θ in $[xr(\mathfrak{l}), yA]$ such that $\theta x \mid r(\mathfrak{l})=yi$ by [10], p. 23, Proposition 16. 5. Since A is quasi-injective, θ is extended in an element s in S. Hence, $y-sx \in lr(\mathfrak{l})=\mathfrak{l}$. Therefore, $y \in \mathfrak{l}+Sx$.

Corollary. *Let A be as above. Then $lr(\mathfrak{l})=\mathfrak{l}$ for any finitely generated left ideal $\mathfrak{l}$ in S_A.* (Dual of [6], Lemma 2.6 or [13], Theorem 1. 1).

Lemma 2. *Let A be a quasi-injective object in* $\mathbf{A}$. *If A satisfies the condition in the theorem, then every co-idempotent sub-object B $(\neq A)$ of A is contained in a proper direct summand of A.*

Proof. The proof is a dual of [6], Proposition 2. 3. However, we shall give the proof for the sake of completeness. Let $B=r(\mathfrak{l}')=r(\mathfrak{l}'^2)$ for a left ideal $\mathfrak{l}'$ in $S=[A, A]$. From the assumption we can take a maximal sub-object C among C' such that $A \supsetneqq C' \supseteq B$ and $C'=r(\mathfrak{l})=r(\mathfrak{l}^2)$. Since $\mathfrak{l}^2 \neq 0$, we can choose x in $\mathfrak{l}$ which has properties; $\mathfrak{l}x \neq 0$ and $r(x)$ is maximal among $r(y)$ such that $\mathfrak{l}y \neq 0$, $y \in \mathfrak{l}$. If $\mathfrak{l}\mathfrak{l}x=0$, $\operatorname{Im} x \subseteq r(\mathfrak{l}^2)=r(\mathfrak{l})$, and $\mathfrak{l}x=0$. Therefore, there exists y in $\mathfrak{l}$

such that $\mathfrak{I}yx\neq 0$. Since $r(yx)\supseteq r(x)$, $r(yx)=r(x)$ by the maximality of $r(x)$. Hence, $Syx=Sx$ by Lemma 1. Therefore, there exists a in $\mathfrak{I}$ such that $ax=x$. If a is not idempotent, then $0\neq\mathfrak{I}'=\{z \mid \in\mathfrak{I}, zx=0\}\subsetneqq\mathfrak{I}$. Further $r(\mathfrak{I}')\supset \mathrm{Im}\, x$ and $r(\mathfrak{I})\not\supset \mathrm{Im}\, x$. Hence, $\mathfrak{I}'$ is nilpotent by the maximality of C. Thus, we can find a non-zero idempotent e in $\mathfrak{I}$. Hence, $r(\mathfrak{I})\subseteq r(e)=\mathrm{Im}\,(1-e)$.

Proof of the theorem. Since every direct summand of A is an annihilator object, A is a directsum of finite number of indecomposable objects. First we assume that A is indecomposable. Let $\mathfrak{I}$ be a proper ideal in S. Then $r(\mathfrak{I}^n)=r(\mathfrak{I}^{2n})$ for some integer n by the assumption. Hence, $r(\mathfrak{I}^n)=A$ by Lemma 2. Therefore, $\mathfrak{I}$ is nilpotent, which implies that S is a semi-primary ring with unique maximal ideal. In general case, we can use the standard argument as in the proof of [6], Proposition 2. 4.

Corollary. *Let A be a quasi-injective and quasi-projective object in* **A**. *If A is noetherian, S_A is right artinian.*

Proof. S_A is semi-primary by Theorem 1 and right noetherian by [6], Proposition 2, 7. Hence, S_A is right artinian.

Proposition 1. *Let A be a quasi-injective object in* **A**. *Then the following statements are equivalent.*

1) *S_A is left noetherian.*
2) *A is artinian with respect to annihilator sub-objects.* (cf. [13] and [3]).

Proof. 1)→2). It is clear. 2)→1). The set of all finitely generated left ideals in S_A is noetherian from 2) and Lemma 1. Hence, S_A is left noetherian.

Corollary. *Let A be a quasi-injective object in* **A**. *If A is artinian and noetherian with respect to annihilator sub-objects, then S_A is left artinian.*

Proof. S_A is semi-primary by Theorem 1 and left noetherian by Proposition 1. Therefore, S_A is left artinian.

2. In cases of modules

In this section, we assume that a ring R has the identity element and every right R-module is unitary.

Proposition 2. *Let M be a quasi-injective right R-module and $S=\mathrm{Hom}_R(M, M)$. Then M is noetherian as a left S-module if and only if M is noetherian with respect to annihilator submodules for sub-sets in R.*

Proof. We assume the later condition in the proposition. Then R is artinian with respect to annihilator right ideals for sub-sets in M. Let T be an S-submodule in M. We take a minimal one $r(T')$ among $r(T^*)$, where T^* runs through all finitely generated S-submodules in T. Let t be any element in T. Then $r(St+T')=r(T')$ by the minimality of T'. Hence, $St+T'=T'$ by [9], Theorem 2. 1.

Corollary 1. *Let R be a right artinian ring and M a quasi-injective right R-module. Then M is a noetherian S-module. Furthermore, if M is artinian (or noetherian) as R-modules, then S is left artinian and M has a finite composition length as S-modules.*

Proof. The first part is clear. If M is artinian, then S is left noetherian by Proposition 1. Let J be the Jacobson radical of S. Then $J^nM=J^{n+1}M$ for some n. Since M is S-noetherian, $J^nM=0$. Hence, $J^n=0$ and S is semi-primary, since S/J is a regular ring in the sense of Von Neumann, (see [4]). Therefore, S is left artinian. The last part is clear from the above and the first part.

Corollary 2. ([2]). *Let R be a right noetherian and self-injective as a right R-module. Then R is left and right artinian* (QF-ring).

Proof. R is a projective injective right R-module. Hence, R is right artinian by Corollary to Theorem 1. Therefore, R is left artinian by the above corollary.

According to Azumaya [1], we define a *weakly distinguished* R-module T as follows: for any R-submodules $T_1 \supset T_2$ in T such that T_1/T_2 is R-irreducible, $\mathrm{Hom}_R(T_1/T_2, T) \neq 0$. It is clear that if T is an R-cogenerator, then T is weakly distinguished. Furthermore, if T is quasi-injective, T is weakly distinguished if and only if $l(T_1) \subsetneqq l(T_2)$ for any R-submodules $T_1 \supsetneqq T_2$ or equivalently, $rl(T')=T'$ for any R-submodule T' of T, (cf. [1], Proposition 6).

Lemma 3. *Let M be a right R-quasi-injective and noetherian with respect to annihilator R-submodules for sub-sets in S, where $S=\mathrm{Hom}_R(M, M)$. We assume that S satisfies a condition: for any left ideals $\mathfrak{l}$ and $\mathfrak{l}'$ in S*

$$(*) \qquad r(\mathfrak{l} \cap \mathfrak{l}') = r(\mathfrak{l}) + r(\mathfrak{l}') .$$

Then S is left artinian.

Proof. Since S is semi-primary, S contains the non-zero left socle T, say $T=\sum_i \oplus \mathfrak{l}_i$, where $\mathfrak{l}_i$'s are minimal left ideals. Put $L_i=\sum_{j \geqslant i} \oplus \mathfrak{l}_j$. Then $L_1 \supset L_2 \supset L_3 \supset \cdots$ and $r(L_1) \subseteq r(L_2) \subseteq r(L_3) \subseteq \cdots$. Hence, $r(L_n)=r(L_{n+1})$ for some n by the assumption. We assume $L_n \neq 0$. Then $L_n=\mathfrak{l}_n \oplus L_{n+1}$, $r(L_n) \subseteq r(\mathfrak{l}_n)$ and $M=r(\mathfrak{l}_n \cap L_{n+1})=r(\mathfrak{l}_n)+r(L_{n+1})$, which is a contradiction. Hence, $T=\sum_{i=1}^m \oplus \mathfrak{l}_i$. Put $M_1=r(T)$, Since T is finitely generated, $l(M_1)=T$ by Lemma 1. Furthermore, T is a two-sided ideal and hence, M_1 is a left S-module, which implies M_1 is a quasi-injective R-module by [9], Theorem 1.2. Put $S_1=\mathrm{Hom}_R(M_1, M_1)$. Then we have a natural epimorphism φ of S to S_1 with Ker $\varphi=l(M_1)=T$ and hence, M_1 is noetherian with respect to annihilator R-submodules. Put T_1 the left socle of S_1, say $T_1=\sum_i \oplus \bar{\mathfrak{l}}_{1i}\supseteq$ where $\mathfrak{l}_{1i}\supseteq T$ and $\bar{\mathfrak{l}}_{1i}=\mathfrak{l}_{1i}/T$ is irreducible. Then $M_1=r(T)=r(\mathfrak{l}_{1n}\cap L_{1n})=r(\mathfrak{l}_{1n})+r(L_{1n})$. Hence, we know from the same argument in the above that $T_1=\sum_{i=1}^{m}\oplus \mathfrak{l}_{1i}$. Repeating this we have a series of ideals $S\supset T_n\supset T_{n-1}\supset\cdots\supset T_1\supset 0$ such that T_i/T_{i-1} is the left socle of S/T_{i-1} which has a finite composition length. Now S is semi-primary and hence, $N^{n-i}\subseteq T_i$, where $N^{n-1}\neq 0$ and $N^n=0$. Therefore, S is left artinian.

Theorem 2. *Let M be R-weakly distinguished and quasi-injective and* $S=\mathrm{Hom}_R(M, M)$. *Then the following two conditions are equivalent.*

1) *S is left noetherian.*
2) *M is artinian as an R-module.*

And 1) *or* 2) *implies*

3) *M is S-injective.*

Furthermore, if M is noetherian with respect to annihilator submodules for sub-sets in S, then 3) *implies* 1) *and* 2) *and S is left artinian and M is R-noetherian.*

Proof. 1)→2). It is clear from the remark before Lemma 3. 2)→1). It is clear from Proposition 1. 1)→3). We assume that S is left noetherian. Let $\mathfrak{l}_1, \mathfrak{l}_2$ be left ideals in S. Then $r(\mathfrak{l}_1)+r(\mathfrak{l}_2))=lr(\mathfrak{l}_1)\cap lr(\mathfrak{l}_2)=\mathfrak{l}_1\cap\mathfrak{l}_2$ by Corollary to Lemma 1. Hence, $r(\mathfrak{l}_1)+r(\mathfrak{l}_2)=r(\mathfrak{l}_1\cap\mathfrak{l}_2)$ by the above remark. Now, we shall show by the induction on the number of generators of left ideals in S that M satisfies the Bear's condition, (it is essentially due to [8]). Let $\mathfrak{l}=Sx$, Then $l(xM)=l_S(x)=\{y\mid\in S, yx=0\}$ and $r(l_S(x))\supseteq xM$. Hence, $r(l_S(x))=xM$. Let f be an element in $\mathrm{Hom}_S(\mathfrak{l}, M)$, then $f(x)\in r(l_S(x))=xM$. Hence, there exists m in M such that $f(x)=xm$. Let $\mathfrak{l}=\sum_{i=1}^{n} Sx_i$ and $\mathfrak{l}_1=\sum_{i=1}^{n-1} Sx_i$. From an exact sequence: $0\to\mathfrak{l}_1\to\mathfrak{l}\to\mathfrak{l}/\mathfrak{l}_1\overset{\varphi}{\approx} Sx_n/(Sx_n\cap\mathfrak{l}_1)\to 0$, we have the exact sequence: $\mathrm{Hom}_S(\mathfrak{l}_1, M)\leftarrow\mathrm{Hom}_S(\mathfrak{l}, M)\leftarrow\mathrm{Hom}_S(\mathfrak{l}/,\mathfrak{l}_1\ M)\overset{[\varphi,M]}{\approx}\mathrm{Hom}_S(Sx_n/(Sx_n\cap\mathfrak{l}), M)\leftarrow 0$. Let f be in $\mathrm{Hom}_S(\mathfrak{l}, M)$. Then there exists m in M such that $f(x)=xm$ for $x\in\mathfrak{l}_1$ by the hypothesis of the induction. We define an element f_m in $\mathrm{Hom}_S(\mathfrak{l}, M)$ by setting $f_m(x)=xm$ for $x\in\mathfrak{l}$. Then $g=f-f_m\in\mathrm{Hom}_S(\mathfrak{l}/\mathfrak{l}_1, M)$. Since $\mathrm{Hom}_S(Sx_n, M)\leftarrow\mathrm{Hom}_S(Sx_n/Sx_n\cap\mathfrak{l}_1, M)$ is monomorphic, there exists m' in M such that $g(\varphi^{-1}(\overline{sx}_n))=sx_nm'$, where $\overline{sx}_n$ means a residue class of sx_n in $Sx_n/(Sx_n\cap\mathfrak{l}_1)$. Hence, $m'\in r(Sx_n\cap\mathfrak{l}_1)=r(Sx_n)+\mathrm{r}(\mathfrak{l}_1)$. Let $m'=m_1+m_2$, $m_1\in r(x_n)$, $m_2\in r(\mathfrak{l}_1)$ and define f_{m_2} as above. Then for any $x=x_1+x_2$ in $\mathfrak{l}$ ($x_1\in\mathfrak{l}_1$, $x_2\in Sx_n$), $g(x)=g(\varphi^{-1}(x_2))=x_2m'=x_2m_2=f_{m_2}(x)$. Therefore, $f=f_{m+m_2}$. 3)→1).

Let M be S-injective and $\mathfrak{l}_1$, $\mathfrak{l}_2$ be left ideals in S. Then we have an exact sequence: $0 \leftarrow \mathrm{Hom}_S(S/(\mathfrak{l}_1 \cap \mathfrak{l}_2), M) \leftarrow \mathrm{Hom}_S(S/\mathfrak{l}_1, M) \oplus \mathrm{Hom}_S(S/\mathfrak{l}_2, M)$, which means that M satisfies the condition $(*)$. Hence, S is left artinian from Lemma 3. The last part is clear, since S is artinian by Theorem 1.

Corollary. *Let M and S be as in Theorem 2. If M is R-artinian, then any S-R bi-submodule N of M is $S/l(N)$-injective.*

Proof. Let N be an S-R submodule of M. Then $N=rl(N)$ and $l(N)$ is a two-sided ideal in S. Put $\bar{S}=S/l(N)$. Then $\bar{S}=\mathrm{Hom}_R(N, N)$ and N satisfies the same conditions as M by [9], Theorem 1. 1. Hence, N is $\bar{S}$-injective by Theorem 2.

Theorem 3. *Let M be an S-R bi-module such that* $\mathrm{Hom}_R(M, M)=S$ *and* $\mathrm{Hom}_S(M, M)=R$. *Furthermore, we assume that M is S- and R-injective, respectively. Then the following two statements are equivalent.*

1) *M is R-noetherian,*

2) *S is left artinian.*

And 1) *or* 2) *implies that M is R-artinian. Thus, if M is R- and S-noetherian or if R and S are right and left artinian, respectively, then M gives us a duality between the category of finitely generated right R-modules and the category of finitely generated left S-modules in the sense of Morita.*

Proof. 1)→2). Since S satisfies the condition $(*)$ of Lemma 3, S is left artinian. 2)→1). It is obtained by Corollary to Proposition 2. Now, we assume 1) or 2). Let T be an R-submodule, then $T=rl(T)$ by [9], Theorem 2. 1. Hence, M is R-artinian, since S is left noetherian. The last part is clear from [11], Theorem 6. 3, v.

REMARK. Let M and S be as the first half in Theorem 2. Then the injectivity of M as an S-module does not imply the fact that S is left noetherian. Furthermore, if R is commutative, then a fact that M is R-noetherian implies that M is R-artinian, (see Proposition 2 in [7]). However, the converse is not true in general.

Finally, we shall give an example of injective noetherian but not artinian modules. Let K be a field and $I=Z^+ \cup \alpha$ the set of indices, where Z^+ is the set of positive integers. Let R be the ring of upper tri-angular matrices over K with indices I, (α is the last index and α-column consists of all column finite). Let e_{ij} be matrix units in R and put $M=e_{11}R$. Then $M \approx \mathrm{Hom}_K(Re_{\alpha\alpha}, Ke_{1\alpha})$. Hence, M is R-injective. It is clear that M is R-noetherian but not R-artinian.

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