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ON VANISHING THEOREMS OF SQUARE-INTEGRABLE $\bar{\partial}$-COHOMOLOGY SPACES ON HOMOGENEOUS KAHLER MANIFOLDS

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1. Introduction

Let G be a connected non-compact semi-simple Lie group. We assume that G has a complex form G^C and a compact Cartan subgroup H. The quotient manifold $D=G/H$ carries a G-invariant complex structure and a G-invariant hermitian metric. Then, corresponding to each character λ of H, one can construct a homogeneous hermitian line bundle $\mathcal{L}_\lambda=G\times_H C$ over D. Let $H_2^q(\mathcal{L}_\lambda)$ be the q-th square-integrable $\bar{\partial}$-cohomology space with coefficients in the bundle $\mathcal{L}_\lambda$, i.e. the Hilbert space of all square-integrable $\mathcal{L}_\lambda$-valued harmonic $(0, q)$-forms on D. P.A. Griffiths and W. Schmid [4] have obtained some vanishing theorems for these cohomology spaces, assuming that the character λ is sufficiently non-singular.

Now the manifold D does not necessarily admit a G-invariant Kähler metric. In fact, P.A. Griffiths and W. Schmid used a non-Kähler hermitian metric on D. The purpose of this paper is to prove certain vanishing theorems for these $\bar{\partial}$-cohomology spaces under the assumption that D has a G-invariant Kähler metric. The main result is Theorem 2 in §7. In some cases, our result is considerably better than the one given in [4]. (cf. §7. Example)

In §2, we recall some facts about Lie algebras and homogeneous vector bundles. In §3 and following sections, we assume further that the Riemannian symmetric space G/K is hermitian symmetric, where K is a maximal compact subgroup of G containing H. Under this assumption, we introduce canonically an invariant complex structure and an invariant Kähler metric on the manifold D. Next, we shall define in §4 the q-th square-integrable $\bar{\partial}$-cohomology space $H_2^q(\mathcal{L}_\lambda)$ on D with coefficients in $\mathcal{L}_\lambda$. Also we shall give explicit formulas for the differential operator $\bar{\partial}$ and the inner product on the space of all compactly supported $\mathcal{L}_\lambda$-valued C^∞-forms on D.

In [1], A. Andreotti and E. Vesentini expressed the Laplace-Beltrami operator $\Box$ on a hermitian manifold in terms of the metric connection and showed that this expression of $\Box$ becomes simpler if the manifold is Kählerian.

In §5, we construct the metric connection in the bundle $\mathcal{L}_\lambda$ and the Riemannian connection of D, applying Wang's results about invariant connections. Moreover, in §6, we express the operators $\bar{\partial}$, δ and $\square$ in terms of these connections. From the fact that the metric on D is Kählerian, we get a simple explicit formula for the operator $\square$ (cf. §6. Proposition 2). In §7, we shall prove the main vanishing theorem. In this proof, we use the criterion for the vanishing of square-integrable $\bar{\partial}$-cohomology spaces which has been established in [1].

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2. Preliminaries

Let G be a connected non-compact semi-simple Lie group. We denote by $\mathfrak{g}_0$ the Lie algebra of left invariant vector fields on G and by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. Throughout this paper, we assume that G has a compact Cartan subgroup H. Let K be a maximal compact subgroup of G which contains H. Let $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be the subalgebras of $\mathfrak{g}_0$ corresponding to the subgroups K and H, and $\mathfrak{k}$ and $\mathfrak{h}$ the complexifications of $\mathfrak{k}_0$ and $\mathfrak{h}_0$ respectively. For each $x\in\mathfrak{g}$, we denote by $\bar{x}$ the image of x under the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$. Let Δ be the set of all non-zero roots of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$. Then, the Lie algebra $\mathfrak{g}$ decomposes into the direct sum

$$\mathfrak{g} = \mathfrak{h}\oplus\sum_{\alpha\in\Delta}\mathfrak{g}^{\alpha}\,, \tag{2.1}$$

where we put

$$\mathfrak{g}^{\alpha} = \{x\in\mathfrak{g}\,|\,[h, x] = \langle\alpha, h\rangle x \qquad \text{for all} \quad h\in\mathfrak{h}\}\,.$$

Since H is compact, each root $\alpha\in\Delta$ takes purely imaginary values on $\mathfrak{h}_0$. Thus, we may consider Δ as a subset of the dual space $\mathfrak{h}_{\boldsymbol{R}}{}^*$ of $\mathfrak{h}_{\boldsymbol{R}}=\sqrt{-1}\,\mathfrak{h}_0$, and we have

$$\bar{\mathfrak{g}}^{\alpha} = \mathfrak{g}^{-\alpha} \qquad \text{for all} \quad \alpha\in\Delta\,.$$

Now, let B be the Killing form of $\mathfrak{g}$. We denote by $(\ ,\ )$ the natural inner product on $\mathfrak{h}_{\boldsymbol{R}}{}^*$ obtained from the restriction of B on $\mathfrak{h}_{\boldsymbol{R}}$. Put

$$\mathfrak{p} = \{x\in\mathfrak{g}\,|\,B(x, y) = 0 \qquad \text{for all} \quad y\in\mathfrak{k}\}\,.$$

Then we have

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p},\ \ [\mathfrak{k}, \mathfrak{p}]\subset\mathfrak{p},\ \ [\mathfrak{p}, \mathfrak{p}]\subset\mathfrak{k}\,. \tag{2.2}$$

A root α is called compact or non-compact according as $\mathfrak{g}^{\alpha}\subset\mathfrak{k}$ or $\mathfrak{g}^{\alpha}\subset\mathfrak{p}$. We denote by $\Delta_{\mathfrak{k}}$(resp. $\Delta_{\mathfrak{p}}$) the set of all compact (resp. non-compact) roots. Then we have

$$\Delta = \Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}} .$$

For each $\alpha \in \Delta$, let h_α be the element of $\mathfrak{h}$ such that

$$B(h, h_\alpha) = \alpha(h) \qquad \text{for all} \quad h \in \mathfrak{h} . \tag{2.3}$$

Then we can choose root vectors $e_\alpha \in \mathfrak{g}^\alpha$ ($\alpha \in \Delta$) satisfying the following conditions:

$$\begin{aligned} (2.4) \quad & 1) \quad [e_\alpha, e_{-\alpha}] = h_\alpha , \\ & 2) \quad [e_\alpha, e_\beta] = 0 \qquad \text{if} \quad \alpha+\beta \neq 0 \text{ and } \alpha+\beta \notin \Delta , \\ & 3) \quad [e_\alpha, e_\beta] = N_{\alpha,\beta} e_{\alpha+\beta} \qquad \text{if} \quad \alpha+\beta \in \Delta , \\ & 4) \quad \bar{e}_\alpha = \varepsilon_\alpha e_{-\alpha} , \end{aligned}$$

where the $N_{\alpha,\beta}$'s are non-zero real constants, and $\varepsilon_\alpha = -1$ if $\alpha \in \Delta_{\mathfrak{k}}$ and $\varepsilon_\alpha = 1$ if $\alpha \in \Delta_{\mathfrak{p}}$ [5]. Moreover, the $N_{\alpha,\beta}$'s satisfy following equalities:

$$\begin{aligned} (2.5) \qquad & N_{-\alpha,-\beta} = -N_{\alpha,\beta} \\ & N_{-\alpha,-\beta} = N_{-\beta,\alpha+\beta} = N_{\alpha+\beta,-\alpha} . \end{aligned}$$

For convenience, we define $N_{\alpha,\beta} = 0$ if $\alpha+\beta \neq 0$ and $\alpha+\beta \notin \Delta$. We denote by $\{\omega^\alpha \mid \alpha \in \Delta\}$ the left-invariant 1-forms on G which are dual to $\{e_\alpha \mid \alpha \in \Delta\}$.

We consider the quotient manifold $D = G/H$. In the decomposition (2.1) of $\mathfrak{g}$, we put

$$\mathfrak{n} = \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha, \quad \mathfrak{n}_0 = \mathfrak{n} \cap \mathfrak{g}_0 .$$

Then, we have

$$\mathfrak{g}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0, \quad [\mathfrak{h}_0, \mathfrak{n}_0] \subset \mathfrak{n}_0 , \tag{2.6}$$

and D is a reductive homogeneous space. The tangent space of D at the point $o = eH$ may be identified with the subspace $\mathfrak{n}_0$ of $\mathfrak{g}_0$, where e denotes the identity element of G. Now, let $\pi: H \to GL(E)$ be a representation of H in a complex vector space E. We denote by E_π the homogeneous vector bundle over D associated with the representation π of H. Let $A^p(E_\pi)$ be the space of E_π-valued C^∞ p-forms on D. A form in $A^p(E_\pi)$ can be identified with an E-valued C^∞ p-form φ on G satisfying the conditions

$$\begin{cases} \theta(h)\varphi = -\pi(h)\varphi \\ i(h)\varphi = 0 \end{cases} \qquad \text{for all} \quad h \in \mathfrak{h} , \tag{2.7}$$

where $\theta(h)$ and $i(h)$ denote the operator of Lie derivation and interior product by the vector field h and π is the representation of $\mathfrak{h}$ in E induced by the representation π of H [7]. Let $C^\infty(G)$ be the space of all complex-valued

C^∞-functions on G. Let $\mathfrak{n}^*$ be the dual space of $\mathfrak{n}$ and $\overset{p}{\wedge}\mathfrak{n}^*$ the p-th exterior product of $\mathfrak{n}^*$. For an ordered p-tuple $C=(\lambda_1, \cdots, \lambda_p)$ of roots, we put

$$\omega^C = \omega^{\lambda_1}\wedge\cdots\wedge\omega^{\lambda_p} .$$

Let Λ be a set of ordered p-tuples such that $\{\omega^C | C\in\Lambda\}$ forms a basis of $\overset{p}{\wedge}\mathfrak{n}^*$. The vector space $C^\infty(G)\otimes E\otimes\overset{p}{\wedge}\mathfrak{n}^*$ is generated by monomials $F\omega^C$ with $F\in C^\infty(G)\otimes E$ and $C\in\Lambda$. By (2.7), the space $A^p(E_\pi)$ can be identified with the subspace of $C^\infty(G)\otimes E\otimes\overset{p}{\wedge}\mathfrak{n}^*$ consisting of all elements $\varphi=\sum_{C\in\Lambda}F_C\omega^C$ satisfying the condition

$$hF_C = -\pi(h)F_C+\langle |C|, h\rangle F_C . \tag{2.8}$$

for all $C\in\Lambda$ and $h\in\mathfrak{h}$, where $|C|=\lambda_1+\cdots+\lambda_p$ and hF_C denotes the differentiation of the function F_C by the vector field h. In particular, the space $A^0(E_\pi)$ is identified with the subspace of $C^\infty(G)\otimes E$ consisting of all elements $F\in C^\infty(G)\otimes E$ such that

$$hF = -\pi(h)F \qquad \text{for all} \quad h\in\mathfrak{h} .$$

3. Homogeneous Kähler manifolds

Let G be a connected non-compact semi-simple Lie group with a compact Cartan subgroup H. In the following, we assume that there is a complex Lie group G^C with Lie algebra $\mathfrak{g}$ which contains G as a Lie subgroup corresponding to the subalgebra $\mathfrak{g}_0$.

We introduce an ordering for the roots and denote by Δ_+ the set of all positive roots with respect to this ordering. Put

$$\mathfrak{n}_+ = \sum_{\alpha\in\Delta_+}\mathfrak{g}^\alpha, \quad \mathfrak{n}_- = \sum_{\alpha\in\Delta_+}\mathfrak{g}^{-\alpha} .$$

Then we have

$$\begin{aligned} &\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{n}_+\oplus\mathfrak{n}_- \\ &\bar{\mathfrak{n}}_+ = \mathfrak{n}_- \quad \bar{\mathfrak{n}}_- = \mathfrak{n}_+ \\ &[\mathfrak{h}, \mathfrak{n}_+]\subset\mathfrak{n}_+, \quad [\mathfrak{n}_+, \mathfrak{n}_+]\subset\mathfrak{n}_+ . \end{aligned} \tag{3.1}$$

For the quotient manifold $D=G/H$, the complexified tangent space of D at the point o may be identified with the vector space $\mathfrak{n}=\mathfrak{n}_+\oplus\mathfrak{n}_-$, and then, by (3.1), D has a G-invariant complex structure such that the holomorphic tangent space of D at o corresponds to $\mathfrak{n}_+$. This complex structure of D can also be obtained in the following way. Let B be the Borel subgroup of G^C corresponding to the Borel subalgebra $\mathfrak{h}\oplus\mathfrak{n}_-$ of $\mathfrak{g}$. The group G acts on the homogeneous complex manifold G^C/B. Since we have

$$\begin{aligned}\mathfrak{g}_0\cap(\mathfrak{h}\oplus\mathfrak{n}_-)&=\mathfrak{g}_0\cap(\mathfrak{h}\oplus\mathfrak{n}_-)\cap(\overline{\mathfrak{h}\oplus\mathfrak{n}_-})\\&=\mathfrak{g}_0\cap\mathfrak{h}=\mathfrak{h}_0\,,\end{aligned}$$

the Lie algebra of the isotropy subgroup $G\cap B$ is $\mathfrak{h}_0$. Therefore, H is the identity component of the subgroup $G\cap B$ and $G\cap B$ normalizes H. The normalizer of H in G is compact and H is a maximal compact subgroup of the Borel subgroup B. Hence we have

$$G\cap B=H\,,$$

and $D=G/H$ is identified with the G-orbit of eB in G^C/B. Since

$$\dim G^C/B=\dim\mathfrak{n}_+=\dim G/H\,,$$

D is open in G^C/B and D has a G-invariant complex structure as an open submanifold of G^C/B. Then, it is easily seen that the holomorphic tangent space of D at o corresponds to $\mathfrak{n}_+$ [4].

In the following, we will assume that the Riemannian symmetric space G/K is hermitian symmetric. By (2.2), the complexified tangent space of G/K at eK may be identified with $\mathfrak{p}$. Let $\mathfrak{p}_+$(resp. $\mathfrak{p}_-$) be the subspace of $\mathfrak{p}$ corresponding to the holomorphic (resp. anti-holomorphic) tangent space of G/K at eK under this identification. We know that there exists an element h_0 belonging to the center of $\mathfrak{k}_0$ such that

$$(3.2)\qquad [h_0,x]=\begin{cases}\sqrt{-1}\,x & \text{for } x\in\mathfrak{p}_+\\ -\sqrt{-1}\,x & \text{for } x\in\mathfrak{p}_-\,.\end{cases}$$

It follows that

$$\begin{aligned}&[\mathfrak{p}_+,\mathfrak{p}_+]=0,\quad[\mathfrak{p}_-,\mathfrak{p}_-]=0\\&[\mathfrak{k},\mathfrak{p}_+]\subset\mathfrak{p}_+,\quad[\mathfrak{k},\mathfrak{p}_-]\subset\mathfrak{p}_-\end{aligned}$$

and in particular

$$[\mathfrak{h},\mathfrak{p}_+]\subset\mathfrak{p}_+,\quad[\mathfrak{h},\mathfrak{p}_-]\subset\mathfrak{p}_-\,.$$

Hence, we see that for some subset Δ_0 of Δ

$$\mathfrak{p}_+=\sum_{\alpha\in\Delta_0}\mathfrak{g}^{\alpha},\quad\mathfrak{p}_-=\sum_{\alpha\in\Delta_0}\mathfrak{g}^{-\alpha}\,.$$

We may choose an ordering for the roots in such a way that the roots belonging to Δ_0 are all positive, i.e.

$$(3.3)\qquad \Delta_0=\Delta_+\cap\Delta_{\mathfrak{p}}\,.$$

We choose such an ordering once for all, and introduce an invariant complex structure on D defined by this ordering.

Lemma 1. *There exists an element τ of $\mathfrak{h}_R{}^*$ satisfying following conditions:*

$$(3.4)\qquad \begin{cases} (\alpha, \tau)>0 & \text{for all} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}} \\ (\alpha, \tau)<0 & \text{for all} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}. \end{cases}$$

Proof. Since the element h_0 belongs to the center of $\mathfrak{k}_0$, we have

$$\alpha(\sqrt{-1}\,h_0)=0 \qquad \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}}.$$

By (3.2) and (3.3), we have also

$$\alpha(\sqrt{-1}\,h_0)=-1 \qquad \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}.$$

We denote by τ_0 the element of $\mathfrak{h}_R{}^*$ such that

$$B(h, \sqrt{-1}\,h_0)=\tau_0(h) \qquad \text{for all} \quad h\in\mathfrak{h}_R.$$

Then, we obtain

$$(\alpha, \tau_0)=\begin{cases} 0 & \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}} \\ -1 & \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}. \end{cases}$$

On the other hand, we know that for the element $\rho=\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ of $\mathfrak{h}_R{}^*$ we have $(\rho, \alpha)>0$ for $\alpha\in\Delta_+$. Therefore, if we put $\tau=\rho+c\tau_0$ with a sufficiently large constant c, we get

$$(\alpha, \tau)=\begin{cases} (\alpha, \rho)>0 & \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}} \\ (\alpha, \rho)+c(\alpha, \tau_0)<0 & \text{for} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}. \end{cases}$$

q.e.d.

Now, let τ be an element of $\mathfrak{h}_R{}^*$ satisfying the condition (3.4). Using this τ, we shall construct an invariant Kähler metric on D. We define a complex symmetric bilinear form B_τ on $\mathfrak{n}=\sum_{\alpha\in\Delta}\mathfrak{g}^\alpha$ by the following formula:

$$(3.5)\qquad \begin{aligned} &B_\tau(e_\alpha, e_\beta)=B_\tau(e_{-\alpha}, e_{-\beta})=0 \\ &B_\tau(e_\alpha, e_{-\beta})=-\delta_{\alpha,\beta}(\alpha, \tau) \end{aligned}$$

for $\alpha, \beta\in\Delta_+$. Clearly, B_τ is invariant under the adjoint action of H on $\mathfrak{n}$ and the restriction of B_τ on the real subspace $\mathfrak{n}_0=\mathfrak{n}\cap\mathfrak{g}_0$ is a positive-definite symmetric bilinear form on $\mathfrak{n}_0$. Define the endomorphism J on $\mathfrak{n}$ by

$$Jx=\begin{cases} \sqrt{-1}\,x & \text{for} \quad x\in\mathfrak{n}_+ \\ -\sqrt{-1}\,x & \text{for} \quad x\in\mathfrak{n}_-. \end{cases}$$

Then we have also

$$B_\tau(Jx, Jy)=B_\tau(x, y)$$

for all $x, y \in \mathfrak{n}$. Therefore, we can define a G-invariant hermitian metric on D such that the metric on the tangent space of D at o corresponds to $B_\tau|\mathfrak{n}_0$ on $\mathfrak{n}_0$. The Kähler form of this hermitian metric corresponds to the complex 2-form Ω on G given by

$$\Omega = \sum_{\alpha \in \Delta_+} \sqrt{-1}\,(\alpha, \tau)\omega^\alpha \wedge \omega^{-\alpha} .$$

We see easily that

$$d\Omega = 0 .$$

Thus the metric on D induced by B_τ is Kählerian. We denote by g_τ this invariant Kähler metric on D.

REMARK. In the above, we constructed an invariant Kähler metric on D under the assumption that the symmetric space G/K is hermitian and the natural projection $G/H \rightarrow G/K$ is holomorphic. Conversely, it is known that if the manifold D has a G-invariant Kähler metric, then G/K is hermitian and the fibering $G/H \rightarrow G/K$ is holomorphic [2].

4. Homogeneous line bundles and square-integrable $\bar{\partial}$-cohomology spaces

Let $D = G/H$ be the homogeneous complex manifold of $2n$ real dimension with the G-invariant Kähler metric g_τ. Let λ be a character of H. We consider the homogeneous real line bundle $\mathcal{L}_\lambda = G \times_H C$ over D associated with λ. Let B be the Borel subgroup of G^C such that the quotient manifold G^C/B contains D as an open submanifold (cf §3). The character λ can be extended to the unique holomorphic character on B [3]. We can consider the homogeneous complex line bundle $G^C \times_B C$ over G^C/B. Then, the bundle $\mathcal{L}_\lambda$ is isomorphic to the restriction of the bundle $G^C \times_B C$ on D as a real line bundle. Therefore, $\mathcal{L}_\lambda$ has a G-invariant complex structure as an open submanifold of $G^C \times_B C$. We get thus a hermitian line bundle $\mathcal{L}_\lambda$ with a natural hermitian metric in the fibers.

Let $A^{p,q}(\mathcal{L}_\lambda)$ be the space of all $\mathcal{L}_\lambda$-valued C^∞-forms of type (p, q) on D, and $A_0^{p,q}(\mathcal{L}_\lambda)$ the subspace of all compactly supported forms in $A^{p,q}(\mathcal{L}_\lambda)$. The hermitian metric on D defines the complex linear operator $*$ of $A^{p,q}(\mathcal{L}_\lambda)$ into $A^{n-q,n-p}(\mathcal{L}_\lambda)$. On the other hand, the hermitian metric on the fibers of $\mathcal{L}_\lambda$ gives rise to a conjugate linear isomorphism

$$\#\colon\ A^{p,q}(\mathcal{L}_\lambda) \rightarrow A^{q,p}(\mathcal{L}_\lambda^*)$$

where $\mathcal{L}_\lambda^*$ is the complex dual bundle of $\mathcal{L}_\lambda$. We define an inner product $(\ ,\ )$ on $A_0^{0,q}(\mathcal{L}_\lambda)$ by

$$(\varphi, \psi) = \int_D \varphi \wedge * \# \psi$$

for φ, ψ in $A_0^{0,q}(\mathcal{L}_\lambda)$. Let $L_2^{0,q}(\mathcal{L}_\lambda)$ be the completion of $A_0^{0,q}(\mathcal{L}_\lambda)$ with respect to this inner product. The type (0, 1)-component of exterior differentiation defines the differential operator

$$\bar{\partial}\colon A^{0,q}(\mathcal{L}_\lambda) \to A^{0,q+1}(\mathcal{L}_\lambda).$$

Let $\delta\colon A^{0,q}(\mathcal{L}_\lambda) \to A^{0,q-1}(\mathcal{L}_\lambda)$ be the formal adjoint operator of $\bar{\partial}$. We define the Laplace-Beltrami operator $\square$ by

$$\square = \bar{\partial}\delta + \delta\bar{\partial}.$$

Then the space

$$H_2^q(\mathcal{L}_\lambda) = \{\varphi \in L_2^{0,q}(\mathcal{L}_\lambda) \cap A^{0,q}(\mathcal{L}_\lambda) \mid \square\varphi = 0\}$$

is called the q-th square-integrable $\bar{\partial}$-cohomology space of D with coefficients in the bundle $\mathcal{L}_\lambda$. (cf. [1]).

Let $\mathfrak{n}_-^*$ be the dual space of $\mathfrak{n}_-$ and $\overset{q}{\wedge}\mathfrak{n}_-^*$ the q-th exterior product of $\mathfrak{n}_-^*$. We denote by $\{\alpha_1, \cdots, \alpha_n\}$ the set of positive roots Δ_+. For an ordered q-tuple of positive roots $A = (\alpha_{i_1}, \cdots, \alpha_{i_q})$, we put

$$\omega^{-A} = \omega^{-\alpha_{i_1}} \wedge \cdots \wedge \omega^{-\alpha_{i_q}}.$$

Let $\mathfrak{A}$ be the set of all ordered q-tuples $A = (\alpha_{i_1}, \cdots, \alpha_{i_q})$ such that $1 \leqq i_1 < \cdots < i_q \leqq n$. Then the space $C^\infty(G) \otimes \overset{q}{\wedge}\mathfrak{n}_-^*$ is generated by monomials $f\omega^{-A}$ with $f \in C^\infty(G)$ and $A \in \mathfrak{A}$. From the discussion in § 2, the space $A^{0,q}(\mathcal{L}_\lambda)$ is identified with the subspace of $C^\infty(G) \otimes \overset{q}{\wedge}\mathfrak{n}_-^*$. Let $\varphi = \sum_{A \in \mathfrak{A}} f_A \omega^{-A}$ be an element of $C^\infty(G) \otimes \overset{q}{\wedge}\mathfrak{n}_-^*$. Then, according to the condition (2.8), φ belongs to $A^{0,q}(\mathcal{L}_\lambda)$ if and only if the following condition is satisfied:

$$hf_A = \langle -\lambda - |A|, h \rangle f_A \tag{4. 1}$$

for all $A \in \mathfrak{A}$ and $h \in \mathfrak{h}$, where λ is the representation of $\mathfrak{h}$ induced by the character λ of H. Under this identification, the space $A_0^{0,q}(\mathcal{L}_\lambda)$ corresponds to a subspace of $C_0^\infty(G) \otimes \overset{q}{\wedge}\mathfrak{n}_-^*$, where $C_0^\infty(G)$ is the space of all compactly supported functions in $C^\infty(G)$.

Now, we give an expression of the inner product on $A_0^{0,q}(\mathcal{L}_\lambda)$. The bilinear form B_τ on $\mathfrak{n}$ induces the following hermitian inner product B_τ^- on $\mathfrak{n}_-$:

$$\begin{aligned} B_\tau^-(e_{-\alpha}, e_{-\beta}) &= B_\tau(e_{-\alpha}, \bar{e}_{-\beta}) \\ &= \delta_{\alpha,\beta}(-\varepsilon_\alpha(\alpha, \tau)). \end{aligned}$$

From B_τ^-, we obtain the hermitian inner product $(\ ,\ )_-$ on $\overset{q}{\wedge}\mathfrak{n}_-^*$ as follows:

$$(4.2)\qquad \begin{cases} (\omega^{-A}, \omega^{-B})_- = 0 & \text{if } (A) \neq (B) \text{ as sets,} \\ (\omega^{-A}, \omega^{-A})_- = \prod_{\alpha \in A} \left(-\dfrac{1}{\varepsilon_\alpha(\alpha, \tau)} \right). \end{cases}$$

Let dg be a G-invariant volume element of G. Then dg defines an inner product $(\ ,\ )_G$ on $C_0^\infty(G)$. These inner products $(\ ,\ )_-$ and $(\ ,\ )_G$ define an inner product on $C_0^\infty(G) \otimes \overset{q}{\wedge} \mathfrak{n}_-^*$ in a canonical way. In fact, for two elements $\varphi = \sum_{A \in \mathfrak{A}} f_A \omega^{-A}$, $\psi = \sum_{A \in \mathfrak{A}} g_A \omega^{-A}$ in $C_0^\infty(G) \otimes \overset{q}{\wedge} \mathfrak{n}_-^*$, this inner product (φ, ψ) is given by

$$\begin{aligned} (\varphi, \psi) &= \sum_{A \in \mathfrak{A}} (f_A, g_A)_G \cdot (\omega^{-A}, \omega^{-A})_- \\ &= \sum_{A \in \mathfrak{A}} \prod_{\alpha \in A} \left(-\frac{1}{\varepsilon_\alpha(\alpha, \tau)} \right) \cdot \int_G f_A \cdot \overline{g_A}\, dg\,. \end{aligned}$$

The following lemma asserts that if we choose a suitable volume element dg of G, the inner product on $A_0^{0,q}(\mathcal{L}_\lambda)$ is the restriction of this inner product $(\ ,\ )$ on the subspace $A_0^{0,q}(\mathcal{L}_\lambda)$.

Lemma 2. *If we choose a suitable G-invariant volume element dg on G, the inner product of $A_0^{0,q}(\mathcal{L}_\lambda)$ is given by the following formula:*

$$(\varphi, \psi) = \sum_{A \in \mathfrak{A}} \prod_{\alpha \in A} \left(-\frac{1}{\varepsilon_\alpha(\alpha, \tau)} \right) \cdot \int_G f_A \cdot \overline{g_A}\, dg$$

where $\varphi = \sum_{A \in \mathfrak{A}} f_A \omega^{-A}$ and $\psi = \sum_{A \in \mathfrak{A}} g_A \omega^{-A}$ are forms in $A_0^{0,q}(\mathcal{L}_\lambda)$.

Proof. We apply the methods used in the proof of Proposition 5.1 in [7]. Let dv_D be the G-invariant volume element on D determined by the metric g_τ. Then, we can choose invariant volume elements dg on G and dh on H such that

$$\int_G f(g)dg = \int_D \left(\int_H f(gh)dh \right) dv_D$$

for all $f \in C_0^\infty(G)$ ([5], p. 369, Theorem 1.7). Let p: $G \to D$ be the natural projection. Then we have

$$(4.3)\qquad \int_D f' dv_D = \frac{1}{v_H} \int_G f' \circ p\, dg$$

for every compactly supported C^∞-function f' on D, where v_H is the volume of H with respect to dh.

For a root $\alpha_i \in \Delta_+$, we put

$$x_{\alpha_i} = \left(-\frac{1}{\varepsilon_{\alpha_i}(\alpha_i, \tau)} \right)^{1/2} e_{\alpha_i}\,.$$

Then, $\{x_{\alpha_1}, \cdots, x_{\alpha_n}, \overline{x_{\alpha_1}}, \cdots, \overline{x_{\alpha_n}}\}$ is a basis of $\mathfrak{n}$ and we have

$$B_\tau(x_{\alpha_i}, \overline{x_{\alpha_j}}) = \delta_{ij}\,. \tag{4.4}$$

We take a point $p(g)(g\in G)$ in D. By (4.4), $\{p_*(x_{\alpha_1})_g, \cdots, p_*(x_{\alpha_n})_g, p_*(\overline{x_{\alpha_1}})_g, \cdots, p_*(\overline{x_{\alpha_n}})_g\}$ is a basis of the complexified tangent space of D at $p(g)$ such that

$$g_\tau(p_*(x_{\alpha_i})_g, p_*(\overline{x_{\alpha_j}})_g) = \delta_{ij}\,,$$

where p_* is the differential of p. For a sufficiently small neighbourhood U of $p(g)$ in D, we can find $(1, 0)$-forms $\theta^1, \cdots, \theta^n$ and $(0, 1)$-forms $\bar\theta^1, \cdots, \bar\theta^n$ such that

$$\begin{aligned}\theta^i_{p(g)}(p_*(x_{\alpha_j})_g) &= \bar\theta^i_{p(g)}(p_*(\overline{x_{\alpha_j}})_g) = \delta_{ij}\\ \theta^i_{p(g)}(p_*(\overline{x_{\alpha_j}})_g) &= \bar\theta^i_{p(g)}(p_*(x_{\alpha_j})_g) = 0\,.\end{aligned}$$

Let φ and ψ be forms in $A_0^{0,q}(\mathcal{L}_\lambda)$. We denote by $\sum_{A\in\mathfrak{A}} f_A\omega^{-A}$(resp. $\sum_{A\in\mathfrak{A}} g_A\omega^{-A}$) an element of $C_0^\infty(G)\otimes\overset{q}{\wedge}\mathfrak{n}_-{}^*$ which corresponds to φ (resp. ψ) under the identification of $A_0^{0,q}(\mathcal{L}_\lambda)$ with the subspace of $C_0^\infty(G)\otimes\overset{q}{\wedge}\mathfrak{n}_-{}^*$. The forms φ and ψ are written on U in the form

$$\begin{aligned}\varphi &= \sum_{i_1<\cdots<i_q} u_{i_1\cdots i_q}\bar\theta^{i_1}\wedge\cdots\wedge\bar\theta^{i_q}\\ \psi &= \sum_{i_1<\cdots<i_q} v_{i_1\cdots i_q}\bar\theta^{i_1}\wedge\cdots\wedge\bar\theta^{i_q}\,.\end{aligned}$$

Then, we have

$$\begin{aligned}u_{i_1\cdots i_q}(p(g)) &= \varphi_{p(g)}(p_*(\overline{x_{\alpha_{i_1}}})_g, \cdots, p_*(\overline{x_{\alpha_{i_q}}})_g)\\ &= \nu(g, (\textstyle\sum f_A\omega^{-A})_g((\overline{x_{\alpha_{i_1}}})_g, \cdots, (\overline{x_{\alpha_{i_q}}})_g))\\ &= \prod_{j=1}^{q}\left(-\frac{1}{\varepsilon_{\alpha_{i_j}}(\alpha_{i_j}, \tau)}\right)^{1/2}\cdot\nu(g, f_{(\alpha_{i_1}, \cdots, \alpha_{i_q})}(g))\end{aligned} \tag{4.5}$$

where ν is the projection of $G\times C$ onto $\mathcal{L}_\lambda = G\times_H C$. Similarly, we have also

$$v_{i_1\cdots i_q}(p(g)) = \prod_{j=1}^{q}\left(-\frac{1}{\varepsilon_{\alpha_{i_j}}(\alpha_{i_j}, \tau)}\right)^{1/2}\nu(g, g_{(\alpha_{i_1}, \cdots, \alpha_{i_q})}(g))\,. \tag{4.6}$$

On the other hand, by the definition of the inner product $(\ ,\ )$ on $A_0^{0,q}(\mathcal{L}_\lambda)$, we have

$$\begin{aligned}(\varphi, \psi) &= \int_D \varphi\wedge\#*\psi\\ &= \int_D \sum_{i_1<\cdots<i_q}(u_{i_1\cdots i_q}, v_{i_1\cdots i_q})_{\mathcal{L}_\lambda}dv_D\end{aligned}$$

where $(\ ,\ )_{\mathcal{L}_\lambda}$ is the inner product on the fibers of $\mathcal{L}_\lambda$. By (4.5) and (4.6), we get

$$(u_{i_1\cdots i_q}, v_{i_1\cdots i_q})_{\mathcal{L}_\lambda}(p(g)) = \prod_{j=1}^{q}\left(-\frac{1}{\varepsilon_{\alpha_{i_j}}(\alpha_{i_j}, \tau)}\right) f_{(\alpha_{i_1}\cdots\alpha_{i_q})}(g)\, \overline{g_{(\alpha_{i_1}\cdots\alpha_{i_q})}(g)}\,.$$

Therefore, by (4.3), we obtain

$$\begin{aligned}(\varphi, \psi) &= \sum_{A\in\mathfrak{A}}\prod_{\alpha\in A}\left(-\frac{1}{\varepsilon_\alpha(\alpha, \tau)}\right)\cdot\int_D f_A\cdot\overline{g_A}dv_D \\ &= \sum_{A\in\mathfrak{A}}\prod_{\alpha\in A}\left(-\frac{1}{\varepsilon_\alpha(\alpha, \tau)}\right)\cdot\int_G f_A\cdot\overline{g_A}\frac{1}{v_H}dg\,.\end{aligned}$$

Thus, taking $\frac{1}{v_H}dg$ as a G-invariant volume element of G, we obtain the lemma. q.e.d.

For later use, we give an expression of the operator $\bar{\partial}$ due to [4]. First, we define some operators. Since we have

$$[\mathfrak{n}_-, \mathfrak{n}_-]\subset\mathfrak{n}_-\,,$$

the vector space $\mathfrak{n}_-$ is an $\mathfrak{n}_-$-module under the adjoint representation, and so $\mathfrak{n}_-{}^*$ is also an $\mathfrak{n}_-$-module. Then the action of $e_{-\alpha}(\alpha\in\Delta_+)$ on $\mathfrak{n}_-{}^*$ is given by

$$e_{-\alpha}\omega^{-\beta} = \begin{cases} N_{-\alpha,\beta}\,\omega^{\alpha-\beta} & \text{if } \beta-\alpha\in\Delta_+ \\ 0 & \text{otherwise.}\end{cases} \tag{4.7}$$

This action of $\mathfrak{n}_-$ on $\mathfrak{n}_-{}^*$ is extended to $\overset{q}{\wedge}\mathfrak{n}_-{}^*$. On the other hand, for $x\in\mathfrak{n}_+$ and $y\in\mathfrak{n}_-$, we put

$$x\cdot y = [x, y]_{\mathfrak{n}_-}$$

where $[x, y]_{\mathfrak{n}_-}$ is the $\mathfrak{n}_-$-component of $[x, y]$. Then, since $\mathfrak{h}\oplus\mathfrak{n}_+$ is an $\mathfrak{n}_+$-module, $\mathfrak{n}_-$ becomes an $\mathfrak{n}_+$-module, and so $\mathfrak{n}_-{}^*$ is an $\mathfrak{n}_+$-module. The action of $e_\alpha(\alpha\in\Delta_+)$ on $\mathfrak{n}_-{}^*$ is given by

$$e_\alpha\,\omega^{-\beta} = N_{\alpha,\beta}\,\omega^{-\alpha-\beta}\,. \tag{4.8}$$

This action of $\mathfrak{n}_+$ on $\mathfrak{n}_-{}^*$ is also extended to $\overset{q}{\wedge}\mathfrak{n}_-{}^*$. In the case $q=0$, we define $e_\alpha c=0$ and $e_{-\alpha}c=0$ for $c\in C=\overset{0}{\wedge}\mathfrak{n}_-{}^*$. Moreover, we define the operators

$$\begin{aligned} e(\omega^{-\alpha});\ &\overset{q}{\wedge}\mathfrak{n}_-{}^*\rightarrow\overset{q+1}{\wedge}\mathfrak{n}_-{}^* \\ i(\omega^{-\alpha});\ &\overset{q}{\wedge}\mathfrak{n}_-{}^*\rightarrow\overset{q-1}{\wedge}\mathfrak{n}_-{}^*\end{aligned}$$

by the following formulas:

$$e(\omega^{-\alpha})\omega^{-A} = \omega^{-\alpha}\wedge\omega^{-A} \tag{4.9}$$

$$(4.10)\qquad \begin{cases} i(\omega^{-\alpha})=0 & \text{on } C=\overset{0}{\wedge}\mathfrak{n}_-^*\,, \\ i(\omega^{-\alpha})\omega^{-A}=0 & \text{if } \alpha\notin A\,, \\ i(\omega^{-\alpha})(\omega^{-\alpha}\wedge\omega^{-A})=\omega^{-A} & \text{if } \alpha\notin A\,. \end{cases}$$

Now, returning to the holomorphic line bundle $\mathcal{L}_\lambda$, from the definition of the complex structure of $\mathcal{L}_\lambda$, we obtain the formula

$$(4.11)\qquad \begin{aligned}\bar\partial(f\omega^{-A}) &= \sum_{\alpha\in\Delta_+}(e_{-\alpha}f)\omega^{-\alpha}\wedge\omega^{-A}+\frac{1}{2}\sum_{\alpha\in\Delta_+}f\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-A} \\ &= \sum_{\alpha\in\Delta+}(1\otimes e(\omega^{-\alpha}))((e_{-\alpha}f)\omega^{-A}+\frac{1}{2}fe_{-\alpha}\omega^{-A})\end{aligned}$$

for each monomial $f\omega^{-A}\in A^{0,q}(\mathcal{L}_\lambda)$, where 1 denotes the identity operator in $C^\infty(G)$.

5. Connections

Let $D=G/H$ be the homogeneous complex manifold with the G-invariant Kähler metric g_τ induced by B_τ, and let $\mathcal{L}_\lambda\to D$ be the homogeneous hermitian line bundle defined by the character λ of H. In this section, we will discuss the metric connection in the bundle $\mathcal{L}_\lambda$ and the Riemannian connection of D.

We consider the bundle $\mathcal{L}_\lambda\to D$. By the reductive decomposition (2.6) of the Lie algebra $\mathfrak{g}$, we can define a canonical G-invariant connection in the principal bundle $G\to D=G/H$. This connection in $G\to D$ induces a connection in the associated line bundle $\mathcal{L}_\lambda\to D$. We denote by ∇_λ: $A^0(\mathcal{L}_\lambda)\to A^1(\mathcal{L}_\lambda)$ the covariant differentiation with respect to this connection. It is easy to see that for a C^∞-section f: $G\to C$ of $\mathcal{L}_\lambda$, $\nabla_\lambda f$ is given by

$$(5.1)\qquad \nabla_\lambda f=\sum_{\alpha\in\Delta}e_\alpha f\otimes\omega^\alpha\,.$$

REMARK. The connection ∇_λ in $\mathcal{L}_\lambda$ is the metric connection in the hermitian vector bundle $\mathcal{L}_\lambda$ i.e. the connection of type (1, 0) such that for C^∞ sections f, f' of $\mathcal{L}_\lambda$ we have

$$d(f,f')_{\mathcal{L}_\lambda}=(\nabla_\lambda f,f')_{\mathcal{L}_\lambda}+(f,\nabla_\lambda f')_{\mathcal{L}_\lambda}$$

where d is the exterior differential operator and $(\ ,\ )_{\mathcal{L}_\lambda}$ is the hermitian inner product on the fibers of $\mathcal{L}_\lambda$ [4].

Now, we consider the tangent bundle $T(D)$ of D with the Kähler metric g_τ on the fibers. The bundle $T(D)$ may be identified with the homogeneous vector bundle $G\times_H\mathfrak{n}_0$ over D associated with the adjoint representation of H in $\mathfrak{n}_0$. We denote by ν the canonical projection of $G\times\mathfrak{n}_0$ onto $T(D)=G\times_H\mathfrak{n}_0$. Let $L_{g_0}(g_0\in G)$ be the action of g_0 on D, then L_{g_0} induces the transformation $(L_{g_0})_*$ on the bundle $T(D)$ and we have

$$(L_{g_0})_*(\nu(g, x)) = \nu(g_0 g, x)$$

for $(g, x)\in G\times\mathfrak{n}_0$. Let $P(D)$ be the frame bundle of D. We fix a basis of $\mathfrak{n}_0$ and identify the set of all frames of $\mathfrak{n}_0$ with $GL(\mathfrak{n}_0)$. Then the bundle $P(D)$ can be identified with the homogeneous principal bundle $G\times_H GL(\mathfrak{n}_0)$ defined as follows: The group H acts on $G\times GL(\mathfrak{n}_0)$ by

$$(g, M)h = (gh, Ad(h^{-1})\cdot M)$$

for $(g, M)\in G\times GL(\mathfrak{n}_0)$ and $h\in H$. The space $G\times_H GL(\mathfrak{n}_0)$ is the quotient space $(G\times GL(\mathfrak{n}_0))/H$. We denote by μ the natural projection of $G\times GL(\mathfrak{n}_0)$ onto $P(D)=G\times_H GL(\mathfrak{n}_0)$. The transformation $(L_{g_0})_*(g_0\in G)$ on $P(D)$ induced by L_{g_0} on D is given by

$$(L_{g_0})_*(\mu(g, M)) = \mu(g_0 g, M)$$

for $(g, M)\in G\times GL(\mathfrak{n}_0)$. We fix a frame $u_0=\mu(e, 1)$ at the point $o\in D$. We will now apply the following lemma due to Wang ([6], II, p. 191, Theorem 2.1).

Lemma 3. *There is a one-to-one correspondence between the set of G-invariant connections in the bundle $P(D)$ and the set of linear mappings $\Lambda_{\mathfrak{n}_0}$; $\mathfrak{n}_0\to\mathfrak{gl}(\mathfrak{n}_0)$ such that*

$$\Lambda_{\mathfrak{n}_0}(Ad(h)x) = Ad(h)\Lambda_{\mathfrak{n}_0}(x)Ad(h)^{-1} \tag{5.2}$$

for $h\in H$ and $x\in\mathfrak{n}_0$, where Ad is the adjoint representation of H on $\mathfrak{n}_0$. A linear mapping $\Lambda_{\mathfrak{n}_0}$ satisfying (5.2) *corresponds to the invariant connection whose connection form ω is given by*

$$\omega_{u_0}(\tilde{x}) = \begin{cases} ad(x) & \text{if } x\in\mathfrak{h}_0 \\ \Lambda_{\mathfrak{n}_0}(x) & \text{if } x\in\mathfrak{n}_0 \end{cases} \tag{5.3}$$

where $\tilde{x}$ is the vector field on $P(D)$ defined by the 1-parameter group of transformations $(L_{\exp tx})_*$.

Let $\Lambda_{\mathfrak{n}_0}$ be a linear mapping satisfying (5.2). The connection in $P(D)$ corresponding to $\Lambda_{\mathfrak{n}_0}$ induces the connection in the bundle $T(D)$. We denote by $\nabla_{\Lambda_{\mathfrak{n}_0}}$; $A^0(T(D))\to A^1(T(D))$ the operator of the covariant differentiation with respect to this connection. The operator $\nabla_{\Lambda_{\mathfrak{n}_0}}$ is complex-linearly extended to the operator of $A^0(T(D)^C)$ into $A^1(T(D)^C)$. The complexified tangent bundle $T(D)^C$ may be identified with the homogeneous vector bundle $G\times_H\mathfrak{n}$ associated with the adjoint representation of H in $\mathfrak{n}$. Therefore the space $A^p(T(D)^C)$ is identified with a subspace of $C^\infty(G)\otimes\mathfrak{n}\otimes\overset{p}{\wedge}\mathfrak{n}^*$.

Lemma 4. *Let F; $G\to\mathfrak{n}$ be a section of $T(D)^C$. Then $\nabla_{\Lambda_{\mathfrak{n}_0}}F\in A^1(T(D)^C)$ is given by*

$$\nabla_{\Lambda_{\mathfrak{n}_0}} F = \sum_{\alpha \in \Delta} (e_\alpha F + \Lambda_{\mathfrak{n}_0}(e_\alpha)F) \otimes \omega^\alpha \tag{5.4}$$

where $\Lambda_{\mathfrak{n}_0}$ *is extended to the complex linear mapping of* $\mathfrak{n}$ *into* $\mathfrak{gl}(\mathfrak{n})$.

Proof. Let x be a vector field in $\mathfrak{n}_0$, and $\tilde{x}$ (resp. x_D) be a vector field on $P(D)$ (resp. D) defined by the 1-parameter group of transformations $(L_{\exp tx})_*$ (resp. $L_{\exp tx}$). The curve $(L_{\exp tx})_*(u_0)$ on $P(D)$ gives rise to the vector $\tilde{x}_{u_0}$ and the curve $(L_{\exp tx})(o)$ on D gives rise to the vector $(x_D)_0$. By Proposition 11.2 [6] II. p. 104, the horizontal lift v_t of the curve $L_{\exp tx}(o)$ such that $v_0 = u_0$ is given by

$$v_t = (L_{\exp tx})_*(u_0) \cdot a_t^{-1}$$

where a_t is the 1-parameter subgroup of $GL(\mathfrak{n}_0)$ generated by $\omega_{u_0}(\tilde{x})$. By (5.3) in Lemma 3, we have

$$\begin{aligned} v_t &= (L_{\exp tx})_*(u_0) \cdot (\exp t\Lambda_{\mathfrak{n}_0}(x))^{-1} \\ &= \mu(\exp tx, (\exp t\Lambda_{\mathfrak{n}_0}(x))^{-1}) . \end{aligned}$$

Thus, when we denote by τ_0^t the parallel displacement of the tangent space $T_{\exp tx \cdot o}(D)$ along the curve $L_{\exp tx}(o)$ from $\exp tx \cdot o$ to o, we have

$$\tau_0^t(\nu(\exp tx, F(\exp tx))) = \nu(o, \exp t\Lambda_{\mathfrak{n}_0}(x) \cdot F(\exp tx))$$

Since we have $p_*(x_e) = (x_D)_0$, by the definition of the covariant differentiation, we get

$$\begin{aligned} (\nabla_{\Lambda_{\mathfrak{n}_0}} F)(x_e) &= \frac{d}{dt} (\exp t\Lambda_{\mathfrak{n}_0}(x) \cdot F(\exp tx))_{t=0} \\ &= (xF)(e) + \Lambda_{\mathfrak{n}_0}(x) \cdot F(e) . \end{aligned}$$

Since the connection is G-invariant, we have

$$\nabla_{\Lambda_{\mathfrak{n}_0}}(F)(x) = xF + \Lambda_{\mathfrak{n}_0}(x) \cdot F .$$

If we extend complex linearly the operator $\nabla_{\Lambda_{\mathfrak{n}_0}}$ to the operator of $A^0(T(D)^C)$ into $A^1(T(D)^C)$, we obtain the formula (5.4).

Lemma 5. *Under the correspondence of Lemma* 3, *the Riemannian connection in* $T(D)$ *is given by the following mapping*

$$\Lambda_{\mathfrak{n}_0}(x) y = \frac{1}{2} [x, y]_{\mathfrak{n}_0} + U(x, y) , \tag{5.5}$$

where $U(x, y)$ *is the symmetric bilinear mapping of* $\mathfrak{n}_0 \times \mathfrak{n}_0$ *into* $\mathfrak{n}_0$ *defined by*

$$2B_\tau(U(x, y), z) = B_\tau(x, [z, y]_{\mathfrak{n}_0}) + B_\tau([z, x]_{\mathfrak{n}_0}, y) \tag{5.6}$$

for all $x, y, z \in \mathfrak{n}_0$. *Here,* $[x, y]_{\mathfrak{n}_0}$ *is the* $\mathfrak{n}_0$*-component of* $[x, y]$ *with respect to the decomposition* (2.6) *of* $\mathfrak{g}_0$.

For the proof, see [6] II, p. 201, Theorem 3.3.

We denote by Λ_τ the linear mapping of $\mathfrak{n}_0$ into $\mathfrak{gl}(\mathfrak{n}_0)$ which gives the Riemannian connection of $T(D)$ in Lemma 5, and by the same letter Λ_τ its extention to the complex linear mapping of $\mathfrak{n}$ into $\mathfrak{gl}(\mathfrak{n})$. Then, by (3.5) we can calculate the mapping Λ_τ and we get

$$(5.7)\qquad \begin{cases} \Lambda_\tau(e_\alpha)e_\beta = \dfrac{(\beta,\tau)}{(\alpha+\beta,\tau)}[e_\alpha, e_\beta] \\ \Lambda_\tau(e_\alpha)e_{-\beta} = [e_\alpha, e_{-\beta}]_{\mathfrak{n}_-} \end{cases}$$
$$\begin{cases} \Lambda_\tau(e_{-\alpha})e_\beta = [e_{-\alpha}, e_\beta]_{\mathfrak{n}_+} \\ \Lambda_\tau(e_{-\alpha})e_{-\beta} = \dfrac{(\beta,\tau)}{(\alpha+\beta,\tau)}[e_{-\alpha}, e_{-\beta}] \end{cases}$$

where α and β are positive roots and $[x, y]_{\mathfrak{n}_+}$ (resp. $[x, y]_{\mathfrak{n}_-}$) is the $\mathfrak{n}_+$(resp. $\mathfrak{n}_-$) component of $[x, y]$. By (5.5) and (5.6), we verify easily the following property of Λ_τ:

$$(5.8)\qquad B_\tau(\Lambda_\tau(x)y, z)+B_\tau(y, \Lambda_\tau(x)z) = 0$$

where $x, y, z\in\mathfrak{n}$. We denote by $\nabla_{T(D)^C}$; $A^0(T(D)^C)\to A^1(T(D)^C)$ the covariant differentiation with respect to this Riemannian connection. By Lemma 4, the operator $\nabla_{T(D)^C}$ is given by

$$\nabla_{T(D)^C}F = \sum_{\alpha\in\Delta}(e_\alpha F+\Lambda_\tau(e_\alpha)F)\otimes\omega^\alpha$$

for a section F; $G\to\mathfrak{n}$ of $T(D)^C$.

Let $\Theta(D)$ be the holomorphic tangent bundle of D. The bundle $T(D)^C$ decomposes into the Whitney sum

$$\begin{aligned} T(D)^C &= \Theta(D)\oplus\overline{\Theta}(D) \\ &= (G\times_H\mathfrak{n}_+)\oplus(G\times_H\mathfrak{n}_-) \end{aligned}$$

where $\overline{\Theta}(D)$ is the conjugate bundle of $\Theta(D)$. Since

$$\Lambda_\tau(x)(\mathfrak{n}_+)\subset\mathfrak{n}_+,\quad \Lambda_\tau(x)(\mathfrak{n}_-)\subset\mathfrak{n}_-$$

for all $x\in\mathfrak{n}$, we have

$$\begin{aligned} &\nabla_{T(D)^C}(A^0(\Theta(D)))\subset A^1(\Theta(D)) \\ &\nabla_{T(D)^C}(A^0(\overline{\Theta}(D)))\subset A^1(\overline{\Theta}(D))\,. \end{aligned}$$

Therefore, the restriction of $\nabla_{T(D)^C}$ on $A^0(\Theta(D))$ (resp. $A^0(\overline{\Theta}(D))$) defines a connection in $\Theta(D)$ (resp. $\overline{\Theta}(D)$) which we denote by ∇_Θ (resp. $\nabla_{\overline{\Theta}}$). The connection $\nabla_{\bar{\Theta}}$ induces a connection $\nabla_{\bar{\Theta}^*}$ in the dual bundle $\bar{\Theta}^*(D)$ of $\bar{\Theta}(D)$ [1]. It is easy to see that, for a section F^*; $G \to \mathfrak{n}_-^*$ of $\bar{\Theta}^*(D)$, $\nabla_{\bar{\Theta}^*}F^*$ is given by

$$(5.9)\qquad \nabla_{\bar{\Theta}^*}F^* = \sum_{\alpha\in\Delta}(e_\alpha F^* - {}^t\Lambda_\tau(e_\alpha)F^*)\otimes\omega^\alpha\,,$$

where the linear mapping ${}^t\Lambda_\tau(e_\alpha)$; $\mathfrak{n}_-^* \to \mathfrak{n}_-^*$ is the transposed mapping of $\Lambda_\tau(e_\alpha)|_{\mathfrak{n}_-}$; $\mathfrak{n}_- \to \mathfrak{n}_-$ and given by

$$(5.10)\qquad \begin{cases} {}^t\Lambda_\tau(e_\alpha)\omega^{-\beta} = -e_\alpha\omega^{-\beta} \\ {}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta} = \dfrac{(\alpha-\beta,\,\tau)}{(\beta,\,\tau)}e_{-\alpha}\omega^{-\beta} \end{cases}$$

for $\alpha, \beta \in \Delta_+$.

In the above, we have constructed the connections ∇_λ in $\mathcal{L}_\lambda$ and $\nabla_{\bar{\Theta}^*}$ in $\bar{\Theta}^*(D)$. These connections give rise to a connection in the bundle $\mathcal{L}_\lambda\otimes\overset{q}{\wedge}\bar{\Theta}^*(D)$, where $\overset{q}{\wedge}\bar{\Theta}^*(D)$ is the q-th exterior product of the bundle $\bar{\Theta}^*(D)$ [1]. We shall denote this connection by

$$\nabla;\ A^0(\mathcal{L}_\lambda\otimes\overset{q}{\wedge}\bar{\Theta}^*(D)) \to A^1(\mathcal{L}_\lambda\otimes\overset{q}{\wedge}\bar{\Theta}^*(D))\,.$$

Then, for an element $f\omega^{-A}$ of $A^{0,q}(\mathcal{L}_\lambda) = A^0(\mathcal{L}_\lambda\otimes\overset{q}{\wedge}\bar{\Theta}^*(D))$, we get

$$(5.11)\qquad \nabla(f\omega^{-A}) = \sum_{\alpha\in\Delta}(e_\alpha f\omega^{-A} - f({}^t\Lambda_\tau(e_\alpha)\omega^{-A}))\otimes\omega^\alpha\,,$$

where the mapping ${}^t\Lambda_\tau(e_\alpha)$; $\overset{q}{\wedge}\mathfrak{n}_-^* \to \overset{q}{\wedge}\mathfrak{n}_-^*$ is the natural extension of the endomorphism ${}^t\Lambda_\tau(e_\alpha)$; $\mathfrak{n}_-^* \to \mathfrak{n}_-^*$. In the following sections, we shall use this connection ∇.

6. Computation of the Laplace-Beltrami operator

We retain the notation introduced in the preceding sections. In this section, we will give an expression of the Laplace-Beltrami operator $\square = \bar{\partial}\delta + \delta\bar{\partial}$. To begin with, we give expressions of the operators $\bar{\partial}$ and δ in terms of the connection ∇ in §5. For each $e_\alpha \in \mathfrak{g}$, we define a linear mapping

$$\nabla_{e_\alpha};\ C^\infty(G)\otimes\overset{q}{\wedge}\mathfrak{n}_-^* \to C^\infty(G)\otimes\overset{q}{\wedge}\mathfrak{n}_-^*$$

by the following formula:

$$(6.1)\qquad \nabla_{e_\alpha}(f\omega^{-A}) = (e_\alpha f)\omega^{-A} - f({}^t\Lambda_\tau(e_\alpha)\omega^{-A})\,.$$

Proposition 1. *Let $f\omega^{-A}$ be a form in $A^{0,q}(\mathcal{L}_\lambda)$. Then we have*

$$(6.2)\qquad \bar{\partial}(f\omega^{-A}) = \sum_{\alpha\in\Delta_+}(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}(f\omega^{-A})$$

$$(6.3)\qquad \delta(f\omega^{-A}) = \sum_{\alpha\in\Delta_+}\frac{1}{(\alpha,\,\tau)}(1\otimes i(\omega^{-\alpha}))\nabla_{e_\alpha}(f\omega^{-A})\,.$$

Proof. By (5.10) we have

$$\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta}) = \sum_{\alpha<\beta} \frac{(\beta-\alpha,\tau)}{(\beta,\tau)}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-\beta}$$

for $\alpha, \beta\in\Delta_+$. If we replace $\beta-\alpha$ by α in the right side, by (2.5) and (4.7), we get also

$$\begin{aligned}\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta}) &= \sum_{\alpha<\beta}\frac{(\alpha,\tau)}{(\beta,\tau)}\omega^{\alpha-\beta}\wedge e_{\alpha-\beta}\omega^{-\beta}\\ &= \sum_{\alpha<\beta}\frac{(\alpha,\tau)}{(\beta,\tau)}N_{\alpha-\beta,\beta}\,\omega^{\alpha-\beta}\wedge\omega^{-\alpha}\\ &= \sum_{\alpha<\beta}\frac{(\alpha,\tau)}{(\beta,\tau)}\omega^{-\alpha}\wedge N_{-\alpha,\beta}\,\omega^{\alpha-\beta}\\ &= \sum_{\alpha<\beta}\frac{(\alpha,\tau)}{(\beta,\tau)}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-\beta}.\end{aligned}$$

Hence, we get

$$2\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta}) = \sum_{\alpha\in\Delta_+}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-\beta}.$$

This formula can be extended to the formula for $\omega^{-A}\in \overset{q}{\wedge}\mathfrak{n}_-^*$ and we have

$$\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-A}) = \frac{1}{2}\sum_{\alpha\in\Delta_+}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-A}.$$

Then, by (4.11) and (6.1), we obtain

$$\begin{aligned}\bar{\partial}(f\omega^{-A}) &= \sum_{\alpha\in\Delta_+}(e_{-\alpha}f)\omega^{-\alpha}\wedge\omega^{-A} + \frac{1}{2}f\Big(\sum_{\alpha\in\Delta_+}\omega^{-\alpha}\wedge e_{-\alpha}\omega^{-A}\Big)\\ &= \sum_{\alpha\in\Delta_+}(e_{-\alpha}f)e(\omega^{-\alpha})\omega^{-A} + f\sum_{\alpha\in\Delta_+} e(\omega^{-\alpha})(-{}^t\Lambda_\tau(e_{-\alpha})\omega^{-A})\\ &= \sum_{\alpha\in\Delta_+}(1\otimes e(\omega^{-\alpha}))((e_{-\alpha}f)\omega^{-A} - f({}^t\Lambda_\tau(e_{-\alpha})\omega^{-A}))\\ &= \sum_{\alpha\in\Delta_+}(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}(f\omega^{-A}).\end{aligned}$$

This proves (6.2).

In order to obtain the expression (6.3) of δ, we construct the adjoint operators of the operator $e_{-\alpha}$ on $C_0^\infty(G)$ and the operator $e(\omega^{-\alpha})$ and ${}^t\Lambda_\tau(e_{-\alpha})$ on $\overset{q}{\wedge}\mathfrak{n}_-^*$. For two functions $f, g\in C_0^\infty(G)$, we have

$$\int_G e_{-\alpha}(f\cdot\bar{g})dg = 0$$

([7], Lemma 5.1). Thus, we see that

$$(e_{-\alpha}f, g)_G = (f, -\varepsilon_\alpha e_\alpha g)_G \tag{6.4}$$

where $(\, , \,)_G$ is the inner product on $C_0^\infty(G)$ defined in §4. On the other hand, by easy computations, we get

$$(e(\omega^{-\alpha})\omega^{-A}, \omega^{-B})_- = \left(\omega^{-A}, -\frac{1}{\varepsilon_\alpha(\alpha, \tau)} i(\omega^{-\alpha})\omega^{-B}\right)_- \tag{6.5}$$

where $(\, , \,)_-$ is the inner product on $\overset{q}{\wedge}\mathfrak{n}_-^*$ introduced in §4. Also, by the definition (5.7) of $\Lambda_\tau(e_\alpha)$, we have

$$\overline{\Lambda_\tau(e_\alpha)y} = \varepsilon_\alpha \Lambda_\tau(e_{-\alpha})\bar{y}$$

for each $y \in \mathfrak{n}$ and $\alpha \in \Delta_+$. Since the operator $\Lambda_\tau(e_\alpha)$ satisfies the formula (5.8), we obtain

$$\begin{aligned} B_\tau^-(\Lambda_\tau(e_{-\alpha})x, y) &= B_\tau(\Lambda_\tau(e_{-\alpha})x, \bar{y}) \\ &= B_\tau(x, -\Lambda_\tau(e_{-\alpha})\bar{y}) \\ &= B_\tau(x, -\overline{\varepsilon_\alpha \Lambda_\tau(e_\alpha)y}) \\ &= B_\tau^-(x, -\varepsilon_\alpha \Lambda_\tau(e_\alpha)y) . \end{aligned}$$

for $x, y \in \mathfrak{n}_-$. It follows that we have

$$({}^t\Lambda_\tau(e_{-\alpha})\omega^{-A}, \omega^{-B})_- = (\omega^{-A}, -\varepsilon_\alpha {}^t\Lambda_\tau(e_\alpha)\omega^{-B})_- . \tag{6.6}$$

By the formulas (6.1), (6.2), (6.4)—(6.6) and Lemma 2, the formal adjoint operator δ of $\bar{\partial}$ is given by

$$\begin{aligned} \delta(f\omega^{-A}) &= \sum_{\alpha \in \Delta_+} \frac{1}{(\alpha, \tau)} (1 \otimes i(\omega^{-\alpha}))((e_\alpha f)\omega^{-A} - f({}^t\Lambda_\tau(e_\alpha)\omega^{-A})) \\ &= \sum_{\alpha \in \Delta_+} \frac{1}{(\alpha, \tau)} (1 \otimes i(\omega^{-\alpha}))\nabla_{e_\alpha}(f\omega^{-A}) . \end{aligned}$$

q.e.d.

Now, if $\alpha, \beta \in \Delta_+$, we have the following relations among operators on $\overset{q}{\wedge}\mathfrak{n}_\pm^*$:

$$e(\omega^{-\alpha})e(\omega^{-\beta}) = -e(\omega^{-\beta})e(\omega^{-\alpha}) \tag{6.7}$$

$$e(\omega^{-\alpha})i(\omega^{-\beta}) + i(\omega^{-\beta})e(\omega^{-\alpha}) = \delta_{\alpha,\beta} \tag{6.8}$$

$$[e_\alpha, e(\omega^{-\beta})] = e(e_\alpha \omega^{-\beta}) \tag{6.9}$$

$$[e_\alpha, i(\omega^{-\beta})] = -i(e_{-\alpha}\omega^{-\beta}) \tag{6.10}$$

$$e_\alpha \omega^{-A} = \sum_{\beta \in \Delta_+} e(e_\alpha \omega^{-\beta}) i(\omega^{-\beta})\omega^{-A} . \tag{6.11}$$

All these are easily proved [4], and the equalities (6.9)–(6.11) hold also when we replace α by $-\alpha$.

Lemma 6. *For roots α, $\beta\in\Delta_+$, we have the following relations:*

$$[{}^t\Lambda_\tau(e_\alpha),\ e(\omega^{-\beta})] = e({}^t\Lambda_\tau(e_\alpha)\omega^{-\beta}) \tag{6.12}$$

$$[{}^t\Lambda_\tau(e_\alpha),\ i(\omega^{-\beta})] = i(e_{-\alpha}\,\omega^{-\beta}) \tag{6.13}$$

$$[{}^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})] = \frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}\, i(e_\alpha\,\omega^{-\beta}) \tag{6.14}$$

$${}^t\Lambda_\tau(e_\alpha)\omega^{-A} = \sum_{\beta\in\Delta_+} e({}^t\Lambda_\tau(e_\alpha)\omega^{-\beta})\, i(\omega^{-\beta})\omega^{-A}\,. \tag{6.15}$$

The relations (6.12) *and* (6.15) *hold also when we replace α by $-\alpha$.*

Proof. The equalities (6.12) and (6.15) are easily proved and (6.13) follows from (5.10) and (6.10). We will prove the relation (6.14) on $\overset{q}{\wedge}\mathfrak{n}_-^*$ by the induction on q. For a 1-form $\omega^{-\gamma}$, we have

$$\begin{aligned} {}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})\omega^{-\gamma} &= 0 \\ e_{-\alpha}\, i(\omega^{-\beta})\omega^{-\gamma} &= 0\,. \end{aligned}$$

Hence, by (5.10) and (6.10), we get

$$\begin{aligned} [{}^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]\,\omega^{-\gamma} &= -i(\omega^{-\beta})\,{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma} \\ &= -\frac{(\alpha-\gamma,\ \tau)}{(\gamma,\ \tau)}\, i(\omega^{-\beta})e_{-\alpha}\omega^{-\gamma} \\ &= -\frac{(\alpha-\gamma,\ \tau)}{(\gamma,\ \tau)}\, i(e_\alpha\,\omega^{-\beta})\omega^{-\gamma}\,. \end{aligned}$$

On the other hand, by (4.8) we have

$$\begin{aligned} i(e_\alpha\,\omega^{-\beta})\omega^{-\gamma} &= N_{\alpha,\beta}\, i(\omega^{-\alpha-\beta})\omega^{-\gamma} \\ &= \begin{cases} 0 & \text{if} \quad \alpha+\beta\neq\gamma\,, \\ N_{\alpha,\beta} & \text{if} \quad \alpha+\beta=\gamma\,. \end{cases} \end{aligned}$$

Therefore, we obtain the equality

$$[{}^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]\,\omega^{-\gamma} = \frac{(\beta,\ \tau)}{(\alpha+\beta,\ \tau)}\, i(e_\alpha\omega^{-\beta})\omega^{-\gamma}\,.$$

Now, assume that the equality (6.14) holds on the space $\overset{q-1}{\wedge}\mathfrak{n}_-^*$. By (6.8) and (6.12) we have

$$\begin{aligned} &[{}^t\Lambda_\tau(e_{-\alpha}),\ i(\omega^{-\beta})]e(\omega^{-\gamma}) \\ &= {}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})e(\omega^{-\gamma}) - i(\omega^{-\beta})\,{}^t\Lambda_\tau(e_{-\alpha})e(\omega^{-\gamma}) \\ &= {}^t\Lambda_\tau(e_{-\alpha})\delta_{\beta,\gamma} - {}^t\Lambda_\tau(e_{-\alpha})e(\omega^{-\gamma})i(\omega^{-\beta}) - i(\omega^{-\beta})e(\omega^{-\gamma})\,{}^t\Lambda_\tau(e_{-\alpha}) - i(\omega^{-\beta})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}) \end{aligned}$$

$$= -e(\omega^{-\gamma})^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta}) - e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$+ e(\omega^{-\gamma})i(\omega^{-\beta})^t\Lambda_\tau(e_{-\alpha}) - i(\omega^{-\beta})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})$$
$$= -e(\omega^{-\gamma})[{}^t\Lambda_\tau(e_{-\alpha}), i(\omega^{-\beta})] - \frac{(\alpha-\gamma, \tau)}{(\gamma, \tau)} N_{-\alpha,\gamma}\{e(\omega^{\alpha-\gamma})i(\omega^{-\beta}) + i(\omega^{-\beta})e(\omega^{\alpha-\gamma})\}$$
$$= -e(\omega^{-\gamma})[{}^t\Lambda_\tau(e_{-\alpha}), i(\omega^{-\beta})] - \frac{(\alpha-\gamma, \tau)}{(\gamma, \tau)} N_{-\alpha,\gamma}\delta_{\gamma-\alpha,\beta}.$$

Since

$$N_{-\alpha,\alpha+\beta} = N_{\alpha,\beta}, \tag{6.16}$$

we get

$$[{}^t\Lambda_\tau(e_{-\alpha}), i(\omega^{-\beta})]e(\omega^{-\gamma}) = -e(\omega^{-\gamma})[{}^t\Lambda_\tau(e_{-\alpha}), i(\omega^{-\beta})] + \frac{(\beta, \tau)}{(\alpha+\beta, \tau)} N_{\alpha,\beta}\delta_{\alpha+\beta,\gamma}.$$

By the assumption and (6.8), for a q-form $\omega^{-\gamma}\wedge\omega^{-A}$ with $\omega^{-A}\in\overset{q-1}{\wedge}\mathfrak{n}_-^*$, we obtain

$$[{}^t\Lambda_\tau(e_{-\alpha}), i(\omega^{-\beta})](\omega^{-\gamma}\wedge\omega^{-A})$$
$$= -e(\omega^{-\gamma})[{}^t\Lambda_\tau(e_{-\alpha}), i(\omega^{-\beta})]\omega^{-A} + \frac{(\beta, \tau)}{(\alpha+\beta, \tau)} N_{\alpha,\beta}\,\delta_{\alpha+\beta,\gamma}\omega^{-A}$$
$$= \frac{(\beta, \tau)}{(\alpha+\beta, \tau)}\{-e(\omega^{-\gamma})i(e_\alpha\omega^{-\beta})\omega^{-A} + N_{\alpha,\beta}\,\delta_{\alpha+\beta,\gamma}\,\omega^{-A}\}$$
$$= \frac{(\beta, \tau)}{(\alpha+\beta, \tau)} i(e_\alpha\,\omega^{-\beta})e(\omega^{-\gamma})\omega^{-A}$$
$$= \frac{(\beta, \tau)}{(\alpha+\beta, \tau)} i(e_\alpha\,\omega^{-\beta})(\omega^{-\gamma}\wedge\omega^{-A}).$$

This proves (6.14) and the lemma is proved. q.e.d.

We recall also following equalities proved in [4]:

$$\sum_{\beta<\alpha} N_{\beta,\alpha-\beta}N_{-\beta,\alpha} = (2\rho-\alpha, \alpha) \tag{6.17}$$

where $\rho=\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$ ([4], p. 266, Lemma 3.1), and

$$(e_{-\alpha}e_\gamma - e_\gamma e_{-\alpha})\omega^{-\beta} - [e_{-\alpha}, e_\gamma]\omega^{-\beta} = \begin{cases} 0 & \text{if } \alpha<\beta \\ (\gamma, \beta)\omega^{-\gamma} & \text{if } \alpha=\beta \\ N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\,\omega^{\alpha-\beta-\gamma} & \text{if } \alpha>\beta \end{cases} \tag{6.18}$$

for all α, β, $\gamma\in\Delta_+$ ([4], p. 281).

Proposition 2. *Let $f\omega^{-A}$ be a form in $A^{0,q}(\mathcal{L}_\lambda)$. Then*

$$\square(f\omega^{-A})=\sum_{\alpha\in\Delta_+}\frac{1}{(\alpha,\tau)}\nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A})+\Big(\sum_{\alpha\in A}\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}\Big)f\omega^{-A}.$$

Proof. By (4.7) we have $e_{-\alpha}\omega^{-\alpha}=0$ and thus

$$^t\Lambda_\tau(e_{-\alpha})\omega^{-\alpha}=0.$$

Hence, by (6.12) and (6.13) in Lemma 6, $e(\omega^{-\alpha})$ commutes with $^t\Lambda_\tau(e_{-\alpha})$ and $i(\omega^{-\beta})$ with $^t\Lambda_\tau(e_\beta)$. In particular, $1\otimes i(\omega^{-\beta})$ commutes with ∇_{e_β}. Using Proposition 1 and (6.8) we have

$$\begin{aligned}\square(f\omega^{-A}) &= (\bar{\partial}\delta+\delta\bar{\partial})(f\omega^{-A})\\ &=\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}\nabla_{e_\beta}(1\otimes i(\omega^{-\beta}))(f\omega^{-A})\\ &\quad+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}\nabla_{e_\beta}(1\otimes i(\omega^{-\beta}))(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}(f\omega^{-A})\\ &=\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}(1\otimes e(\omega^{-\alpha}))\nabla_{e_{-\alpha}}\nabla_{e_\beta}(1\otimes i(\omega^{-\beta}))(f\omega^{-A})\\ &\quad+\sum_{\alpha\in\Delta_+}\frac{1}{(\alpha,\tau)}\nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A})\\ &\quad-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}\nabla_{e_\beta}(1\otimes e(\omega^{-\alpha}))(1\otimes i(\omega^{-\beta}))\nabla_{e_{-\alpha}}(f\omega^{-A}).\end{aligned}$$

By the definition (6.1) of $\nabla_{e_{-\alpha}}$ and ∇_{e_β}, we get

$$\begin{aligned}\square(f\omega^{-A}) &=\sum_{\alpha\in\Delta_+}\frac{1}{(\alpha,\tau)}\nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A})\\ &\quad+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}(e_{-\alpha}e_\beta f-e_\beta e_{-\alpha}f)e(\omega^{-\alpha})i(\omega^{-\beta})\omega^{-A}\\ &\quad-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e_\beta fe(\omega^{-\alpha})({}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})-i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))\omega^{-A}\\ &\quad-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e_{-\alpha}f(e(\omega^{-\alpha}){}^t\Lambda_\tau(e_\beta)-{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha}))i(\omega^{-\beta})\omega^{-A}\\ &\quad+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}f(e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta})\\ &\quad-{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))\omega^{-A}.\end{aligned}$$

By (6.14), the third term can be written as follows:

$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\alpha+\beta,\tau)}e_\beta fe(\omega^{-\alpha})i(e_\alpha\omega^{-\beta})\omega^{-A},$$

and by (5.10), (6.12) the fourth term as follows:

$$-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e_{-\alpha}fe(e_\beta\omega^{-\alpha})i(\omega^{-\beta})\omega^{-A}.$$

On the other hand, from the formula

$$[e_\alpha, e_{-\alpha}] = h_\alpha, \quad [e_{-\alpha}, e_\beta] = N_{-\alpha,\beta}e_{\beta-\alpha},$$

we have

$$\begin{aligned}
&\sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)}(e_{-\alpha}e_\beta f - e_\beta e_{-\alpha} f)e(\omega^{-\alpha})i(\omega^{-\beta})\omega^{-A} \\
&= -\sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} h_\alpha f e(\omega^{-\alpha})i(\omega^{-\alpha})\omega^{-A} \\
&\quad + \sum_{0<\alpha<\beta} \frac{1}{(\beta, \tau)} e_{\beta-\alpha} f e(\omega^{-\alpha}) i(N_{-\alpha,\beta}\omega^{-\beta})\omega^{-A} \\
&\quad + \sum_{\alpha>\beta>0} \frac{1}{(\beta, \tau)} e_{\beta-\alpha} f e(N_{-\alpha,\beta}\omega^{-\alpha}) i(\omega^{-\beta})\omega^{-A}.
\end{aligned}$$

Here, we replace $\beta-\alpha$ by β in the second term and $\beta-\alpha$ by $-\alpha$ in the third term. Then, by (2.5) and (4.8), we have

$$\begin{aligned}
&\sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)}(e_{-\alpha}e_\beta f - e_\beta e_{-\alpha} f)e(\omega^{-\alpha})i(\omega^{-\beta})\omega^{-A}. \\
&= -\sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} h_\alpha f e(\omega^{-\alpha})i(\omega^{-\alpha})\omega^{-A} \\
&\quad + \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\alpha+\beta, \tau)} e_\beta f e(\omega^{-\alpha}) i(e_\alpha\omega^{-\beta})\omega^{-A} \\
&\quad + \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} e_{-\alpha} f e(e_\beta\omega^{-\alpha}) i(\omega^{-\beta})\omega^{-A}.
\end{aligned}$$

By (4.1) in §4, the function f; $G\to C$ satisfies the following property:

$$h_\alpha f = -(\lambda+|A|, \alpha)f \qquad \text{for all} \quad \alpha\in\Delta_+.$$

Therefore, if we put

$$\begin{aligned}
\text{(6.19)} \qquad \mathfrak{R}_\tau\omega^{-A} = \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta, \tau)} &(e(\omega^{-\alpha})^t\Lambda_\tau(e_{-\alpha})^t\Lambda_\tau(e_\beta)i(\omega^{-\beta}) \\
&- {}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta})^t\Lambda_\tau(e_{-\alpha}))\omega^{-A},
\end{aligned}$$

we have

$$\begin{aligned}
\text{(6.20)} \qquad \Box(f\omega^{-A}) = &\sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} \nabla_{e_\alpha}\nabla_{e_{-\alpha}}(f\omega^{-A}) \\
&+ \Big(\sum_{\alpha\in A} \frac{(\alpha, \lambda+|A|)}{(\alpha, \tau)}\Big) f\omega^{-A} + f\cdot\mathfrak{R}_\tau\omega^{-A}.
\end{aligned}$$

It remains to compute the operator $\mathfrak{R}_\tau$: $\overset{q}{\wedge}\mathfrak{n}_-^* \to \overset{q}{\wedge}\mathfrak{n}_-^*$. We begin with the operator $\Re_\tau \cdot e(\omega^{-\gamma})$ for $\gamma \in \Delta_+$. Using (6.7)–(6.15), we will exchange the operator $e(\omega^{-\gamma})$ with the operators $e(\omega^{-\alpha})$, $i(\omega^{-\alpha})$, ${}^t\Lambda_\tau(e_\alpha)$ and ${}^t\Lambda_\tau(e_{-\alpha})$ $(\alpha \in \Delta_+)$ one by one. By (6.8) and (6.12) we have

$$\begin{aligned}
\Re_\tau \cdot e(\omega^{-\gamma}) &= \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta,\tau)}(e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta}) \\
&\quad - {}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))\cdot e(\omega^{-\gamma}) \\
&= -\sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta,\tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)e(\omega^{-\gamma})i(\omega^{-\beta}) \\
&\quad + \sum_{\alpha\in\Delta_+} \frac{1}{(\gamma,\tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\gamma) \\
&\quad - \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta,\tau)} {}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta})\, e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}) \\
&\quad - \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta,\tau)} {}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta})e(\omega^{-\gamma}){}^t\Lambda_\tau(e_{-\alpha}) .
\end{aligned}$$

By (5.10) and (6.8), we get

$$\begin{aligned}
i(\omega^{-\beta})\, e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}) &= \frac{(\alpha-\gamma,\tau)}{(\gamma,\tau)} N_{-\alpha,\gamma} i(\omega^{-\beta})e(\omega^{\alpha-\gamma}) \\
&= \frac{(\alpha-\gamma,\tau)}{(\gamma,\tau)} N_{-\alpha,\gamma}(-e(\omega^{\alpha-\gamma})\, i(\omega^{-\beta}) + \delta_{\gamma-\alpha,\beta}) \\
&= -e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta}) - \frac{(\beta,\tau)}{(\gamma,\tau)} N_{-\alpha,\gamma}\delta_{\gamma,\alpha+\beta}
\end{aligned}$$

provided that $\alpha < \gamma$. By (4.8) and (5.10), we have also

$$ {}^t\Lambda_\tau(e_\gamma)\omega^{-\beta} = -e_\gamma\omega^{-\beta} = e_\beta\omega^{-\gamma} = -{}^t\Lambda_\tau(e_\beta)\omega^{-\gamma} ,$$

and thus by (6.15) we have

$$\begin{aligned}
{}^t\Lambda_\tau(e_\gamma) &= \sum_{\beta\in\Delta_+} e({}^t\Lambda_\tau(e_\gamma)\omega^{-\beta})i(\omega^{-\beta}) \\
&= -\sum_{\beta\in\Delta_+} e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta}) .
\end{aligned}$$

Therefore using again (4.8) and (6.12) we get

$$\begin{aligned}
\Re_\tau e(\omega^{-\gamma}) &= -\sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta,\tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha})e(\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta}) \\
&\quad - \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\beta,\tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta}) \\
&\quad - \sum_{\alpha,\beta\in\Delta_+} \frac{1}{(\gamma,\tau)} e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}\,{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})\\
&+\sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_{\gamma-\alpha})e(\omega^{-\alpha})\\
&-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}\,{}^t\Lambda_\tau(e_\beta)e(\omega^{-\gamma})e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})\\
&-\sum_{\alpha\in\Delta_+}\frac{1}{(\gamma,\tau)}\,{}^t\Lambda_\tau(e_\gamma)e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha})\\
=&\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\gamma})e(\omega^{-\alpha}){}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta})\\
&-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta)i(\omega^{-\beta})\\
&-\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})\\
&-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma}){}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})\\
&-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma}){}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})\\
&+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha}){}^t\Lambda_\tau(e_\beta)e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})\\
&+\sum_{\alpha,\beta\in\Delta_+}e({}^t\Lambda_\tau(e_\beta)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})\\
&+\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\beta})i(\omega^{-\beta})\\
&+\sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}e({}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\alpha})\\
&-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})\\
&+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})\\
&+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})\\
&-\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e({}^t\Lambda_\tau(e_\gamma)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta})i(\omega^{-\beta})\,.
\end{aligned}
$$

Changing suitably the order of the terms in the above equality, by (6.19), we get

$$
\mathfrak{R}_\tau\cdot e(\omega^{-\gamma}) = e(\omega^{-\gamma})\cdot\mathfrak{R}_\tau+\sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}e({}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\alpha})
$$

$$-\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})$$
$$-\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})({}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})-i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))$$
$$+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e({}^t\Lambda_\tau(e_\beta)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$+\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\alpha})({}^t\Lambda_\tau(e_\beta)e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})-e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta))i(\omega^{-\beta})$$
$$+\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{1}{(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\beta})\,i(\omega^{-\beta})$$
$$-\sum_{\alpha\in\Delta_+}\frac{1}{(\gamma,\tau)}e({}^t\Lambda_\tau(e_\gamma)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta})i(\omega^{-\beta})\,.$$

Now, by (5.10) and (6.17)

$$\text{(6.21)}\qquad \sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}e({}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\alpha})=\frac{1}{(\gamma,\tau)}\Big(\sum_{0<\alpha<\gamma}N_{-\alpha,\gamma}N_{\gamma-\alpha,\alpha}\Big)e(\omega^{-\gamma})$$
$$=\frac{(2\rho-\gamma,\gamma)}{(\gamma,\tau)}e(\omega^{-\gamma})\,.$$

By (5.10), we have

$$ {}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)\omega^{-\gamma}=-N_{\beta,\gamma}{}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta-\gamma}$$
$$=\begin{cases}\dfrac{(\alpha-\beta-\gamma,\tau)}{(\beta+\gamma,\tau)}N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}\omega^{\alpha-\beta-\gamma} & \text{if}\quad \alpha<\beta+\gamma\\ 0 & \text{otherwise.}\end{cases}$$

Hence, as for the third term in the above equality, we obtain

$$\text{(6.22)}\qquad -\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha}){}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(\omega^{-\beta})$$
$$=-\sum_{\substack{0<\alpha<\beta+\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$
$$=-\sum_{\substack{0<\alpha<\beta\\ \alpha\neq\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$
$$-\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \alpha\neq\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$

$$+\sum_{\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\beta})e(N_{\gamma,\beta}N_{-\beta,\beta+\gamma}\omega^{-\gamma})i(\omega^{-\beta})$$
$$+\sum_{\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\gamma})e(N_{\gamma,\beta}N_{-\gamma,\beta+\gamma}\omega^{-\beta})i(\omega^{-\beta})\,.$$

If we use (5.10) and (6.14) and replace $\alpha+\beta$ by β in the fourth term, we have

$$\sum_{\alpha,\beta\in\Delta_+}\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})({}^t\Lambda_\tau(e_{-\alpha})i(\omega^{-\beta})-i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha}))$$
$$=\sum_{\alpha,\beta\in\Delta_+}\frac{(\beta+\gamma,\tau)}{(\gamma,\tau)(\alpha+\beta,\tau)}e(\omega^{-\alpha})e({}^t\Lambda_\tau(e_\beta)\omega^{-\gamma})i(e_\alpha\omega^{-\beta})$$
$$=-\sum_{\alpha,\beta\in\Delta_+}\frac{(\beta+\gamma,\tau)}{(\gamma,\tau)(\alpha+\beta,\tau)}e(\omega^{-\alpha})e(N_{\beta,\gamma}N_{\alpha,\beta}\omega^{-\beta-\gamma})i(\omega^{-\alpha-\beta})$$
$$=\sum_{\substack{0<\alpha<\beta\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{\beta-\alpha,\gamma}N_{\alpha,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\,.$$

Hence, from (2.5), the fourth term is equal to

$$\sum_{\substack{0<\alpha<\beta\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\,.$$

As for the fifth term, we use (5.10) and replace $\alpha+\beta$ by α. Then we get

$$\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e({}^t\Lambda_\tau(e_\beta)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})i(\omega^{-\beta})$$
$$=\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha-\beta})e(-N_{\beta,\alpha}N_{-\alpha,\gamma}\omega^{\alpha-\gamma})i(\omega^{-\beta})$$
$$=\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(-N_{\beta,\alpha-\beta}N_{\beta-\alpha,\gamma}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\,.$$

Hence, from (2.5), the fifth term is equal to

$$\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\,.$$

Therefore, the sum of the fourth and fifth term is equal to

$$\sum_{\substack{0<\alpha<\beta\\ \alpha\neq\gamma,\,\beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\tag{6.23}$$
$$+\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \alpha\neq\gamma,\,\beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})$$

$$+\sum_{\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\gamma})e(N_{-\gamma,\beta}N_{\beta-\gamma,\gamma}\omega^{-\beta})i(\omega^{-\beta}).$$

By (5.10) and (6.12), we have

$$\begin{aligned}
&e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}){}^t\Lambda_\tau(e_\beta)-{}^t\Lambda_\tau(e_\beta)e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})\\
&=e({}^t\Lambda_\tau(e_\beta){}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma})\\
&=\begin{cases}\dfrac{(\alpha-\gamma,\tau)}{(\gamma,\tau)}e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma}) & \text{if } \alpha<\gamma\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

By (5.10), we have also

$$e(N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\beta})=e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})$$

provided that $\alpha<\gamma$. Thus, the sum of the sixth and seventh term is equal to

$$-\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta}).$$

In the eighth term, we use (5.10) and replace $\alpha+\gamma$ by α. Then by (2.5) we get

$$\begin{aligned}
&-\sum_{\alpha\in\Delta_+}\frac{1}{(\gamma,\tau)}e({}^t\Lambda_\tau(e_\gamma)\omega^{-\alpha})e({}^t\Lambda_\tau(e_{-\alpha})\omega^{-\beta})i(\omega^{-\beta})\\
&=-\sum_{\substack{0<\alpha<\beta\\ \beta\in\Delta_+}}\frac{(\alpha-\beta,\tau)}{(\gamma,\tau)(\beta,\tau)}e(-N_{\gamma,\alpha}\omega^{-\alpha-\gamma})e(N_{-\alpha,\beta}\omega^{\alpha-\beta})i(\omega^{-\beta})\\
&=-\sum_{\substack{\gamma<\alpha<\beta+\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\gamma,\tau)(\beta,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta}).
\end{aligned}$$

Therefore, the sum of the sixth, seventh and eighth term is equal to

$$\begin{aligned}
(6.24)\qquad &-\sum_{\substack{0<\alpha<\beta\\ \alpha\neq\gamma,\beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\\
&-\sum_{\substack{\beta<\alpha<\beta+\gamma\\ \alpha\neq\gamma,\beta\in\Delta_+}}\frac{(\alpha-\beta-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}e(\omega^{-\alpha})e(N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha}\omega^{\alpha-\beta-\gamma})i(\omega^{-\beta})\\
&+\sum_{\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\beta})e(N_{-\beta,\gamma}N_{\beta,\gamma-\beta}\omega^{-\gamma})i(\omega^{-\beta}).
\end{aligned}$$

Now, we use the equality (6.18). In the case $\alpha<\beta$ we have

$$(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}-N_{-\alpha,\beta}N_{\gamma,\beta-\alpha})\omega^{\alpha-\beta-\gamma}-N_{-\alpha,\gamma}N_{\gamma-\alpha,\beta}\omega^{\alpha-\beta-\gamma}=0.$$

Since $e_{-\alpha}\omega^{-\beta}=0$ for $\alpha>\beta$, in the case $\alpha>\beta$ we have also

$$(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}+N_{-\alpha,\gamma}N_{\beta,\gamma-\alpha})\omega^{\alpha-\beta-\gamma}=N_{-\alpha,\beta}N_{\gamma,\beta-\alpha}\omega^{\alpha-\beta-\gamma}$$

i.e. $(N_{\gamma,\beta}N_{-\alpha,\beta+\gamma}-N_{-\alpha,\beta}N_{\gamma,\beta-\alpha})\omega^{\alpha-\beta-\gamma}-N_{-\alpha,\gamma}N_{\gamma-\alpha,\beta}\omega^{\alpha-\beta-\gamma}=0.$

From (6.21)–(6.24), it follows that

$$\begin{aligned}\mathfrak{R}_\tau\cdot e(\omega^{-\gamma}) &= e(\omega^{-\gamma})\cdot\mathfrak{R}_\tau+\frac{(2\rho-\gamma,\gamma)}{(\gamma,\tau)}e(\omega^{-\gamma})\\ &\quad+\sum_{\beta\in\Delta_+}\frac{1}{(\beta,\tau)}e(\omega^{-\beta})e((N_{\gamma,\beta}N_{-\beta,\beta+\gamma}-N_{-\beta,\gamma}N_{\gamma-\beta,\beta})\omega^{-\gamma})i(\omega^{-\beta})\\ &\quad-\sum_{\beta\in\Delta_+}\frac{1}{(\gamma,\tau)}e(\omega^{-\gamma})e((N_{\beta,\gamma}N_{-\gamma,\beta+\gamma}-N_{-\gamma,\beta}N_{\beta-\gamma,\gamma})\omega^{-\beta})i(\omega^{-\beta}).\end{aligned}$$

On the other hand, by (6.18) in the case $\alpha=\beta$ we have

$$\begin{aligned}(N_{\gamma,\beta}N_{-\beta,\beta+\gamma}-N_{-\beta,\gamma}N_{\gamma-\beta,\beta})\omega^{-\gamma} &= (\gamma,\beta)\omega^{-\gamma}\\ (N_{\beta,\gamma}N_{-\gamma,\beta+\gamma}-N_{-\gamma,\beta}N_{\beta-\gamma,\gamma})\omega^{-\beta} &= (\beta,\gamma)\omega^{-\beta}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}(6.25)\quad \mathfrak{R}_\tau\cdot e(\omega^{-\gamma}) &= e(\omega^{-\gamma})\cdot\mathfrak{R}_\tau+\frac{(2\rho-\gamma,\gamma)}{(\gamma,\tau)}e(\omega^{-\gamma})\\ &\quad-\sum_{\beta\in\Delta_+}(\beta,\gamma)\left(\frac{1}{(\beta,\tau)}+\frac{1}{(\gamma,\tau)}\right)e(\omega^{-\gamma})e(\omega^{-\beta})i(\omega^{-\beta}).\end{aligned}$$

Now, we compute $\mathfrak{R}_\tau\omega^{-A}$. For a 1-form $\omega^{-\gamma}$, by (5.10), (6.17) and (6.19) we have

$$\begin{aligned}\mathfrak{R}_\tau\omega^{-\gamma} &= -\sum_{\alpha,\beta\in\Delta_+}\frac{1}{(\beta,\tau)}{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta}){}^t\Lambda_\tau(e_{-\alpha})\omega^{-\gamma}\\ &= -\sum_{\substack{0<\alpha<\gamma\\ \beta\in\Delta_+}}\frac{(\alpha-\gamma,\tau)}{(\beta,\tau)(\gamma,\tau)}N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_\beta)e(\omega^{-\alpha})i(\omega^{-\beta})\omega^{\alpha-\gamma}\\ &= \sum_{0<\alpha<\gamma}\frac{1}{(\gamma,\tau)}N_{-\alpha,\gamma}{}^t\Lambda_\tau(e_{\gamma-\alpha})\omega^{-\alpha}\\ &= \frac{1}{(\gamma,\tau)}\Big(\sum_{0<\alpha<\gamma}N_{-\alpha,\gamma}N_{\gamma-\alpha,\alpha}\Big)\omega^{-\gamma}\\ &= \frac{(2\rho-\gamma,\gamma)}{(\gamma,\tau)}\omega^{-\gamma}.\end{aligned}$$

Using the induction on the number of elements in A and applying (6.25), we obtain easily

$$\mathfrak{R}_\tau\omega^{-A}=\left\{\sum_{k=1}^{q}\frac{(2\rho-\alpha_{i_k},\alpha_{i_k})}{(\alpha_{i_k},\tau)}-\sum_{1\leq k<l\leq q}(\alpha_{i_k},\alpha_{i_l})\left(\frac{1}{(\alpha_{i_k},\tau)}+\frac{1}{(\alpha_{i_l},\tau)}\right)\right\}\omega^{-A}$$

$$= \left\{ \sum_{k=1}^{q} \frac{(2\rho, \alpha_{i_k})}{(\alpha_{i_k}, \tau)} - \sum_{1\leq k<l\leq q} \frac{(\alpha_{i_k}, \alpha_{i_l})}{(\alpha_{i_k}, \tau)} \right\} \omega^{-A}$$
$$= \left\{ \sum_{\alpha\in A} \frac{(2\rho, \alpha)}{(\alpha, \tau)} - \sum_{\alpha,\beta\in A} \frac{(\alpha, \beta)}{(\alpha, \tau)} \right\} \omega^{-A}$$

where $A=(\alpha_{i_1}, \cdots, \alpha_{i_q})$. Therefore, by (6.20) the Laplace-Beltrami operator $\square$ is expressed as follows:

$$\square(f\omega^{-A}) = \sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} \nabla_{e_\alpha} \nabla_{e_{-\alpha}} (f\omega^{-A}) + C_A \cdot f\omega^{-A}$$

where C_A is a constant depending on A. In fact, we get

$$C_A = \sum_{\alpha\in A} \frac{(\alpha, \lambda + \sum_{\beta\in A} \beta)}{(\alpha, \tau)} + \sum_{\alpha\in A} \frac{(2\rho, \alpha)}{(\alpha, \tau)} - \sum_{\alpha,\beta\in A} \frac{(\alpha, \beta)}{(\alpha, \tau)}$$
$$= \sum_{\alpha\in A} \frac{(\alpha, \lambda+2\rho)}{(\alpha, \tau)} .$$

q.e.d.

7. Vanishing theorems of square-integrable $\bar{\partial}$-cohomology spaces

We retain the notation introduced in the previous sections. Using proposition 2 we will give vanishing theorems of the square-integrable $\bar{\partial}$-cohomology space $H_2^q(\mathcal{L}_\lambda)$. The following lemma is due to [1] Proposition 8.

Lemma. 7. *Let q be an integer such that $0\leqq q\leqq n=\dim D$. If there exists a constant $c>0$ such that for every $\varphi\in A_0^{0,q}(\mathcal{L}_\lambda)$ we have the inequality*

$$(\square\varphi, \varphi) \geqq c(\varphi, \varphi) ,$$

then we have $$H_2^q(\mathcal{L}_\lambda) = (0) .$$

For each character λ, put

$$\Delta_\lambda = \{\alpha\in\Delta \mid \varepsilon_\alpha(\alpha, \lambda) > 0\} .$$

Let q_λ be the number of all elements in $\Delta_+ \cap \Delta_\lambda$. Then we have the following vanishing theorem about 0-th square-integrable $\bar{\partial}$-cohomology space $H_2^0(\mathcal{L}_\lambda)$.

Theorem 1. *Assume that q_λ is not zero, i.e. there exists an element $\alpha\in\Delta_+$ such that $\varepsilon_\alpha(\alpha, \lambda)>0$. Then we have $H_2^0(\mathcal{L}_\lambda)=0$.*

Proof. Let f be a section in $A_0^{0,0}(\mathcal{L}_\lambda)$. From Proposition 2, we have

$$\square f = \sum_{\alpha\in\Delta_+} \frac{1}{(\alpha, \tau)} e_\alpha e_{-\alpha} f .$$

Since $h_\alpha=[e_\alpha, e_{-\alpha}]$, we get

$$\square f=\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\frac{1}{(\alpha,\tau)}h_\alpha f+\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\frac{1}{(\alpha,\tau)}e_{-\alpha}e_\alpha f+\sum_{\alpha\in\Delta_+-\Delta_\lambda}\frac{1}{(\alpha,\tau)}e_\alpha e_{-\alpha}f\,.$$

A section $f\colon G\to C$ satisfies the following formula:

$$h_\alpha f=-\lambda(h_\alpha)f=-(\alpha,\lambda)f\,.$$

By (6.4), the formal adjoint operator of e_α with respect to the inner product $(\ ,\ )_G$ in $C_0^\infty(G)$ is $-\varepsilon_\alpha e_{-\alpha}$ (cf. the proof of Proposition 1). Hence we have

$$(\square f,f)=\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\Big(-\frac{(\lambda,\alpha)}{(\alpha,\tau)}\Big)(f,f)_G+\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\Big(\frac{-\varepsilon_\alpha}{(\alpha,\tau)}\Big)(e_\alpha f,e_\alpha f)_G$$
$$+\sum_{\alpha\in\Delta_+-\Delta_\lambda}\Big(\frac{-\varepsilon_\alpha}{(\alpha,\tau)}\Big)(e_{-\alpha}f,e_{-\alpha}f)_G\,.$$

Here, $\dfrac{-\varepsilon_\alpha}{(\alpha,\tau)}$ is positive for every positive root α. Therefore we have

$$(\square f,f)\geqq\Big(\sum_{\alpha\in\Delta_+\cap\Delta_\lambda}\Big(-\frac{(\lambda,\alpha)}{(\alpha,\tau)}\Big)\Big)(f,f)\,.$$

If α belongs to $\Delta_+\cap\Delta_\lambda$, we have

$$-\frac{(\alpha,\lambda)}{(\alpha,\tau)}>0\,.$$

Thus, if we assume that $\Delta_+\cap\Delta_\lambda$ is not empty, the bundle $\mathcal{L}_\lambda$ satisfies the condition of Lemma 7 for $q=0$, and we have $H_2^0(\mathcal{L}_\lambda)=(0)$. q.e.d.

REMARK. This theorem follows also from the expression of $\square$ on p. 282 of [4] instead of our Proposition 2.

For general q-th $\bar\partial$-cohomology spaces, we get the following main theorem.

Theorem 2. *Let q be an integer such that $0<q\leqq n$. Assume that for any q-tuple A of positive roots the scalar $c_A=\sum_{\alpha\in A}\dfrac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}$ is positive. Then we have $H_2^q(\mathcal{L}_\lambda)=(0)$.*

Proof. Let $\varphi=\sum_{A\in\mathfrak{A}}f_A\omega^{-A}$ be a form in $A_0^{0,q}(\mathcal{L}_\lambda)$. By (6.4) and (6.6), we have

$$(\nabla_{e_\alpha}\varphi,\varphi)=-\varepsilon_\alpha(\varphi,\nabla_{e_{-\alpha}}\varphi)$$

where $(\ ,\ )$ is the inner product in $C_0^\infty(G)\otimes\overset{q}{\wedge}\mathfrak{n}_-^*$. From Proposition 2, we get

$$(\square\varphi,\varphi)=\sum_{\alpha\in\Delta_+}\Big(-\frac{\varepsilon_\alpha}{(\alpha,\tau)}\Big)(\nabla_{e_{-\alpha}}\varphi,\nabla_{e_{-\alpha}}\varphi)$$
$$+\sum_{A\in\mathfrak{A}}\Big(\sum_{\alpha\in A}\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}\Big)(f_A\omega^{-A},f_A\omega^{-A})\,.$$

Put

$$c = \min_{A\in\mathfrak{A}} c_A = \min_{A\in\mathfrak{A}}\Big(\sum_{\alpha\in A}\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}\Big).$$

Since $-\dfrac{\varepsilon_\alpha}{(\alpha,\tau)}$ is positive for every positive root α, we have

$$(\Box\varphi,\varphi)\geqq c(\varphi,\varphi).$$

From the assumption, c is positive. Therefore, by Lemma 7 we obtain the theorem. q.e.d.

We note that the criterion for the vanishing in this theorem depends on the choice of τ.

Corollary 1. *Assume that*

$$\begin{cases}(\alpha,\lambda+2\rho)>0 & \textit{for}\quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}}\\ (\alpha,\lambda+2\rho)<0 & \textit{for}\quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}.\end{cases}$$

Then we have $H_2^q(\mathcal{L}_\lambda)=(0)$ *for all* $q\geqq 1$.

Proof. By the assumption, we have

$$c_\alpha=\frac{(\alpha,\lambda+2\rho)}{(\alpha,\tau)}>0\qquad\text{for all}\quad \alpha\in\Delta_+.$$

Since $c_A=\sum_{\alpha\in A}c_\alpha$, c_A is positive for any q-tuple A provided that $q\geqq 1$. By Theorem 2, we obtain the corollary. q.e.d.

We consider the case $q=0$. From Corollary 1, if we have $(\alpha,\lambda)<-(\alpha,2\rho)$ for all $\alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}$, we obtain $H_2^q(\mathcal{L}_\lambda)=0$ for $q\neq 0$.

Now, let $\mathcal{L}_\lambda^*$ be the dual line bundle of $\mathcal{L}_\lambda$ and $\Theta^*(D)$ be the dual bundle of the holomorphic tangent bundle of D. By Theorem 1.2 in [8], we obtain the Seere's duality

$$H_2^q(\mathcal{L}_\lambda)\simeq H_2^{n-q}(\mathcal{L}_\lambda^*\otimes\overset{n}{\wedge}\Theta^*(D)).\tag{7.1}$$

On the other hand, the bundle $\mathcal{L}_\lambda^*$ is the homogeneous line bundle associated with the charactor λ^{-1} of H. The bundle $\overset{n}{\wedge}\Theta^*(D)$ is the homogeneous line bundle $G\times_H\overset{n}{\wedge}\mathfrak{n}_+^*$ associated with the representation $\overset{n}{\wedge}Ad_+^*$ induced from the adjoint representation Ad_+ of H in $\mathfrak{n}_+$. Therefore

$$\mathcal{L}_\lambda^*\otimes\overset{n}{\wedge}\Theta^*(D)=\mathcal{L}_{\lambda^{-1}\otimes\overset{n}{\wedge}Ad_+^*}.$$

The differential of the charactor $\lambda^{-1}\otimes\overset{n}{\wedge}Ad_+^*$ is $-\lambda-2\rho$. By Theorem 1, 2 and (7.1) we obtain the following corollaries.

Corollary 2. *If we assume that $q_{-\lambda-2\rho}$ is not zero i.e. there exist a root $\alpha\in\Delta_+$ such that $\varepsilon_\alpha(\alpha, \lambda+2\rho)<0$, we have $H_2^n(\mathcal{L}_\lambda)=(0)$.*

Corollary 3. *Let q be an integer such that $0\leqq q<n$. Assume that for any q-tuple A of positive roots the scalar $d_A=\sum_{\alpha\in A}\dfrac{(\alpha, \lambda)}{(\alpha, \tau)}$ is negative. Then we have $H_2^{n-q}(\mathcal{L}_\lambda)=(0)$.*

From Corollary 3, we have also the following.

Corollary 4. *We assume that*

$$\begin{aligned}(\alpha, \lambda)&<0 \quad \text{for all} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{k}}\\ (\alpha, \lambda)&>0 \quad \text{for all} \quad \alpha\in\Delta_+\cap\Delta_{\mathfrak{p}}\end{aligned}$$

i.e. $q_\lambda=n$. Then, we have $H_2^q(\mathcal{L}_\lambda)=(0)$ for all $q\leqq n-1$.

EXAMPLE. Let $G=SU(2, 1)$ and T be the subgroup of G consisting of all matrices

$$U=\begin{pmatrix}u_1 & 0 & 0\\ 0 & u_2 & 0\\ 0 & 0 & u_3\end{pmatrix}$$

where $u_i\in U(1)$ $(i=1, 2, 3)$ and $\det U=1$. We denote by $K=S(U(2)\times U(1))$ the subgroup of G consisting of all matrices

$$\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{U}} & 0\\ & & 0\\ 0 & 0 & v\end{pmatrix}$$

where $U\in U(2)$, $v\in U(1)$ and $\det U\cdot v=1$. Then, H is a compact Cartan subgroup of G and K is a maximal compact subgroup containing H. The complexification of the Lie algebra of G is $\mathfrak{g}=\mathfrak{sl}(3, C)$ and the subalgebra $\mathfrak{h}$ is given by

$$\mathfrak{h}=\left\{\begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix}\middle|\ \lambda_i\in C,\ \lambda_1+\lambda_2+\lambda_3=0\right\}.$$

The root system of $\mathfrak{g}$ with respect to the Cartan subalgebra $\mathfrak{h}$ is given by

$$\Lambda=\{\lambda_i-\lambda_j\,|\,i\neq j,\ 1\leqq i, j\leqq 3\}\ .$$

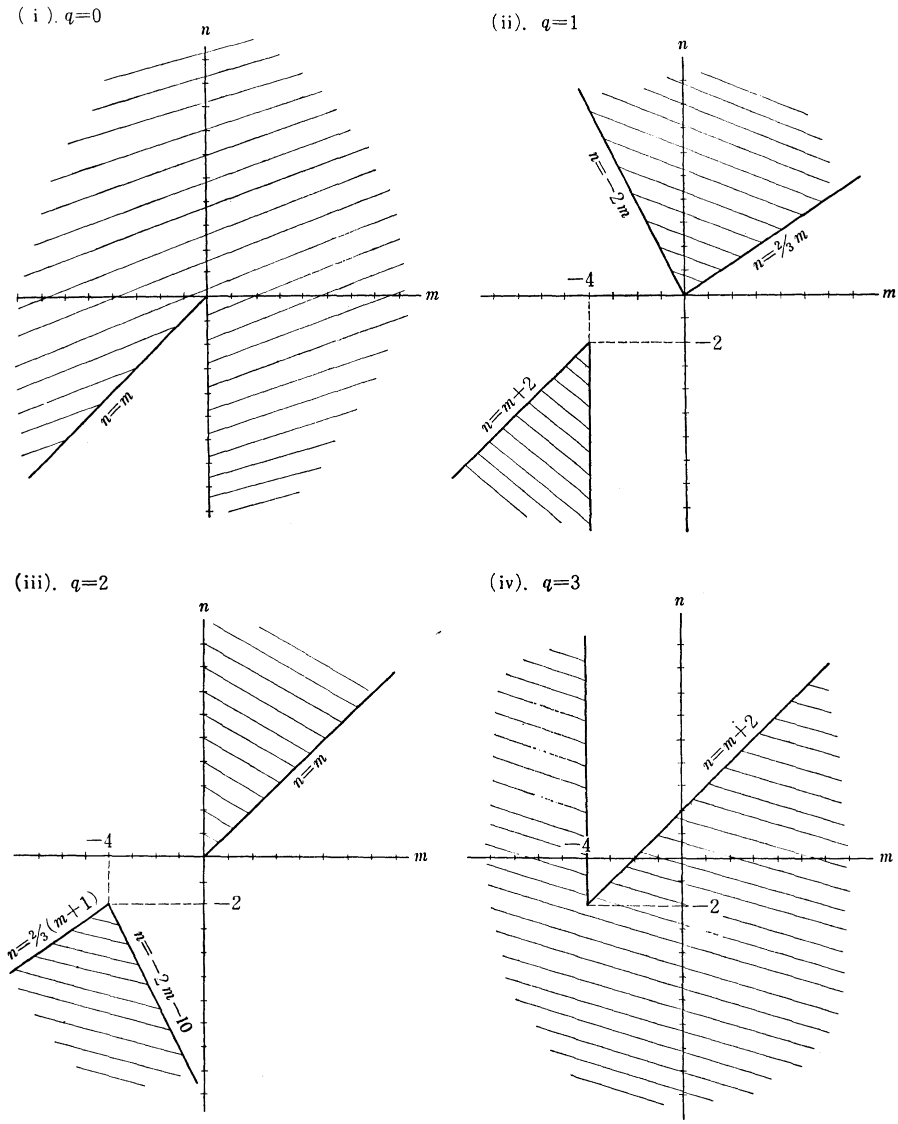
We choose a fundamental root system $\{\lambda_1-\lambda_2, \lambda_2-\lambda_3\}$ and take an ordering for the roots corresponding to this system. Then, the positive root set is $\Delta_+=\{\lambda_1-\lambda_2, \lambda_2-\lambda_3, \lambda_1-\lambda_3\}$ and we have

$$\Delta_+ \cap \Delta_{\mathfrak{k}} = \{\lambda_1 - \lambda_2\}$$
$$\Delta_+ \cap \Delta_{\mathfrak{p}} = \{\lambda_2 - \lambda_3,\ \lambda_1 - \lambda_3\}\ .$$

Let a be a positive real constant, and put

$$h_\tau = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & a \end{pmatrix} \in \mathfrak{h}_{\boldsymbol{R}}\ .$$

Figure 1.



Let τ be an element of $\mathfrak{h}_R{}^*$ corresponding to $h_\tau \in \mathfrak{h}_R$ with respect to the Killing form of $\mathfrak{g}$. Then, the element τ satisfies the condition (3.4). Hence, we have the homogeneous complex manifold $D=G/H$ with the invariant Kähler metric g_τ. The space $\mathfrak{h}_R{}^*$ is generated by λ_1 and λ_2 over R. The set of all elements of $\mathfrak{h}_R{}^*$ which are the differentials of charactors of H is given by

$$\left\{\lambda \in \mathfrak{h}_R{}^* \,\middle|\, \frac{2(\lambda, \alpha)}{(\alpha, \alpha)} \in Z \quad \text{for all} \quad \alpha \in \Delta\right\}$$

$$\text{i.e.} \quad \{\lambda = m\lambda_1 + n\lambda_2 \mid m, n \in Z\}.$$

We can consider a character λ of H as a lattice point in R^2. As for the vanishing of the cohomology space $H^q_2(\mathcal{L}_\lambda)$, our theorems give the following figures (cf. Figure 1). Here, the space $H^q_2(\mathcal{L}_\lambda)$ vanishes for all characters belonging to the shadowed domains.

On the other hand, the vanishing theorems in [4] are written as follows: There exists a positive constant η such that, if the character λ statisfies $|(\lambda, \alpha)| > \eta$ for every $\alpha \in \Delta$, the space $H^q_2(\mathcal{L}_\lambda)$ vanishes for all $q \neq q_\lambda$. In the case of this example, we can see that a positive constant η must be larger than 12 and the above condition on λ is equivalent to the following inequalities:

$$|m| > 6\eta, \quad |n| > 6\eta, \quad |m-n| > 6\eta .$$

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