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ON REALIZATION OF KIRBY-SIEBENMANN'S OBSTRUCTIONS BY 6-MANIFOLDS

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1. Introduction

Let M^n be a closed topological manifold. By Kirby-Siebenmann ([5], [6]), an obstruction to triangulate M^n is defined as an element of $H^4(M^n : Z_2)$, provided $n \geqslant 5$. We will denote this obstruction by $k(M)$. In this paper, we will consider the following problem.

Problem. Let M_0^n be a closed PL manifold. For a given non-zero element $\eta \in H^4(M_0^n : Z_2)$, do there exist a nontriangulable manifold M^n and a homotopy equivalence $f: M_0^n \to M^n$ such that $f^*k(M^n)=\eta$? Here, $f^* : H^4(M^n : Z_2) \to H^4(M_0^n : Z_2)$ is the isomorphism induced by f.

Since there exists a non-triangulable manifold M^6 which is homotopy equivalent to $S^4 \times S^2$ ([5], Introduction p.v), this problem for $M_0^n = S^4 \times S^2$ has an affirmative answer. In some cases, however, the problem has a negative answer. For example, Dr. S. Fukuhara has proved the following ([3]); let M^5 be a closed (possibly non-triangulable) topological manifold which is homotopy equivalent to $S^4 \times S^1$, then M^5 is really homeomorphic to $S^4 \times S^1$.

When M_0^6 is a closed manifold with $\pi_1(M_0^6)$ is free and $H^3(M_0^6 : Z_2)=0$, the problem will be answered affirmatively. And the problem for $M_0^n = S^4 \times S^{n-4}$ will be solved, provided $n \geqslant 9$. (See Corollary 2.)

The method of this paper can be found in [5] and [9]. The author wishes to express his hearty thanks to Professor K. Kawakubo who showed him a construction of non-triangulable manifold having the homotopy type of CP^3.

2. Six-dimensional case

In dimension six, our results are as follow.

Theorem 1. *Let M_0^6 be a closed PL 6-manifold with $H^3(M_0^6 : Z_2)=0$ and η a non-zero element of $H^4(M_0^6 : Z_2)$ whose Poincaré dual $\bar{\eta}$ is spherical. Then there exist a non-triangulable manifold M^6 and a homotopy equivalence $f: M_0^6 \to M^6$ such that $f^*k(M)=\eta$, where $f^* : H^4(M^6 : Z_2) \to H^4(M_0^6 : Z_2)$ is the isomorphism* *induced by f.*

Corollary 1. *Let M_0^6 be a closed PL 6-manifold. Suppose $H_2(\pi_1(M_0^6):Z_2)=0$ and $H^3(M_0^6:Z_2)=0$. Then, for any non-zero element η in $H^4(M_0^6:Z_2)$, there exist a non-triangulable manifold M^6 and a homotopy equivalence $f:M_0^6\to M^6$ such that $f^*k(M)=\eta$, where $f^*:H^4(M^6:Z_2)\to H^4(M_0^6:Z_2)$ is the isomorphism induced by f.*

In Theorem 1, we cannot drop the assumption that the Poincaré dual $\bar{\eta}$ of η is spherical. Hence, in Corollary 1, we cannot drop the assumption about the fundamental group of M_0^6. The following proposition shows both.

Proposition 1. *Let M^6 be a closed topological manifold. Suppose M^6 has the same homotopy type of $S^4\times S^1\times S^1$, then M^6 is triangulable.*

First, we prove Corollary 1 assuming Theorem 1.

Proof of Corollary 1. By the theorem of Hopf (see [1], p. 356), the fact that $H_2(\pi_1(M_0^6):Z)=0$ implies that any element of $H_2(M_0^6:Z_2)$ is spherical. This reduces Corollary 1 to Theorem 1.

To prove Theorem 1, we need some lemmas. The following is proved in [5].

Lemma 1. *Let E^{n-1} be a closed simply-connected PL manifold such that $H^3(E^{n-1}:Z_2)\neq 0$ and that the Bockstein homomorphism $\beta:H^3(E^{n-1}:Z_2)\to H^4(E^{n-1}:Z)$ is trivial. If $n\geqslant 6$, then there exists a homeomorphism $h_0:E^{n-1}\to E^{n-1}$ which is homotopic to the identity but never isotopic to a PL homeomorphism.*

For completeness, we supply the proof of Lemma 1.

Proof of Lemma 1. Since $H^3(E^{n-1}:Z_2)\neq 0$ and $n\geqslant 6$, there exists a *PL* structure Θ on E^{n-1} which is not isotopic to the original *PL* structure on E^{n-1} ([5], [6]). Since E^{n-1} is simply-connected and the Bockstein homomorphism $\beta:H^3(E^{n-1}:Z_2)\to H^4(E^{n-1}:Z)$ is trivial, there exists a *PL* homeomorphism $g:E^{n-1}\to E_\Theta^{n-1}$ which is homotopic to the identity by D. Sullivan ([7], [10]). Put $h_0=$"identity"$\circ g$, where "identity" $:E_\Theta^{n-1}\to E^{n-1}$ is a homeomorphism defined by "identity" $(x)=x$. Then clearly h_0 is homotopic to the identity. If h_0 is isotopic to a *PL* homeomorphism, then "identity" $:E_\Theta^{n-1}\to E^{n-1}$ is also isotopic to a *PL* homeomorphism, for g is a *PL* homeomorphism. This is a contradiction to the choice of Θ. Therefore h_0 is never isotopic to a *PL* homeomorphism. This proves the lemma.

Lemma 2. *Let E^{n-1} be a PL manifold which is a fibration with fibre S^3 over a simply-connected closed manifold N^{n-4} such that $H^4(N^{n-4}:Z)=H^4(N^{n-4}:Z_2)$*

$=0$. *If* $n \geqslant 6$, *then there exists a homeomorphism* $h_0 : E^{n-1} \to E^{n-1}$ *which is homotopic to the identity but never isotopic to a PL homeomorphism.*

REMARK. If we put $h = h_0 \times \mathrm{id.} : E^{n-1} \times R \to E^{n-1} \times R$, then h is also never isotopic to a PL homeomorphism by stability $\pi_3(TOP_m, PL_m) = \pi_3(TOP/PL)$ ([5], [6]).

Proof of Lemma 2. Note that E^{n-1} is simply-connected. By Lemma 1, we need only prove that $H^3(E^{n-1} : Z_2)$ is nontrivial and that the Bockstein homomorphism $\beta : H^3(E^{n-1} : Z_2) \to H^4(E^{n-1} : Z)$ is trivial.

Applying the generalized Gysin cohomology exact sequence to the fibration $E^{n-1} \to N^{n-4}$ with fibre S^3, we obtain the following exact sequence :

$$\begin{aligned} &H^3(E^{n-1} : G) \to H^0(N^{n+4} : G) \to H^4(N^{n-4} : G) \\ &\to H^4(E^{n-1} : G) \to H^1(N^{n-4} : G) \end{aligned}$$

where the coefficient group G is Z or Z_2. By hypothesis, $H^4(N^{n-4} : Z) = H^4(N^{n-4} : Z_2) = 0$ and $H^1(N^{n-4} : Z) = \mathrm{Hom}\,(H_1(N^{n-4} : Z), Z) = 0$. Therefore, $H^3(E^{n-1} : Z_2)$ is non-trivial and $H^4(E^{n-1} : Z)$ is trivial. This proves the lemma.

Proof of Theorem 1. Since $\bar{\eta}$ is spherical, there exists a continuous map $S^2 \to M_0^6$ representing $\bar{\eta} \in H_2(M_0^6 : Z_2)$. By general position, we can assume that this S^2 is PL embedded in M_0^6. By Haefliger-Wall [4], S^2 has a normal PL disk bundle $D(\nu)$ in M_0^6.

Clearly, Int $D(\nu) - S^2$ is PL homeomorphic to $\partial D(\nu) \times R$. Put $\partial D(\nu) = E^5$, then by Lemma 2 and Remark we can find a homeomorphism $h : E^5 \times R \to E^5 \times R$ which is homotopic to the identity but never isotopic to a PL homeomorphism. Clearly $M_0^6 - S^2$ contains $E^5 \times R$ as an open PL collar of the end at S^2. Then M_0^6 can be written obviously as $(M_0^6 - S^2) \bigcup_{id_{E \times R}} \mathrm{Int}\, D(\nu)$.

Let M^6 be a topological manifold $(M_0^6 - S^2) \bigcup_h \mathrm{Int}\, D(\nu)$ obtained by pasting Int $D(\nu)$ to $M_0^6 - S^2$ by the above homeomorphism $h : E^5 \times R \to E^5 \times R$. Let $H_0 : E^5 \times I \to E^5$ be a homotopy connecting h_0 to the identity. Put $H = H_0 \times \mathrm{id} : (E^5 \times R) \times I \to E^5 \times R$. Consider the adjunction space $\mathfrak{M} = ((M_0^6 - S^2) \times I) \bigcup_H \mathrm{Int}\, D(\nu)$ obtained by pasting $(M_0^6 - S^2) \times I$ to Int $D(\nu)$ by the continuous map $H : (E^5 \times R) \times I \to E^5 \times R$. Then, clearly, $\mathfrak{M}$ is homeomorphic to the adjunction space $(M_0^6 - \mathrm{Int}\, D(\nu)) \times I \bigcup_{H_0} D(\nu)$ obtained by pasting together $(M_0^6 - \mathrm{Int}\, D(\nu)) \times I$ and $D(\nu)$ by the continuous map $H_0 : E^5 \times I \to E^5$. Then, we can see that $\mathfrak{M}$ has both M_0^6 and M^6 as deformation retracts. (see [8], p. 21, Adjunction Lemma.) Define a homotopy equivalence $f : M_0^6 \to M^6$ to be the composition of the following maps.

$$M_0^6 \xrightarrow[\text{inclusion}]{} \mathfrak{M} \xrightarrow[\text{deformation retraction}]{} M^6$$

Next, we will show that M^6 is non-triangulable. Suppose M^6 is triangulable. Both $(M_0^6-S^2)$ and $\operatorname{Int} D(\nu)$ are open PL submanifolds of M^6. We denote these submanifolds with induced PL structures from M^6 by $(M_0^6-S^2)_\alpha$ and $(\operatorname{Int} D(\nu))_\beta$. Then the composition of

$$\begin{aligned}&\text{"identity"} : (E^5\times R)_{\alpha|E^5\times R} \to E^5\times R\,,\\ &h : E^5\times R \to E^5\times R \qquad \text{and}\\ &\text{"identity"} : E^5\times R \to (E^5\times R)_{\beta|E^5\times R}\end{aligned}$$

is a PL homeomorphism. On the other hand, by the following diagram, we see that $H^3(M_0^6-S^2 : Z_2)=0$.

$$\begin{array}{ccccccc} H_3(M_0^6 : Z_2) & \to & H_3(M_0^6, S^2 : Z_2) & \to & H_2(S^2 : Z_2) & \to & H_2(M_0^6 : Z_2)\\ \wr\| & & \wr\| & & \wr\| & & \cup\!\!\!\!\!\;| \\ H^3(M_0^6 : Z_2) & \to & H^3(M_0^6-S^2 : Z_2) & & Z_2\ni 1 & \longrightarrow & \tilde{\eta}\neq 0\\ \| & & & & & & \\ 0 & & & & & & \end{array}$$

where the horizontal sequence is exact and the vertical maps are Poincaré and Alexander dualities. Therefore, α is concordant to the original PL structure on $M_0^6-S^2$ and hence $\alpha|E^5\times R$ is concordant to the original PL structure on $E^5\times R$ ([5], [6]). This means that "identity" : $(E^5\times R)_{\alpha|E^5\times R}\to E^5\times R$ is isotopic to a PL homeomorphism. In a similar way, we have that "identity" : $E^5\times R\to(E^5\times R)_{\beta|E^5\times R}$ is isotopic to a PL homeomorphism. Then h itself is isotopic to a PL homeomorphism which is a contradiction. Therefore M^6 must be non-triangulable.

Note that $M^6-S^2=M_0^6-S^2$ is triangulable. Then the naturality of Kirby-Siebenmann's obstruction with respect to inclusion maps of open submanifolds and the following commutative diagram imply that S^2 in M^6 represents the Poincaré dual of $k(M)$ in $H_2(M^6 : Z_2)$.

$$\begin{array}{ccccc} H_2(S^2 : Z_2) & \to & H_2(M^6 : Z_2) & \to & H_2(M^6, S^2 : Z_2)\\ \wr\| & & \wr\| & & \wr\| \\ H^4(M^6, M^6-S^2 : Z_2) & \to & H^4(M^6 : Z_2) & \to & H^4(M^6-S^2 : Z_2)\\ \wr\| & & \cup\!\!\!\!\!\;| & & \cup\!\!\!\!\!\;| \\ Z_2\ni 1 & \longrightarrow & k(M) & \longrightarrow & 0 \end{array}$$

where the horizontal sequences are exact and the vertical isomorphisms are Poincaré and Alexander dualities. Now, it is clear that $f^*k(M^6)=\eta$, this proves the theorem.

Proof of Proposition 1. By virtue of a topological version ([8]) of fibering theorem due to F.T. Farrell [2], M^6 is a fibering over a circle, since $\mathrm{Wh}(\pi_1(M^6))$

$=0$. Therefore there exists a submanifold N^5 of M^6 and a homeomorphism $g: N^5 \to N^5$ such that the mapping torus of g is homeomorphic to M^6. Since N^5 has the homotopy type of $S^4 \times S^1$, N^5 is really homeomorphic to $S^4 \times S^1$ by S. Fukuhara [3]. Since $H^3(S^4 \times S^1 : Z_2)=0$, any homeomorphism of $S^4 \times S^1$ onto itself is isotopic to a PL homeomorphism ([5], [6]). Therefore M^6 is triangulable. This proves the proposition.

3. Higher dimensional case

In higher dimensional case, we can only obtain a weaker result.

Theorem 2. *Let M_0^n be a closed PL manifold of dimension $n \geqslant 6$ with $H^3(M_0^n : Z_2)=0$. Suppose η is a non-zero element of $H^4(M_0^n : Z_2)$ whose Poincaré dual $\bar{\eta}$ in $H_{n-4}(M_0^n : Z_2)$ is represented by a simply-connected $(n-4)$-submanifold N^{n-4} with $H^4(N^{n-4} : Z)=H^4(N^{n-4} : Z_2)=H^3(N^{n-4} : Z_2)=0$. Then there exist a non-triangulable manifold M^n and a homotopy equivalence $f: M_0^n \to M^n$ such that $f^*k(M^n)=\eta$.*

As an application of Theorem 2, we can obtain a number of non-triangulable manifolds which are homotopy equivalent to some PL manifolds.

Corollary 2. *Let N^{n-4} be a closed 4-connected PL manifold and L^4 a simply-connected 4-manifold. If $n \geqslant 9$, then there exists a non-triangulable manifold which has the homotopy type of $L^4 \times N^{n-4}$.*

Proof of Theorem 2. By the assumption, there exists a $(n-4)$-submanifold N^{n-4} of M_0^n representing $\bar{\eta}$. Let $D(\nu)$ be a normal block bundle of N^{n-4} in M_0^n. Put $E^{n-1}=\partial D(\nu)$, then by Lemma 2 and Remark, there exists a homeomorphism $h: E^{n-1} \times R \to E^{n-1} \times R$ which is homotopic to the identity but never isotopic to a PL homeomorphism. As before, put $M^n=(M_0^n - N^{n-4}) \bigcup_h \text{Int } D(\nu)$. Then the rest of the proof is exactly same as that of Theorem 1.

Proof of Corollary 2. By the preceding arguments, we have only to show that $H^3(L^4 \times N^{n-4} : Z_2)=0$. By the Künneth formula and the Poincaré duality, we have the following:

$$
\begin{aligned}
&H^3(L^4 \times N^{n-4} : Z_2) \\
&= H^3(N^{n-4} : Z_2) \oplus [H^2(L^4 : Z) \otimes H^1(N^{n-4} : Z_2)] \oplus [H^2(L^4 : Z) * H^2(N^{n-4} : Z_2)] \\
&= 0
\end{aligned}
$$

This proves the corollary.

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