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ON THE JANKO'S SIMPLE GROUP OF ORDER 175560

HIROYOSHI YAMAKI

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1. Introduction

Let $\mathfrak{J}(11)$ be the Janko's simple group of order 175560 presented in [1] and $\mathfrak{A}_m$ be the alternating group of degree m. In his papers [1], [2] Janko characterized the non-solvable group having the centralizer of an involution in the center of a Sylow 2-subgroup isomorphic to the splitting central extension of a group of order 2 by $\mathfrak{A}_4$ or $\mathfrak{A}_5$. His result is that such a non-solvable group containing no normal subgroup of index 2 must be isomorphic to $P\Gamma L(2,8)$ or $\mathfrak{J}(11)$. The purpose of this note is to sharpen his results [1], [2]. Namely we want to prove the following theorem.

Theorem. *Let $\mathfrak{G}$ be a finite non-solvable group with the following two properties*:

a) *$\mathfrak{G}$ has no normal subgroup of index* 2,

b) *$\mathfrak{G}$ contains an involution J in the center of a Sylow 2-subgroup of $\mathfrak{G}$ such that the centralizer $C_{\mathfrak{G}}(J)=\langle J\rangle\times\mathfrak{X}_m$, where $\mathfrak{X}_m$ is isomorphic to $\mathfrak{A}_m$.*

Then one of the following holds:

1) *$m=4$ and $\mathfrak{G}$ is isomorphic to $P\Gamma L(2, 8)$,*

2) *$m=5$ and $\mathfrak{G}$ is isomorphic to $\mathfrak{J}(11)$.*

REMARK. Our proof depends on Janko's theorems [1], [2] and by his results it is sufficient to prove that $m=4$ or 5.

2. Proof of the Theorem

Put $m=4n+r$, where $0\leqslant r\leqslant 3$. Assume that n is greater than 1. Then the group $\mathfrak{A}_m$ contains involutions $\tilde{X}_i$, $\tilde{X}'_i$ $(1\leqslant i\leqslant n)$ and $\tilde{Y}_j$ $(1\leqslant j\leqslant n-1)$ with the cycle decompositions

$$\begin{aligned}\tilde{X}_i&=(4i-3,\ 4i-2)\ (4i-1,\ 4i)\\ \tilde{X}'_i&=(4i-3,\ 4i-1)\ (4i-2,\ 4i)\\ \tilde{Y}_j&=(4j-3,\ 4j-2)\ (4j+1,\ 4j+2).\end{aligned}$$

In the isomorphism from $\mathfrak{A}_m$ to $\mathfrak{X}_m$, let the images of the elements $\tilde{X}_i$, $\tilde{X}'_i$ and

$\tilde{Y}_j$ be X_i, X'_i and Y_j, respectively. Put $\mathfrak{X}=\langle X_i, X'_j \mid 1 \leqslant i, j \leqslant n\rangle$ and $\mathfrak{Y}=\langle Y_j \mid 1 \leqslant i \leqslant n-1\rangle$. Then $\mathfrak{X}$ and $\mathfrak{Y}$ are 2-groups and $\mathfrak{Y}$ normalizes $\mathfrak{X}$. Hence $\mathfrak{X}\mathfrak{Y}$ is a 2-group. By the definition we have $Y_j^{-1}X'_jY_j=X_jX'_j$ and $Y_j^{-1}X'_{j+1}Y_j=X_{j+1}X'_{j+1}$, and then $\langle X_i \mid 1 \leqslant i \leqslant n\rangle$ is the commutator subgroup $(\mathfrak{X}\mathfrak{Y})'$ of $\mathfrak{X}\mathfrak{Y}$. Put $C_i=X_1X_2\cdots X_i$ for $1 \leqslant i \leqslant n$. Then we may assume that $\{C_i \mid 1 \leqslant i \leqslant n\}$ is the set of the representatives of the conjugacy classes of involutions in $\mathfrak{T}_m$. Let $\mathfrak{D}$ be a Sylow 2-subgroup of $\mathfrak{G}$ contained in $C_{\mathfrak{G}}(J)$ and containing $\langle J\rangle \times \mathfrak{X}\mathfrak{Y}$. Hence the group $\mathfrak{D}'$ contains C_n and the center $Z(\mathfrak{D})$ of $\mathfrak{D}$ contains J and C_n. These facts are also true if $n=1$ and $r=2$ or 3.

Assume by way of contradiction that n is greater than 1, or $n=1$ and $r=2$ or 3. For $1 \leqslant i \leqslant n-1$, C_i is the square of an element of order 4 in $\mathfrak{T}_m$. Since $\mathfrak{G}$ has no normal subgroup of index 2, it follows from a transfer lemma of Thompson [3] that J must fuse with C_n in $\mathfrak{G}$. Note that J is not a square of an element of order 4. Therefore Burnside's argument implies that J must fuse with C_n in the normalizer $N_{\mathfrak{G}}(\mathfrak{D})$ of $\mathfrak{D}$. This is impossible because $\mathfrak{D}'$ contains C_n but does not J. Thus we get a contradiction and hence $n=1$ and $r=0$ or 1, that is, $m=4$ or 5. Applying the results of Janko [1], [2], $\mathfrak{G}$ is isomorphic to $P\Gamma L(2,8)$ or $\mathfrak{J}(11)$, respectively.

The proof of our theorem is complete.

OSAKA UNIVERSITY

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