

## MODULES OVER BOUNDED DEDEKIND PRIME RINGS

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The purpose of this paper is to generalize some results on abelian groups to the case of modules over bounded Dedekind prime rings. After several definitions (section 1), we give, in section 2, some properties on bounded Dedekind prime rings. In section 3, we investigate the structure and properties of modules over bounded Dedekind prime rings. For finitely generated modules, we give a complete structure theorem (Theorems 3.1 and 3.38). Any torsion module is a direct sum of primary modules (Theorem 3.2) and so the study of torsion modules is reduced to that of primary modules. In Theorem 3.33, we give a necessary and sufficient condition for a primary module to be direct sum of cyclic modules. A module is called decomposable if it is isomorphic to a direct sum of uniform right ideals and cyclic modules. Then it can be shown that any submodule of a decomposable module is also decomposable (Theorem 3.36). This is a generalization of the result on modules over commutative Dedekind domains. We define, in section 2, the concept of divisible modules and show that any divisible module is a direct sum of indecomposable divisible modules (Theorem 3.18). In particular, we show that an indecomposable divisible  $R$ -module is either isomorphic to a minimal right ideal of  $Q$  or a module of type  $P^\infty$  (Lemma 3.16 and Theorem 3.17), where  $Q$  is the quotient ring of the bounded Dedekind prime ring  $R$ . We also study the ring of endomorphisms of divisible indecomposable modules and give a complete structure theorem for those rings (Theorem 3.21).

Most of the results in this paper were announced without proofs in [17].

### 1. Definitions and notations

In this paper, all rings have identity and are associative and modules are unitary. Ideals always mean two-sided ideals. A ring  $R$  is called a *Goldie ring* if  $R$  satisfies the ascending chain condition annihilator right (left) ideals and has no infinite direct sum of non-zero right (left) ideals. Let  $R$  be a prime Goldie ring. Then, by virtue of Goldie's theorem [8],  $R$  has a quotient ring  $Q$  which is a simple artinian ring. A prime Goldie ring  $R$  is called a *Dedekind ring* if  $R$

is a maximal order in  $Q$  and every essential one-sided ideal of  $R$  is projective (see [22]), or equivalently [5]  $R$  is a hereditary Noetherian prime ring with no proper idempotent two-sided ideals. Again, by Goldie's theorem, a one-sided ideal  $I$  of  $R$  is essential if and only if  $I$  contains a regular element of  $R$ .  $R$  is *bounded* if every essential one-sided ideal of  $R$  contains a non-zero ideal. Let  $M$  be a right  $R$ -module. We say that  $m \in M$  is a *torsion element* if there is a regular element  $c$  in  $R$  such that  $mc=0$ . Since  $R$  satisfies the Ore condition, the set of torsion elements of  $M$  is a submodule  $T \subseteq M$ . And  $M/T$  is evidently torsion-free (has no torsion elements). Let  $x$  be an element of  $M$ . Then we define  $O(x) = \{r \in R \mid xr=0\}$  and say that  $O(x)$  is an *order right ideal* of  $x$ . Analogously, for a submodule  $N$  of  $M$ , we define  $O(N) = \{r \in R \mid Nr=0\}$  and say that  $O(N)$  is an *order ideal* of  $N$ . Let  $P$  be a prime ideal of  $R$  and let  $M$  be a torsion  $R$ -module. Then we say that  $M$  is *primary* ( $P$ -primary) if  $O(x)$  contains a power of  $P$  for every element  $x$  in  $M$ . A submodule  $S$  of an  $R$ -module  $M$  is said to be *pure* if  $Sc = S \cap Mc$  for every regular element  $c$  in  $R$ . In particular,  $S$  is said to be *strongly pure* (s-pure) if  $Sr = S \cap Mr$  for every element  $r$  in  $R$ . Then the following properties hold: (i) Any direct summand is strongly pure. (ii) A (strongly) pure submodule of a (strongly) pure submodule is (strongly) pure. (iii) The torsion submodule is pure. (iv) If  $M/S$  is torsion-free, then  $S$  is pure. We define an  $R$ -module  $M$  to be *divisible* if  $Mc = M$  for all regular element  $c$  in  $R$ . If an  $R$ -module  $M$  is  $n$ -dimensional in the sense of Goldie [8], then we write  $n = \dim M$ . A submodule  $U$  of an  $R$ -module  $M$  is *uniform* if any two non-zero submodules of  $U$  have a non-zero intersection. A right ideal  $I$  of a ring  $R$  is *uniform* if  $I$  is a uniform submodule when considered as a right  $R$ -module.  $J$  or  $J(R)$  always denotes the Jacobson radical of the ring  $R$ . The ring  $R$  is *local* if  $R/J$  is artinian and  $\bigcap_{s=1}^{\infty} J^s = 0$ .  $R$  is a *discrete valuation ring* if  $R$  is a local and pri-pli-domain (i.e., a principal right and left ideal domain) and  $R/J$  is a division ring. Finally, if  $R$  is any ring, then  $(R)_n$  will always denote the ring of all  $n \times n$  matrices over  $R$  and  $e_{i,j}$  will denote the matrix with 1 in the  $(i, j)$  position and zero elsewhere.

## 2. Bounded Dedekind prime rings

In this section,  $R$  will denote a bounded Dedekind prime ring and  $Q$  will denote the quotient ring of  $R$ . Let  $P$  be a non-zero prime ideal of  $R$ . Now we put  $\mathcal{C}(P) = \{c \in R \mid cx \in P \Rightarrow x \in P\}$ . Then each  $c$  in  $\mathcal{C}(P)$  has an inverse  $c^{-1}$  in  $Q$ . We denote the subring of  $Q$  generated by the elements of the form  $\{ac^{-1} \mid a \in R, c \in \mathcal{C}(P)\}$  by  $R_P$  and  $R_P$  is called a *local ring* of  $R$  with respect to  $P$ . Following [1], we put  $A(P) = \{q \in Q \mid qB \subseteq R \text{ for some non-zero ideal } B \text{ of } R \text{ not contained in } P\}$ . We call  $A(P)$  an *Asano's local ring* of  $R$  with respect to  $P$ .

**Lemma 2.1.** *Let  $P$  be a non-zero prime ideal of a bounded Dedekind prime ring  $R$ . Then*

- (i)  $R_P = A(P)$ .
- (ii)  $R$  satisfies the right and left Ore condition with respect to  $C(P)$  and  $R_P$  is a bounded Dedekind prime local ring which is a pri-*pli*-ring.
- (iii)  $R_P = (L)_n$ , where  $L$  is a local pri-*pli*-domain.

Proof. (i) and (ii) follow from Lemmas 2.12, 3.4 and Theorem 2.14 of [20].  
 (iii) follows from Corollary 3.10 of [20].

REMARK. The domain  $L$  in Lemma 2.1 is an Asano order in its division quotient ring. Hence if  $J(L) = p_0L$ , where  $p_0 \in L$ , then  $J(L) = Lp_0$ .

Let  $R_P$  be the local ring of  $R$  with respect to  $P$  and let  $P' = J(R_P)$ . Then  $R_P/P^n$  is an artinian ring by Theorem 1.3 of [6] and the mapping:  $R_P/P^{n+1} \rightarrow R_P/P^n$  defined by  $q + P^{n+1} \rightarrow q + P^n$ , for  $q \in R_P$ , is an  $R_P$ -homomorphism. By Goldie's theorem [9], the inverse limit  $\hat{R}_P$  of the ring  $R_P/P^n$ ,  $n = 1, 2, \dots$ , is a complete local ring. We call  $\hat{R}_P$  the completion of  $R_P$  with respect to  $P'$ . Since  $R_P/P^n \cong R/P^n$ , clearly,  $\hat{R}_P$  coincides the completion  $\hat{R}$  of  $R$  with respect to  $P$  (see [9]).

**Lemma 2.2.**  $\hat{R}_P$  is a bounded local Dedekind prime ring which is a pri-*pli*-ring and  $\hat{R}_P = (D)_n$ , where  $D$  is a complete discrete valuation ring.

Proof. For a convenience, we let  $R$  be a local Dedekind prime ring which is a pri-*pli*-ring and let  $P = J(R)$ . Then by Lemma 2.1,  $R = (L)_n$  and  $J(L) = P_0 = p_0L = Lp_0$  for some  $p_0 \in L$ . Then  $P = (P_0)_n = p_0R = Rp_0$ . Now we shall show that  $\hat{P} = J(\hat{R}) = \hat{p}_0\hat{R}$ , where  $\hat{p}_0 = (0, p_0 + P^2, p_0 + P^3, \dots)$ . It is clear that  $\hat{P} \supseteq \hat{p}_0\hat{R}$ . Let  $\hat{q} = (q_1, q_2, \dots)$  be any element of  $\hat{P}$ , where  $q_n = r_n + P^n \in R/P^n$ . Then clearly  $q_1 = 0$  and  $r_{n+1} \in P$  for every  $n$  (see [9]). Hence  $r_{n+1} = p_0s_{n+1}$ . On the other hand, since  $r_{n+1} + P^n = r_n + P^n$ , we obtain  $r_{n+1} - r_n = p_0(s_{n+1} - s_n) \in P^n$  and thus  $s_{n+1} - s_n \in P^{n-1}$ . Hence  $\hat{q}_1 = (s_2 + P, s_3 + P^2, \dots) \in \hat{R}$  and  $\hat{q} = \hat{p}_0\hat{q}_1 \in \hat{p}_0\hat{R}$ , as desired. In the same way, we obtain  $\hat{P} = \hat{R}\hat{p}_0$ . By Theorem 3.5 of [21],  $\hat{R}$  is either artinian primary or a prime ring. But  $\hat{R}$  is not artinian, because  $\hat{P}^n \supseteq P^n$  and  $P^n \neq 0$  for every  $n$ . Hence  $\hat{R}$  is a prime ring. By Theorem 5.1 of [21] and Theorem 4.5 of [9],  $\hat{R} = (D)_n$ ,  $D$  is a complete discrete valuation ring. Hence  $\hat{R}$  is a pri-*pli*-ring and thus is a Dedekind prime ring. Since  $\hat{P}$  is a Jacobson radical of  $\hat{R}$ ,  $\hat{R}$  is bounded by Theorem 4.13 of [5].

REMARK. If  $D$  is a discrete valuation ring with maximal ideal  $P_0$ , then, by Theorem 3.1 of [11, p 112], every one-sided ideal is two-sided and the ideals of  $D$  are only the powers of  $P_0$ . Let  $P_0 = p_0D = Dp_0$  with  $p_0 \in D$ . Then every element  $d \neq 0$  of  $D$  has a unique representation of the form  $d = p_0^n s = t p_0^n$ , where  $s$  and  $t$  are units in  $D$ .

An idempotent  $e$  in a ring  $R$  is *uniform* if  $eR$  is a uniform right ideal of  $R$ .

**Lemma 2.3.** Let  $P$  be a non-zero prime ideal of a bounded Dedekind prime

ring  $R$  and let  $e, f$  be any uniform idempotents in  $\hat{R}_P$ . Then we obtain: (In (i)~(v),  $\hat{P}=J(\hat{R}_P)$ ).

- (i)  $e\hat{R}_P/e\hat{P}$  is a simple  $\hat{R}_P$ -module.
- (ii)  $e\hat{R}_P/e\hat{P}^n \cong f\hat{R}_P/f\hat{P}^n$  for  $n=0, 1, 2, \dots$ .
- (iii)  $e\hat{R}_P/e\hat{P}^n \cong e\hat{P}^m/e\hat{P}^{m+n}$  for  $n=0, 1, 2, \dots$ .
- (iv) Let  $g$  be an idempotent in  $\hat{R}_P$ . Then  $g$  is a uniform idempotent in  $\hat{R}_P$  if and only if  $\bar{g}=g+\hat{P}$  is a primitive idempotent in  $\hat{R}_P/\hat{P}$ .
- (v) Idempotents in  $\hat{R}_P/\hat{P}$  can be lifted to  $\hat{R}_P$ .

Proof. For a convenience, we let  $R$  be a local, Dedekind prime, complete ring which is a pri-pli-ring and let  $P=J(R)$ . By Lemma 2.2,  $R=(D)_k$ , where  $D$  is a complete and discrete valuation ring with maximal ideal  $P_0$  and  $P=p_0R=Rp_0$  with  $P_0=p_0D=Dp_0$ . Furthermore we put  $\bar{R}=R/P$  and denote the image of  $x$  in  $\bar{R}$  by  $\bar{x}$  for every element  $x$  in  $R$ . Then  $\bar{e}_{11}\bar{R}=e_{11}R/e_{11}P$  is a simple  $R$ -module by Lemma 2 of [15, p 76], because  $D/P_0$  is a division ring. Now the map:  $e_{11}R/e_{11}P^n \rightarrow e_{11}P^m/e_{11}P^{m+n}$  defined by  $e_{11}q+e_{11}P^n \rightarrow e_{11}p_0^m q+e_{11}P^{m+n}$ , for  $q \in R$ , is an  $R$ -isomorphism, because  $e_{11}P^n=e_{11}p_0^n R$ . Hence (i)~(iii) follow from the fact that  $eR \cong e_{11}R$  for every uniform idempotent  $e$  in  $R$ . But it is shown in Theorem 2.2 of [11, p 46]. To prove (iv), suppose that  $g$  is a uniform idempotent. Then, by Lemma 2 of [15, p 76] and (ii),  $\bar{g}$  is a primitive idempotent in  $\bar{R}$ . Conversely, suppose that  $\bar{g}$  is primitive, i.e.,  $\bar{g}\bar{R}$  is a minimal right ideal of  $\bar{R}$  and so  $\bar{g}\bar{R} \cong \bar{e}_{11}\bar{R}$ . Hence  $gR \cong e_{11}R$  by Proposition 1 of [10, p 53] and thus  $g$  is a uniform idempotent in  $R$ .

Since  $\dim(R/P)=\dim R$ , where  $\dim R$  denotes the dimension of  $R$  in the sense of Goldie, (v) immediately follows from Hilfssatz 3.7 of [19].

### 3. Modules over bounded Dedekind prime rings

Let  $R$  be a semi-hereditary prime Goldie ring, let  $Q$  be the quotient ring of  $R$  and let  $M$  be a finitely generated torsion-free  $R$ -module. Then the sequence  $0 \rightarrow M \rightarrow M \otimes_R Q$  is exact and  $M \otimes_R Q$  is  $Q$ -projective, because  $Q$  is a simple artinian ring. So  $M \otimes_R Q$  is a submodule of a finitely generated free  $Q$ -module. Furthermore, since  $M$  is finitely generated,  $M$  is a submodule of a free  $R$ -module. Hence  $M$  is  $R$ -projective by Proposition 6.2 of [3]. On the other hand,  $R$  is a direct sum of a finite number of uniform right ideals by Hilfssatz 3.6 of [19]. Hence, by the same way as the proof in Proposition 6.1 of [3], we have

**Theorem 3.1.** *Let  $R$  be a semi-hereditary prime Goldie ring and let  $M$  be a finitely generated  $R$ -module with torsion submodule  $T$ . Then*

- (i)  $M/T$  is a projective  $R$ -module and is a direct sum of a finite number of uniform right ideals.
- (ii)  $M=T \oplus M/T$ .

From now on,  $R$  will be a bounded Dedekind prime ring which is not an artinian ring and  $Q$  will be the simple artinian quotient ring of  $R$ .

**Theorem 3.2.** *Any torsion module over a bounded Dedekind prime ring is a direct sum of primary submodules.*

Proof. Let  $M$  be a torsion module over a bounded Dedekind prime ring  $R$  and let  $M_P$  be the  $P$ -primary submodule of  $M$  for every prime ideal  $P$ . Then it is clear that the sum  $\sum M_P$  is direct, because  $R$  is an Asano order. Let  $x$  be a non-zero element of  $M$ . Then  $xc=0$  for some regular element  $c$  in  $R$ . Since  $cR$  is an essential right ideal, there exists a non-zero ideal  $A$  such that  $cR \supseteq A$ . Since  $A \neq R$ , there are positive integers  $n_1, n_2, \dots, n_k$  and maximal ideals  $P_i$  ( $i=1, 2, \dots, k$ ) such that

$$A = P_1^{n_1} P_2^{n_2} \dots P_k^{n_k},$$

because  $R$  is an Asano order. It is clear that  $\sum_i P_1^{n_1} \dots P_{i-1}^{n_{i-1}} P_{i+1}^{n_{i+1}} \dots P_k^{n_k} = R$  and  $x(P_1^{n_1} \dots P_{i-1}^{n_{i-1}} P_{i+1}^{n_{i+1}} \dots P_k^{n_k}) \subseteq M_{P_i}$ .

Hence

$$x \in xR = \sum_i x(P_1^{n_1} \dots P_{i-1}^{n_{i-1}} P_{i+1}^{n_{i+1}} \dots P_k^{n_k}) \subseteq M_{P_1} \oplus \dots \oplus M_{P_k}.$$

This completes the proof of Theorem 3.2.

By Theorem 3.2, the study of torsion modules is reduced to that of primary modules.

**Lemma 3.3.** *Let  $P$  be a non-zero prime ideal of a bounded Dedekind prime ring  $R$  and let  $M$  be a  $P$ -primary module. Then  $M$  is in a natural way an  $R_P$ -module.*

Proof. We put  $\mathcal{C} = \{c \in R \mid cx \in P \Rightarrow x \in P\}$ . (i) We shall first show that  $Mc = M$  for every  $c \in \mathcal{C}$ . To prove this, let  $x$  be a non-zero element of  $M$ . Then there is an integer  $n$  such that  $xP^n = 0$ . By Theorem 4.2 of [9] and Proposition 2.5 of [20],  $c + P^n$  is a regular element in  $R/P^n$  for every  $c \in \mathcal{C}$ . Since  $R/P^n$  is an artinian ring,  $c + P^n$  is a unit and thus we obtain  $cR + P^n = R$  and  $Rc + P^n = R$ . We have then

$$x \in xR = x(Rc + P^n) = xRc.$$

Hence  $x = x_1c$  for some  $x_1 \in M$  and thus  $M = Mc$  for every  $c \in \mathcal{C}$ . (ii) We shall prove that if  $xc = 0$ , where  $x \in M$  and  $c \in \mathcal{C}$ , then  $x = 0$ . By (i), there exists an integer  $n$  such that  $xP^n = 0$  and  $cR + P^n = R$ . So if  $xc = 0$ , then  $x \in xR = x(cR + P^n) = 0$ , as desired.

Now if  $0 \neq x \in M$  and  $c \in \mathcal{C}$ , then the solution  $x_1$  of  $x = x_1c$  is unique by (ii), and we can define  $xc^{-1} = x_1$ , it is easily verified that this definition makes  $M$  into

an  $R_P$ -module.

**Lemma 3.4.** (Kaplansky [12]). *Let  $M$  be any module, let  $S$  be a submodule such that  $M/S$  is a direct sum of modules  $U_i$ , and let  $T_i$  be the inverse image in  $M$  of  $U_i$ . Suppose that  $S$  is a direct summand of each  $T_i$ . Then  $S$  is a direct summand of  $M$ .*

**Lemma 3.5.** *Let  $R$  be a bounded Dedekind prime ring, let  $M$  be an  $R$ -module and let  $S$  be a pure submodule such that  $M/S$  is torsion. If  $x_0$  is a non-zero element of  $M/S$ , then there exists an element  $x$  in  $M$ , which maps on  $x_0 \bmod S$ , and  $O(x) = O(x_0)$ .*

Proof. (i) We shall first show that the lemma holds if  $R$  is a right principal prime ring. To prove this we put  $O(x_0) = cR$ , where  $c$  is a regular element of  $R$  and let  $\sigma: M \rightarrow M/S$  be the canonical epimorphism. First choose any  $z$  in  $M$  such that  $\sigma(z) = x_0$ . Then  $\sigma(zc) = x_0c = 0$  and thus  $zc \in S$ . By the purity of  $S$ , there exists an element  $s \in S$  with  $zc = sc$ . Set  $x = z - s$ . Then  $x$  has the desired properties, that is,  $x$  maps on  $x_0 \bmod S$ , and  $O(x) = O(x_0)$ .

(ii) If  $R$  is a bounded Dedekind prime ring, then, by the validity of (i), the proof of the lemma proceeds just like that of Lemma 4 of [12] did.

We shall call an  $R$ -module *decomposable* if it is a direct sum of cyclic modules and uniform right ideals.

By Lemmas 3.4 and 3.5, we have

**Theorem 3.6.** *Let  $R$  be a bounded Dedekind prime ring, let  $M$  be an  $R$ -module and let  $S$  be a pure submodule such that  $M/S$  is decomposable. Then  $S$  is a direct summand of  $M$ .*

Since every proper homomorphic image of a bounded Dedekind prime ring is uniserial, by Theorem 2.54 of [1, p 79] we have

**Theorem 3.7.** *Let  $R$  be a bounded Dedekind prime ring and let  $M$  be an  $R$ -module of bounded order (i.e.,  $Mc = 0$  for some regular element  $c$  of  $R$ ). Then  $M$  is a direct sum of cyclic modules, each of which is an artinian module.*

Since a finitely generated torsion  $R$ -module is of bounded order we obtain the following corollary by Theorems 3.1 and 3.7.

**Corollary 3.8.** *Let  $R$  be a bounded Dedekind prime ring and let  $M$  be a finitely generated  $R$ -module. Then  $M$  is decomposable.*

**Corollary 3.9.** *Let  $R$  be a bounded Dedekind prime ring and let  $M$  be a finitely generated  $R$ -module. If  $S$  is a submodule of  $M$ , then the following three conditions are equivalent:*

- (i)  $S$  is a direct summand of  $M$ ;
- (ii)  $S$  is an  $s$ -pure submodule of  $M$ ;

(iii)  $S$  is a pure submodule of  $M$ .

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are clear. (iii) $\Rightarrow$ (i) follows from Theorem 3.6 and Corollary 3.8.

**Theorem 3.10.** *Let  $R$  be a bounded Dedekind prime ring,  $M$  be an  $R$ -module and let  $S$  be a submodule of  $M$ . Then the following conditions are equivalent:*

- (i)  $S$  is  $s$ -pure in  $M$ ;
- (ii)  $S$  is pure in  $M$ ;
- (iii) If  $N$  is a submodule between  $S$  and  $M$  such that  $N/S$  is decomposable, then  $S$  is a direct summand of  $N$ ;
- (iv) If the system

$$\sum_{j=1}^m x_j r_{ij} = s_i (s_i \in S, r_{ij} \in R, i \in I)$$

with a finite number  $m$  of unknowns is solvable in  $M$ , then it possesses a solution in  $S$ , too;

- (v) Every coset of  $M$  modulo  $S$  contains an element of the same order as this coset.

Proof. (i) $\Rightarrow$ (ii): This is a special case.

(ii) $\Rightarrow$ (iii): This follows from Theorem 3.6.

(iii) $\Rightarrow$ (iv): Assume that the system  $\sum_{j=1}^l x_j r_{ij} = s_i$  is solvable in  $M$  and that  $m_j$  ( $1 \leq j \leq l$ ) is a solution in  $M$ . Let  $N$  be the submodule generated by  $m_j$  and  $S$ . Then, by the assumption and Corollary 3.8,  $N = S \oplus K$ , where  $K$  is a submodule of  $M$ . Now let  $m_j = s'_j + k_j$  ( $1 \leq j \leq l$ ) with  $s'_j \in S$  and  $k_j \in K$ . Then clearly  $s'_j$  is a solution in  $S$ .

(iv) $\Rightarrow$ (v): Let  $m$  be a non-zero element of  $M$  and let  $\bar{m}$  be the image of  $m$  in  $M/S$ . If  $O(\bar{m}) = 0$ , then  $O(m) = 0$ . If  $O(\bar{m}) = K \neq 0$  is a right ideal of  $R$ , then the system

$$mr_i = s_i \in S$$

where  $r_i$  runs all over elements in  $K$ . By the assumption, there exists an element  $s \in S$  such that  $sr_i = s_i$ . We put  $m_1 = m - s$ , then  $\bar{m}_1 = \bar{m}$  and  $O(m_1) \supset K$ , because  $m_1 r_i = mr_i - sr_i = s_i - s_i = 0$  for every  $r_i \in K$ . Therefore we have that  $O(m_1) = K$ .

(v) $\Rightarrow$ (i); If  $mr = s$  ( $m \in M, r \in R, s \in S$ ), then  $O(\bar{m}) \ni r$ .

By the assumption, there exists an element  $m_1 = m - s_1$  ( $s_1 \in S$ ) of  $M$  such that  $m_1 r = 0$  and so  $s_1 r = (m - m_1)r = s$ , as desired.

By Theorem 3.10, we have

**Corollary 3.11.** *Let  $S$  be a submodule of an  $R$ -module  $M$ . Then  $S$  is pure*

in  $M$  if and only if  $SA=MA \cap S$  for every subset  $A$  of  $R$ .

**Theorem 3.12.** *Let  $R$  be a bounded Dedekind prime ring, let  $M$  be an  $R$ -module and let  $S$  be a pure submodule of bounded order. Then  $S$  is a direct summand of  $M$ .*

*Proof.* We put  $P=S+McR$ , where  $c$  is a regular element contained in  $O(S)$ . Then  $S \cap McR=0$ , because  $S$  satisfy (iv) of Theorem 3.10. Hence, by the same argument as in Theorem 5 of [12], we obtain that  $S$  is a direct summand of  $M$ .

By Theorem 3.4 of [16] and Theorem 3.12, we have

**Corollary 3.13.** *Let  $R$  be a bounded Dedekind prime ring, let  $M$  be an  $R$ -module with torsion submodule  $T$ . Suppose that  $T$  is the direct sum of a divisible submodule and a submodule of bounded order. Then  $T$  is a direct summand of  $M$ .*

**Lemma 3.14.** *Let  $M$  be a  $P$ -primary  $R$ -module. Then  $M$  is in a natural way an  $\hat{R}_P$ -module and  $M$  is torsion as an  $\hat{R}_P$ -module.*

*Proof.* Let  $q=(r_1+P', r_2+P'^2 \dots)$  be a non-zero element of  $\hat{R}_P$ , where  $P'=PR_P=R_PP$ ,  $r_i \in R_P$  and  $r_n-r_{n-1} \in P'^{n-1}$ , and let  $x$  be a non-zero element of  $M$ . Since  $M$  is  $P$ -primary, there exists an integer  $n_0$  such that  $xP^{n_0}=0$  and so  $xr_n=xr_{n_0}$  for every  $n \geq n_0$ . Thus if we define  $xq=xr_n (n \geq n_0)$ , it is easily verified that this definition makes  $M$  into an  $\hat{R}_P$ -module. Since  $\hat{P}=P\hat{R}_P=\hat{R}_PP$  by Lemmas 2.1 and 2.2,  $M$  is torsion as an  $\hat{R}_P$ -module.

**Lemma 3.15.** *Let  $R$  be a bounded Dedekind prime ring. Then*

- (i) *Every simple  $R$ -module is primary.*
- (ii) *An  $R$ -module  $M$  is simple and  $P$ -primary if and only if  $M$  is isomorphic to  $e\hat{R}_P/e\hat{P}$  for some uniform idempotent  $e$  in  $\hat{R}_P$ , where  $P$  is a prime ideal of  $R$  and  $\hat{P}=J(\hat{R}_P)$ .*

*Proof.* (i) If a simple  $R$ -module  $M$  is torsion-free, then, by Theorem 3.1,  $M$  is isomorphic to a uniform right ideal of  $R$ . This is a contradiction, because  $R$  is not a simple artinian ring. Hence  $M$  is primary.

(ii) By Lemma 2.3,  $e\hat{R}_P/e\hat{P}$  is a simple and  $P$ -primary module. Conversely, if  $M$  is simple and  $P$ -primary, then  $MP=0$ , because  $M$  is of bounded order. By Lemma 3.14, we can assume without loss of generality that  $R$  is a complete, local, bounded Dedekind prime ring with maximal ideal  $P$ . Then  $M$  is an  $\bar{R}$ -module, where  $\bar{R}=R/P$ . Now let  $x$  be a non-zero element in  $M$ . Then  $M=x\bar{R}$  and the exact sequence  $0 \rightarrow O(x) \rightarrow \bar{R} \rightarrow x\bar{R} \rightarrow 0$  splits, because  $\bar{R}$  is a simple artinian ring. So  $\bar{R}=O(x) \oplus \bar{e}\bar{R}$  and  $\bar{e}\bar{R}$  is a minimal right ideal. By Lemma 2.3, we may assume that  $e$  is a uniform idempotent in  $R$  and thus we obtain  $M \cong \bar{e}\bar{R} \cong eR/eP$ , as desired.



We denote the injective hull of an  $R$ -module  $A$  by  $E(A)$ .

**Lemma 3.16.** *Let  $R$  be a bounded Dedekind prime ring and let  $Q$  be the quotient ring of  $R$ . If  $E$  is an indecomposable injective  $R$ -module, then  $E$  is either isomorphic to a minimal right ideal of  $Q$  or isomorphic to  $E(e\hat{R}_P/e\hat{P})$ , where  $P$  is a prime ideal of  $R$  and  $e$  is a uniform idempotent contained in  $\hat{R}_P$ . Furthermore,  $E(e\hat{R}_P/e\hat{P})$  is  $P$ -primary.*

Proof. It is clear that  $E$  can not be mixed.

(i) If  $E$  is torsion-free and  $x$  is a non-zero element of  $E$ , then  $xR = \sum_{i=1}^n \oplus I_i$  by Theorem 3.1, where  $I_i$  is a uniform right ideal of  $R$  and thus  $E = E(xR) = \sum_{i=1}^n \oplus I_i Q$ . So  $n=1$  and  $E = I_1 Q$  is a minimal right ideal of  $Q$ .

(ii) If  $E$  is torsion, then  $E$  is primary by Theorem 3.2. Suppose that  $E$  is  $P$ -primary and  $x$  is a non-zero element of  $E$ . Then  $xR$  is an  $R/P^n$ -module for some  $n$  and thus  $xR$  is an artinian module, because  $R/P^n$  is artinian. By Lemma 3.15, we obtain  $xR \supseteq e\hat{R}_P/e\hat{P}$  for some uniform idempotent  $e$  contained in  $\hat{R}_P$ . Hence  $E = E(xR) = E(e\hat{R}_P/e\hat{P})$ , as desired.

By Lemmas 3.15 and 3.16, we obtain the following remarks:

REMARKS. (i) Torsion-free simple modules do not exist and the torsion-free indecomposable injective module is unique up to isomorphism.

(ii) The primary simple module is unique up to isomorphism.

(iii) The primary indecomposable injective module is unique up to isomorphism.

Further, we shall give a characterization of  $E(e\hat{R}_P/e\hat{P})$ . Let  $\hat{R}_P = (D)_h$ , let  $\hat{P} = p_0 \hat{R}_P = \hat{R}_P p_0$ , where  $D$  is a discrete valuation ring with maximal ideal  $P_0 = p_0 D = D p_0$  (see Lemma 2.2) and let  $e = e_{11}$ . By Lemma 2.3, the sequence

$$(*) \quad 0 \rightarrow e\hat{R}_P/e\hat{P}^n \xrightarrow{\varphi_n} e\hat{R}_P/e\hat{P}^{n+1}$$

is exact, where  $\varphi_n(eq + e\hat{P}^n) = ep_0 q + e\hat{P}^{n+1}$  for every  $q$  in  $\hat{R}_P$ .

**Theorem 3.17.** *The inductive limit  $E = \varinjlim e\hat{R}_P/e\hat{P}^n$  of the  $R$ -modules  $e\hat{R}_P/e\hat{P}^n$ ,  $n=1, 2, \dots$ , under the homomorphism defined in  $(*)$ , is divisible and is isomorphic to  $E(e\hat{R}_P/e\hat{P})$ . In particular,  $E$  is isomorphic to the injective hull  $E_{\hat{R}_P}(e_P \hat{R}_P/e\hat{P})$  of the  $\hat{R}_P$ -module  $e\hat{R}_P/e\hat{P}$ .*

Proof. We first prove that  $Ee = \varinjlim D/P_0^n$  is a divisible  $D$ -module, where  $P_0 = J(D)$ . Let  $x = y + P_0^n$  be a non-zero element of  $Ee$ , where  $y \in D$  and let  $d$  be a non-zero element of  $D$ . We put  $d = p_0^k s = t p_0^k$  and  $y = p_0^l u = v p_0^l$ , where  $s, t, u$  and  $v$  are units in  $D$  and  $n > l \geq 0$ . Then  $x = p_0^k u + P_0^{n-l+k}$  in  $Ee$ . For the element  $p_0^k u$ , there exists a unit  $w$  in  $D$  such that  $p_0^k u = w p_0^k$  and thus  $x = (wt^{-1} + P_0^{n-l+k})d$ . Hence  $Ee$  is a divisible  $D$ -module. By Theorem 3.4 and

Corollary 2.3 of [16],  $E$  is a divisible  $\hat{R}_P$ -module. Since  $E$  is a maximal essential extension of  $e\hat{R}_P/e\hat{P}$  as an  $R$ -module, we obtain  $E \cong E_{\hat{R}_P}(e\hat{R}_P/e\hat{P}) \cong E(e\hat{R}_P/e\hat{P})$  by E. Eckmann and A. Schopf [4].

REMARK. The module  $E$  in Theorem 3.17 is a natural generalization of the typical divisible, torsion, abelian group  $Z_{p^\infty}$ .

We call the module  $E(e\hat{R}_P/e\hat{P})$  a module of type  $P^\infty$ .

By Theorems 1.4, 2.5 of [18], Theorem 3.4 of [16] and Lemma 3.16 we obtain the following two theorems:

**Theorem 3.18.** *Let  $R$  be a bounded Dedekind prime ring with quotient ring  $Q$ . Then any divisible  $R$ -module is the direct sum of minimal right ideals of  $Q$  and modules of type  $P^\infty$  for various prime ideals  $P$ .*

**Theorem 3.19.** *Any module  $M$  over a bounded Dedekind prime ring possesses a unique largest divisible submodule  $D$ ;  $M = D \oplus K$ , where  $K$  has no divisible submodules.*

**Lemma 3.20.** *Let  $R$  be a complete, local, bounded Dedekind prime ring with unique maximal ideal  $P$  and let  $e$  be a uniform idempotent of  $R$ . Then*

- (i)  $H_n = \text{Hom}_R(eR/eP^n, eR/eP^n) \cong eRe/eP^n e$ .
- (ii)  $eRe/eP^n e$  is completely primary in the sense of [2].

Proof. (i) Let  $\alpha$  be a non-zero element of  $H_n$  and let  $\alpha(e+eP^n) = ere + eP^n$ . Then the mapping  $\theta: H_n \rightarrow eRe/eP^n e$  defined by  $\theta(\alpha) = ere + eP^n e$  is an isomorphism.

(ii) is clear.

**Theorem 3.21.** *Let  $P$  be a prime ideal of a bounded Dedekind prime ring  $R$  and let  $E$  be an  $R$ -module of type  $P^\infty$ . Then the endomorphism ring of  $E$  is isomorphic to  $e\hat{R}_P e$ , where  $e$  is a uniform idempotent in  $\hat{R}_P$ .*

Proof. It is easily seen that  $H = \text{Hom}_R(E, E) = \text{Hom}_{\hat{R}_P}(E, E)$ . Hence we can assume without loss of generality that  $R$  is a complete, local, bounded Dedekind prime ring with unique maximal ideal  $P$ . Furthermore, we may assume that  $R = (D)_k$  and  $e = e_{11}$ , where  $D$  is a discrete valuation ring. Since  $E = \varinjlim eR/eP^n$  by Theorem 3.17,  $E$  is in a natural way left  $eRe$ -module. Since  $\bigcap_n eP^n = 0$ ,  $E$  is a faithful left  $eRe$ -module. Consequently, we can identify  $eRe$  with the subring of  $H$  consisting of multiplications by elements of  $eRe$ . Let  $\alpha$  be a non-zero element of  $H$ . Then since  $\alpha(eR/eP^n)P^n = 0$ , we may assume that  $\alpha_n \in \text{Hom}_R(eR/eP^n, eR/eP^n)$ , where  $\alpha_n = \alpha|_{eR/eP^n}$  and so  $\alpha_n = er_n e + eP^n e$  by Lemma 3.20. Now we put  $\hat{r} = (er_1 e + P, er_2 e + P^2 \dots)$ . Then since  $\alpha_n = \alpha_{n-1}$  on  $eP/eP^n$ , we obtain  $(er_n e - er_{n-1} e)eP \subseteq eP^n$  and thus  $er_n e - er_{n-1} e \in P^{n-1}$ . Hence  $\hat{r} \in R$ , because  $R$  is complete. Since  $\hat{r} - er_n e \in P^n$ , it is easily checked that  $\alpha = \hat{r}$

and  $\hat{r} = \hat{e}r\hat{e}$ , which completes the proof.

**REMARK.** If  $E$  is a torsion-free indecomposable divisible  $R$ -module, then  $E$  is isomorphic to  $eQ$ , where  $e$  is a primitive idempotent in  $Q$ . Thus the ring of endomorphisms of  $E$  is isomorphic to  $eQe$  and so the rings of endomorphisms of indecomposable divisible  $R$ -modules are completely determined.

Let  $R$  be a local, bounded Dedekind prime ring with unique maximal ideal  $P$  and let  $M$  be an  $R$ -module. An element  $x$  in  $M$  has *height  $n$*  if  $x \in MP^n$  and  $x \notin MP^{n+1}$ , it has *infinite height* if  $x \in MP^n$  for every  $n$ . We write  $h(x)$  for the height of  $x$ ; thus  $h(x)$  is a (non-negative) integer or the symbol  $\infty$ . If  $x$  lies in a submodule  $S$  of  $M$ , we may define two heights for  $x$ . When it is necessary to make a distinction, we shall write  $h_S(x)$  and  $h_M(x)$  for the height of  $x$  in  $S$  and  $M$ , respectively. Note that we always have  $h_S(x) \leq h_M(x)$ . If  $h(x)$  and  $h(y)$  are unequal, then  $h(x+y)$  is precisely the smaller of the two. If  $h(x) = h(y)$ , then  $h(x+y) \geq h(x)$ . Let  $D$  be a discrete valuation ring with maximal ideal  $P_0 = p_0D$  and let  $M$  be a  $D$ -module. Then an element  $x$  in  $M$  has height  $n$  if and only if it is divisible by  $p_0^n$  but not by  $p_0^{n+1}$ . Furthermore a submodule  $S$  of  $M$  is pure if and only if  $h_S(x) = h_M(x)$  for every  $x \in S$ .

By the same arguments as in Lemmas 7 and 8 of [13], we have the following two lemmas.

**Lemma 3.22.** *Let  $D$  be a discrete valuation ring with maximal ideal  $P_0 = p_0D$ , let  $M$  be a primary  $D$ -module and let  $S$  be a submodule with no elements of infinite height. Suppose that the elements of order  $P_0$  in  $S$  have the same height in  $S$  as in  $M$ . Then  $S$  is pure.*

**Lemma 3.23.** *Let  $D$  be a discrete valuation ring with maximal ideal  $P_0 = p_0D$  and let  $M$  be a primary  $D$ -module. Suppose that all elements of order  $P_0$  in  $M$  have infinite height. Then  $M$  is divisible.*

An  $R$ -module is said to be *reduced* if it has no non-zero divisible submodules.

**Theorem 3.24.** *Let  $R$  be a bounded Dedekind prime ring and let  $P$  be a prime ideal of  $R$ . If  $M$  is a  $P$ -primary reduced  $R$ -module, then  $M$  possesses a direct summand which is isomorphic to  $e\hat{R}_P/e\hat{P}^n$ , where  $e$  is a uniform idempotent contained in  $\hat{R}_P$ .*

**Proof.** We can assume without loss of generality that  $R$  is a complete, local, bounded Dedekind prime ring with maximal ideal  $P$  and that  $R = (D)_k$ , where  $D (\cong e_{11}Re_{11})$  is a discrete valuation ring with maximal ideal  $P_0 = p_0D$ . By Corollary 2.3 of [16],  $Me_{11}$  is reduced as a  $D$ -module. So, by Lemma 3.23, there exists an element  $xe_{11}$  in  $Me_{11}$  such that  $O(xe_{11}) = P_0$  and  $h(xe_{11}) = r < \infty$ . Now we put  $xe_{11} = ye_{11}p_0^r$  and  $H = ye_{11}D$ . Then it follows from Lemma 3.22

that  $H$  is pure and is of bounded order, because the elements of order  $P_0$  in  $H$  are only the multiples of  $xe_{11}$  by units in  $D$ . Hence, by Theorem 3.12,  $H$  is a direct summand of  $Me_{11}$  and we put  $Me_{11}=H\oplus K$ , where  $K$  is a  $D$ -submodule of  $Me_{11}$ . Since  $Me_{11}=Me_{i1}$ , we have

$$M = Me_{11} \oplus Me_{22} \oplus \cdots \oplus Me_{kk} = \sum_{i=1}^k \oplus Me_{11}e_{1i}.$$

Hence

$$M = \sum_{i=1}^k \oplus (H \oplus K)e_{1i} = He_{11}R \oplus Ke_{11}R.$$

On the other hand, the sequence  $0 \rightarrow O(ye_{11}) \rightarrow D \rightarrow ye_{11}D \rightarrow 0$  is exact and so  $O(ye_{11})=P_0^n$  for some  $n$ . Hence we obtain

$$ye_{11}R \cong e_{11}R / \sum_{i=1}^k e_{11}P_0^n e_{1i} = e_{11}R / e_{11}P^n.$$

This completes the proof of Theorem 3.24.

By Theorems 3.12 and 3.24, we have

**Corollary 3.25.** *Let  $R$  be a bounded Dedekind prime ring and let  $M$  be a reduced  $R$ -module which is not torsion-free. Then  $M$  possesses a direct summand which is isomorphic to  $e\hat{R}_P/e\hat{P}^n$ , where  $P$  is a prime ideal of  $R$  and  $e$  is a uniform idempotent in  $\hat{R}_P$ .*

By Theorem 3.17 and Corollary 3.25, we obtain

**Corollary 3.26.** *An indecomposable module over a bounded Dedekind prime ring  $R$  can not be mixed, i.e., it is either torsion-free or torsion. In the latter case it is either of type  $P^\infty$  or isomorphic to  $e\hat{R}_P/e\hat{P}^n$  for some prime ideal  $P$  of  $R$  and  $e$  is a uniform idempotent in  $\hat{R}_P$ .*

By the same arguments as in Lemmas 10, 11 and 12 of [13], we have the following three lemmas.

**Lemma 3.27.** *Let  $D$  be a discrete valuation ring with maximal ideal  $P_0=p_0D$ . Let  $M$  be a primary  $D$ -module, let  $H$  be a pure submodule and let  $x$  be an element of order  $P_0$  not in  $H$ . Suppose that  $h(x)=n < \infty$  and suppose further that  $h(x+a) \leq h(x)$  for every  $a$  in  $H$  with  $O(a)=P_0$ . If  $K$  is the cyclic submodule generated by  $y$  with  $x=yp_0^n$  and if  $L=H+K$ , then  $L$  is the direct sum of  $H$  and  $K$ , and  $L$  is pure again.*

Let  $M$  be a module over a bounded local Dedekind prime ring and we say that  $M$  is of *bounded height* if there exists a constant  $k$  such that  $h(x) \leq k$  for all  $x$  in  $M$ . A set  $\{x_i\}$  of elements of  $M$  is *pure independent* if the sum  $\sum x_i R$  is direct and pure in  $M$ .

**Lemma 3.28.** *Let  $D$  be a discrete valuation ring with maximal ideal  $P_0$ , let  $M$  be a primary  $D$ -module and let  $A$  be the submodule of elements  $x$  satisfying  $O(x)=P_0$ . Suppose that  $B, C$  are submodules of  $A$ , with  $C \subseteq B \subseteq A$ , and that  $B$  is of bounded height. If  $\{x_i\}$  is a pure independent set satisfying  $\sum_i \oplus x_i D \cap A = C$ , then  $\{x_i\}$  can be enlarged on a pure independent set  $\{y_j\}$  satisfying  $\sum_j \oplus y_j D \cap A = B$ .*

**Lemma 3.29.** *Let  $D$  be a discrete valuation ring with maximal ideal  $P_0 = p_0 D$ , let  $M$  be a primary  $D$ -module and let  $H$  be a pure submodule of  $M$  containing all the elements of order  $P_0$  in  $M$ . Then  $H = M$ .*

**Proposition 3.30.** *Let  $D$  be a discrete valuation ring with maximal ideal  $P_0$ , let  $M$  be a primary  $D$ -module and let  $A$  be the submodule of elements satisfying  $xP_0 = 0$ . Then a necessary and sufficient condition for  $M$  to be a direct sum of cyclic submodules is that  $A$  be the union of an ascending sequence of submodules of bounded height.*

Proof. By the validity of Lemmas 3.28 and 3.29, this follows from the same way as in Theorem 12 of [13].

**Proposition 3.31.** *Let  $P$  be a prime ideal of a bounded Dedekind prime ring  $R$ , let  $\hat{R}_P = (D)_k$  and let  $M$  be a  $P$ -primary module. Then  $M$  is a direct sum of cyclic  $R$ -modules if and only if  $Me_{11}$  is a direct sum of cyclic  $D$ -modules.*

Proof. It is clear that a  $P$ -primary module  $M$  is a direct sum of cyclic  $R$ -modules if and only if it is a direct sum of cyclic  $\hat{R}_P$ -modules. Hence we can assume without loss of generality that  $R$  is a complete, local, bounded Dedekind prime ring and that  $R = (D)_k$ , where  $D$  is a discrete valuation ring. If  $M = \sum_{\sigma} \oplus u_{\sigma} R$ , then  $Me_{11} = \sum_{\sigma} \oplus u_{\sigma} Re_{11}$  and  $u_{\sigma} Re_{11}$  is a finitely generated torsion  $D$ -module. Thus  $u_{\sigma} Re_{11}$  is a direct sum of cyclic  $D$ -modules by Corollary 3.8. Hence  $Me_{11}$  is a direct sum of cyclic  $D$ -modules.

Conversely, suppose that  $Me_{11} = \sum_{\sigma} \oplus u_{\sigma} D$ . Then we have

$$\begin{aligned} M &= Me_{11} \oplus \cdots \oplus Me_{kk} = Me_{11} \oplus \cdots \oplus Me_{11}e_{1k} \\ &= (\sum_{\sigma} \oplus u_{\sigma} D)e_{11} \oplus \cdots \oplus (\sum_{\sigma} \oplus u_{\sigma} D)e_{1k} \\ &= \sum_{\sigma} \oplus u_{\sigma} e_{11}(Re_{11} + \cdots + Re_{1k}) \\ &= \sum_{\sigma} \oplus u_{\sigma} e_{11} R \end{aligned}$$

, which completes the proof of Proposition 3.31.

**Lemma 3.32.** *Let  $P$  be a prime ideal of a bounded Dedekind prime ring  $R$  and let  $\hat{R}_P = (D)_k$ . If  $M$  is a  $P$ -primary module with no elements of infinite height. Then  $Me_{11}$  is a  $D$ -module with no elements of infinite height and if  $M$  is of bounded height, then so is  $Me_{11}$ .*

Proof. This is immediate.

**Theorem 3.33** (*Kulikov's Criterion* [13]). *Let  $P$  be a prime ideal of a bounded Dedekind prime ring  $R$ , let  $M$  be a  $P$ -primary module and let  $A$  be the submodule of elements  $x$  satisfying  $xP=0$ . Then a necessary and sufficient condition for  $M$  to be a direct sum of cyclic  $R$ -modules is that  $A$  be the union of an ascending sequence of submodules of bounded height.*

*Proof.* We can assume without loss of generality that  $R$  is a complete, local, bounded Dedekind prime ring and that  $R=(D)_k$ , where  $D$  is a discrete valuation ring and  $P_0=J(D)$ . The necessity is clear. To prove the sufficiency we suppose  $A=\cup_i A_i$ , where  $A_1\subseteq A_2\subseteq\cdots$ , and each  $A_i$  is of bounded height. Then it is clear that  $Ae_{11}=\{xe_{11}\in Me_{11}\mid xe_{11}P_0=0\}$  and  $A_i e_{11}$  is of bounded height as a  $D$ -module by Lemma 3.32. Hence  $Me_{11}$  is a direct sum of cyclic  $D$ -modules and so  $M$  is a direct sum of cyclic  $R$ -modules by Proposition 3.31.

By Theorem 3.33, we have the following two Corollaries:

**Corollary 3.34** (*Prufer's theorem* [13]). *Let  $R$  be a bounded Dedekind prime ring and let  $M$  be a countable primary  $R$ -module with no elements of infinite height. Then  $M$  is a direct sum of cyclic  $R$ -modules.*

**Corollary 3.35.** *Let  $R$  be a bounded Dedekind prime ring and let  $M$  be a primary  $R$ -module which is a direct sum of cyclic  $R$ -modules. Then any submodule  $N$  of  $M$  is a direct sum of cyclic  $R$ -modules.*

Now, by the validity of Theorems 3.2, 3.6 and Corollary 3.35, the proof of the following theorem proceeds just like that of Theorem 4 of [12] did.

**Theorem 3.36.** *Let  $R$  be a bounded Dedekind prime ring and let  $M$  be a decomposable  $R$ -module. Then any submodule of  $M$  is decomposable.*

**Lemma 3.37.** *Let  $P$  be a prime ideal of a bounded Dedekind prime ring  $R$  and let  $K$  be a cyclic uniform  $P$ -primary module. Then  $O(K)=P^n$  if and only if  $K$  is isomorphic to  $e\hat{R}_P/e\hat{P}^n$ , where  $e$  is uniform idempotent contained in  $\hat{R}_P$ .*

*Proof.* It follows immediately from Theorem 3.24.

Now, let  $M$  be a finitely generated  $R$ -module. Then  $M$  is a direct sum of uniform right ideals and uniform cyclic  $R$ -modules by Corollary 3.8, Theorems 3.1 and 3.24. Furthermore we have

**Theorem 3.38.** *Let  $R$  be a bounded Dedekind prime ring and let  $M$  be a finitely generated  $R$ -module. Then for a decomposition of  $M$  into the direct sum of uniform right ideals and uniform cyclic  $R$ -modules, suppose that:*

- (i) *the number of direct summands of uniform right ideals is  $r$ ,*
- (ii) *the number of  $P$ -primary cyclic summands for a given prime ideal  $P$  is*

$k_p$ , where  $k_p \geq 0$ , and that the orders of these summands are

$$P^{\alpha_{p1}}, P^{\alpha_{p2}}, \dots, P^{\alpha_{pk_p}},$$

where  $\alpha_{p1} \geq \alpha_{p2} \geq \dots \geq \alpha_{pk_p}$ .

For a decomposition of any submodule  $N$  of  $M$  into the direct sum of uniform right ideals and uniform cyclic  $R$ -modules, suppose that:

- (i) the number of direct summands of uniform right ideals is  $s$ ,
  - (ii) the number of  $P$ -primary cyclic summands for given prime ideal  $P$  is  $l_p$ ,
- where  $l_p \geq 0$ , and that the orders of these summands are

$$P^{\beta_{p1}}, P^{\beta_{p2}}, \dots, P^{\beta_{pl_p}},$$

where  $\beta_{p1} \geq \beta_{p2} \geq \dots \geq \beta_{pl_p}$ .

Then

- (a)  $s \leq r$ ,
- (b)  $l_p \leq k_p$  for each prime ideal  $P$ ,
- (c)  $\beta_{pi} \leq \alpha_{pi} (i=1, 2, \dots, l_p)$ ,
- (d)  $r + \sum k_p = \dim M$  and  $s + \sum l_p = \dim N$ .

Proof. By Goldie's dimension [8], (a), (b) and (d) are clear.

(c) We may assume without loss of generality that  $R$  is a complete, local, bounded Dedekind prime ring with maximal ideal  $P$  and that  $M, N$  are  $P$ -primary with  $M \supseteq N$ ,  $k_p = \dim M$  and  $l_p = \dim N$ . By Lemma 3.37, we have  $M = eR/eP^{\alpha_{p1}} \oplus \dots \oplus eR/eP^{\alpha_{pk_p}}$ , where  $e$  is a uniform idempotent in  $R$ . Suppose now that

$$\beta_{p1} \leq \alpha_{p1}, \dots, \beta_{p,j-1} \leq \alpha_{p,j-1}, \text{ but } \beta_{pj} > \alpha_{pj}.$$

Then  $\dim MP^{\alpha_{pj}} \leq j-1$  and  $\dim NP^{\alpha_{pj}} \geq j$ , which is contradiction.

By Theorem 1 of [2], Theorem 3.24 and Lemma 3.20, we have

**Theorem 3.39.** *Let  $M$  be a decomposable primary modules over a bounded Dedekind prime ring. Then  $M$  is a direct sum of uniform cyclic modules and the number of uniform cyclic summands of a given order is an invariant of  $M$ ; these cardinal numbers are a complete set of invariants for  $M$ .*

REMARK. Theorems 3.35 and 3.39 are generalized to the case of torsion modules.

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