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ON THE POTENTIAL TAKEN WITH RESPECT TO COMPLEX-VALUED AND SYMMETRIC KERNELS

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On the fundamental of the potential theory, we have two following theorems well-known as existence theorems. Let $K(P, Q)$ be any real-valued function defined in a locally compact Hausdorf space Ω, lower semi-continuous for any points P and Q, may be $+\infty$ for $P=Q$, always finite for $P\neq Q$ and bounded from above for P and Q belonging to disjoint compact sets of Ω respectively. The potential of a measure μ taken with respect to the kernel $K(P, Q)$ is the function defined as

$$K(P, \mu)=\int K(P, Q)d\mu(Q),$$

which will be called simply the potential of μ. The potential of a positive measure μ with compact support is always well determined as a function lower semi-continuous in Ω and bounded on any compact set disjoint with the support of μ. Let $K(P, Q)$ be symmetric: $K(P, Q)=K(Q, P)$ for any points P and Q. Then, we have two following theorems.

Theorem A. *Let F be any compact set of Ω with positive K-transfinite diameter*[1] *and $f(P)$ be any real-valued function upper semi-continuous and bounded*

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1) In the case where the kernel K is symmetric, that a compact set F of Ω is of positive K-transfinite diameter is defined as follows. The sequence, made from n different points $P_1, P_2, \cdots$ and P_n of F,

$$W_n(F)=\min\frac{\sum_{i<j}K(P_i, P_j)}{\binom{n}{2}}=\min\frac{\sum_{i\neq j}K(P_i, P_j)}{n(n-1)}$$

is monotone increasing for $n\uparrow +\infty$. As is well-known, its limit $W(F)$ is equal to the minimum of energy integrals of positive measures μ with total mass 1 supported by F:

$$W(F)=\min K(\mu, \mu)=\min\iint K(P, Q)d\mu(Q)d\mu(P).$$

When $W(F)$ is finite, F is said to be of positive K-transfinite diameter. Any Borelian set E of Ω will be said to be of positive K-transfinite diameter if it contains a compact set of positive K-transfinite diameter, otherwise said to be of K-transfinite diameter zero. Whenever we consider the potential taken with respect to a kernel K, we should like to suppose that all the open sets of Ω are of positive K-transfinite diameter.

from below defined on F. Then, given any positive number a, there exist a positive measure μ supported by F and a constant γ such that

(1) $\mu(F)=a$,

(2) $K(P, \mu)\geqq f(P)+\gamma$ *on F with a possible exception of a set of K-transfinite diameter zero, and*

(3) $K(P, \mu)\leqq f(P)+\gamma$ *on the support of μ.*

Theorem B. *In the above theorem, suppose the further conditions: $K(P, Q)>0$ and $\inf f(P)>0$ for any points P and Q of F. Then, given any compact set F of Ω with positive K-transfinite diameter, there exists a positive measure μ supported by F such that*

(1) $K(P, \mu)\geqq f(P)$ *on F with a possible exception of a set of K-transfinite diameter zero, and*

(2) $K(P, \mu)\leqq f(P)$ *on the support of μ.*

The former is an extension of the result stated in Frostman's thesis (see [1], p. 65), and the latter is an extension of the result studied in Kametani's paper (see [2]).

In this paper, we are going to extend these theorems for the potential taken with respect to complex-valued and symmetric kernels and to complex-valued measures.

Let $K(P, Q)$ be any complex-valued function defined in a locally compact Hausdorff space Ω. Let $k(P, Q)=\Re K(P, Q)$ be a function lower semi-continuous, may be $+\infty$ for $P=Q$, always finite for $P\neq Q$ and bounded from above for P and Q belonging to disjoint compact sets of Ω respectively, and $n(P, Q)=\Im K(P, Q)$ be a finite and continuous function. We suppose that the kernel K is Hermitian symmetric: $k(P, Q)=k(Q, P)$ and $n(P, Q)=-n(Q, P)$ for any points P and Q. Given any positive numbers a and b and any Borelian set E of Ω, denote by $\mathfrak{M}(a, E)$ the family of all the complex-valued measures supported by E whose real parts are positive measures with total mass a and whose imaginary parts are any positive measures, by $\mathfrak{M}(E, b)$ the family of all the complex-valued measures supported by E whose real parts are any positive measures and whose imaginary parts are positive measures with total mass b, and by $\mathfrak{M}(a, E, b)$ the family of all the complex-valued measures supported by E whose real parts and imaginary parts are positive measures with total mass a and b respectively. We shall study the potential of such measures α

$$K(P, \alpha)=\int K(P, Q)d\alpha(Q),$$

which is well determined whenever both $\Re\alpha$ and $\Im\alpha$ are with compact supports. Then, we have two following theorems.

Theorem 1.[2] *Let F be any compact set of Ω with positive k-transfinite diameter and $F(P)$ be any complex-valued function whose $\Re F(P)$ and $\Im F(P)$ are functions upper semi-continuous and bounded from below defined on F both. Then, given any positive numbers a and b, there exist a measure α of $\mathfrak{M}(a, F, b)$ and a complex number γ such that*

(1) $\Re K(P, \alpha) \geqq \Re\{F(P)+\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero,*

(2) $\Re K(P, \alpha) \leqq \Re\{F(P)+\gamma\}$ *on the support of $\Re\alpha$,*

(3) $\Im K(P, \alpha) \geqq \Im\{F(P)+\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero, and*

(4) $\Im K(P, \alpha) \leqq \Im\{F(P)+\gamma\}$ *on the support of $\Im\alpha$.*

Theorem 2. *In the above theorem, suppose the further conditions: $k(P, Q)>0$, $\inf \Re F(P)>0$ and $\inf \Im F(P)>0$ for any points P and Q of F. Then, given any positive number a such that $a\cdot|n(P, Q)|<\Im F(P)$ for points P and Q of F, there exist a measure α of $\mathfrak{M}(a, F)$ and a real number γ, such that*

(1) $\Re K(P, \alpha) \geqq \Re F\{(P)+\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero,*

(2) $\Re K(P, \alpha) \leqq \Re\{F(P)+\gamma\}$ *on the support of $\Re\alpha$.*

(3) $\Im K(P, \alpha) \geqq \Im F(P)$ *on F with a possible exception of a set of k-transfinite diameter zero, and*

(4) $\Im K(P, \alpha) \leqq \Im F(P)$ *on the support of $\Im\alpha$.*

Similarly, given any positive number b such that $b\cdot|n(P, Q)|<\Re F(P)$ for points P and Q of F, there exist a measure α of $\mathfrak{M}(F, b)$ and an imaginary number γ such that

(1′) $\Re K(P, \alpha) \geqq \Re F(P)$ *on F with a possible exception of a set of k-transfinite diameter zero,*

(2′) $\Re K(P, \alpha) \leqq \Re F(P)$ *on the support of $\Re\alpha$.*

(3′) $\Im K(P, \alpha) \geqq \Im\{F(P)++\gamma\}$ *on F with a possible exception of a set of k-transfinite diameter zero, and*

(4′) $\Im K(P, \alpha) \leqq \Im\{F(P)+\gamma\}$ *on the support of $\Im\alpha$.*

Proof of Theorem 1. For any measure α of $\mathfrak{M}(a, F, b)$, let us consider the quantity

$$G(\alpha)=\iint K(P, Q)d\alpha(Q)d\bar{\alpha}(P)-\int F(P)d\bar{\alpha}(P)-\int \overline{F(P)}d\alpha(P),$$

which is obviously an extension of the Gauss' variation taken with respect to a real-valued kernel and positive measures. Put

2) This result has been written by the author in the journal (in Japanese) edited by the Mathematical Society of Japan, "Sûgaku, vol. 20, no. 2, 1968, pp. 96–97", which was reviewed by Masayuki Ito in the "Math. Review, vol. 39, no. 2, 1970, p. 1674".

$$K(P, Q) = k(P, Q)+in(P, Q)\,, \quad F(P) = f(P)+ig(P) \quad \text{and} \quad \alpha = \mu+i\nu\,.$$

Then, the kernel K being symmetric, we have

$$\begin{aligned} G(\alpha) = & \iint k(P, Q)d\mu(Q)d\mu(P)+\iint k(P, Q)d\nu(Q)d\nu(P) \\ & -\int d\mu(P)\int n(P, Q)\,d\nu(Q)+\int d\nu\,(P)\int n(P, Q)d\mu(Q) \\ & -2\int f(P)d\mu(P)-2\int g(P)d\nu(P)\,. \end{aligned}$$

So, $G(\alpha)$ is always real and $-\infty < G(\alpha) \leqq +\infty$. Put $G^*=\inf G(\alpha)$ for measures α of $\mathfrak{M}(a, F, b)$. F being of positive k-transfinite diameter, G^* is a finite number. Take any sequence $\{\alpha_n\}$ of measures of $\mathfrak{M}(a, F, b)$ such that $G(\alpha_n) \downarrow G^*$, and put $\alpha_n=\mu_n+i\nu_n$. Then, we may consider both $\{\mu_n\}$ and $\{\nu_n\}$ as vaguely convergent sequences by the selection theorem of F. Riesz, if necessary, by extracting their proper subsequences. Let μ and ν be their limiting measures respectively. The measure $\alpha=\mu+i\nu$ is naturally one of $\mathfrak{M}(a, F, b)$. As there hold

$$\iint k(P, Q)d\mu(Q)d\mu(P) \leqq \underline{\lim}_{n\to\infty} \iint k(P, Q)d\mu_n(Q)d\mu_n(P)\,,$$

$$\int d\mu(P)\int n(P, Q)\,d\nu(Q) = \lim_{n\to\infty}\int d\mu_n(P)\int n(P, Q)d\nu_n(Q)$$

and

$$\int f(P)d\mu(P) \geqq \overline{\lim}_{n\to\infty}\int f(P)d\mu_n(P)\,, \text{ etc.,}$$

we have

$$G^* \leqq G(\alpha) \leqq \underline{\lim}_{n\to\infty} G(\alpha,) \leqq G^*\,,$$

which indicates that the measure α minimizes the quantity G among all the measures of $\mathfrak{M}(a, F, b)$. Take any measure $\beta=\sigma+i\tau$ supported by F such that $\alpha+\beta\in\mathfrak{M}(a, F, b)$. Naturally, both σ and τ are real-valued measures with total mass zero supported by F, and both $\mu+\sigma$ and $\nu+\tau$ are positive measures with total mass a and b supported by F respectively. As the measure $\alpha+\varepsilon\beta$ are of $\mathfrak{M}(a, F, b)$ for any positive number ε (<1), we have $G(\alpha)\leqq G(\alpha+\varepsilon\beta)$. The hernel K being symmetric, this induces the inequality

$$\begin{aligned} 0 \leqq\ & 2\varepsilon\int A(P)d\sigma(P)+2\varepsilon\int B(P)d\tau(P) \\ & +\varepsilon^2\left\{\iint k(P, Q)d\sigma(Q)d\sigma(P)+\iint k(P, Q)d\tau(Q)d\tau(P)\right. \\ & \left.-\int d\sigma(P)\int n(P, Q)d\tau(Q)\right\}, \end{aligned}$$

where

$$A(P)=\int k(P, Q)d\mu(Q)-\int n(P, Q)d\nu(Q)-f(P)$$

and

$$B(P)=\int k(P, Q)d\nu(Q)+\int n(P, Q)d\mu(Q)-g(P).$$

Putting

$$\gamma_1=\int A(P)d\mu(P) \quad \text{and} \quad \gamma_2=\int B(P)d\nu(P),$$

we are going to prove that

(1) $A(P)\geqq\frac{\gamma_1}{a}$ on F with a possible exception of a set of k-transfinite diameter zero,

(2) $A(P)\leqq\frac{\gamma_1}{a}$ on the support of μ ,

(3) $B(P)\geqq\frac{\gamma_2}{b}$ on F with a possible exception of a set of k-transfinite diameter zero, and

(4) $B(P)\leqq\frac{\gamma_2}{b}$ on the support of ν.

First, apply that inequality for the case where $\tau\equiv 0$. We have

$$0 \leqq 2\int A(P)d\sigma(P)+\varepsilon\iint k(P, Q)d\sigma(Q)d\sigma(P)$$

for any measure σ such that $\mu+\sigma$ is a positive measure with total mass a supported by F. For any positive number δ, suppose that the set

$$S=\left\{P;\, P\in F,\, A(P)<\frac{\gamma_1}{a}-2\delta\right\}$$

is of positive k-transfinite diameter, then there exists a positive measure σ' of finite energy supported by S whose total mass is equal to the total mass $c(>0)$ of the restricted measure $\mu'(\not\equiv 0)$ of μ to the set

$$T=\left\{P;\, P\in F,\, A(P)>\frac{\gamma_1}{a}-\delta\right\}.$$

Applying the inequality for the measure $\sigma=\sigma'-\mu'$, we have

$$0 \leqq -\delta c+\varepsilon\iint k(P, Q)d\sigma(Q)d\sigma(P).$$

Here, the coefficient of ε is finite, since

$$\iint k(P, Q)d\sigma(Q)d\sigma(P)$$
$$= \iint k(P, Q)d\sigma'(Q)d\sigma'(P) + \iint k(P, Q)d\mu'(Q)d\mu'(P)$$
$$+2\int d\sigma'(P)\int k(P, Q)d\mu'(Q)\,,$$

whose first and second terms are finite both. As to the third term, there holds, by taking a positive number C such that $k(P, Q)+C>0$ for any points P and Q of F,

$$-\infty < \int d\sigma'(P)\int k(P, Q)d\mu'(Q)$$
$$= \int d\sigma'(P)\int k(P, Q)d\mu(Q) - \int d\sigma'(P)\int k(P, Q)(d\mu - d\mu')(Q)$$
$$< \int\left(\int n(P, Q)d\nu(Q) + f(P) + \frac{\gamma_1}{a} - 2\delta\right)d\sigma'(P) + C\cdot c(a-c)\,.$$

$f(P)$ being bounded, we have

$$\iint k(P, Q)d\sigma(Q)d\sigma(P) < +\infty\,.$$

Making $\varepsilon \to 0$, we have a contradiction, which induces that the set S is of k-transfinite diameter zero. Making $\delta \to 0$, we obtain the property (1). Then, we have also the property (2) by recalling that μ has no positive mass on any set of k-transfinite diameter zero and that $\int A(P)\,d\mu(P) = \gamma_1$. Similarly, we shall have the properties (3) and (4). Finally, as we have

$$A(P) = \Re\{K(P, \alpha) - F(P)\} \quad \text{and} \quad B(P) = \Im\{K(P, \alpha) - F(P)\}\,,$$

the measure $\alpha=\mu+i\nu$ and the number $\gamma=\frac{\gamma_1}{a}+i\,\frac{\gamma_2}{b}$ are what the theorem needs. Q.E.D.

Proof of Theorem 2. For any measure α of $\mathfrak{M}(a, F)$, let us consider the Gauss' variation as presented in the proof of Theorem 1

$$G(\alpha) = \iint K(P, Q)d\alpha(Q)d\bar{\alpha}(P) - \int F(P)d\bar{\alpha}(P) - \int \overline{F(P)}d\alpha(P)$$
$$= \iint k(P, Q)d\mu(Q)d\mu(P) + \iint k(P, Q)d\nu(Q)d\nu(P)$$
$$-\int d\mu(P)\int n(P, Q)d\nu(Q) + \int d\nu(P)\int n(P, Q)d\mu(Q)$$
$$-2\int f(P)d\mu(P) - 2\int g(P)d\nu(P)\,.$$

Put $G^*=\inf G(\alpha)$ for measures α of $\mathfrak{M}(a, F)$. First, we are going to show that $-\infty<G^*<\inf G(\mu)$ for measures of $\mathfrak{M}(a, F, 0)$. In fact, there holds

$$G(\mu)+p\varepsilon^2-2(r+aq)\varepsilon \leqq G(\mu+i\varepsilon\nu_1)$$
$$\leqq G(\mu)+\varepsilon^2 \iint k(P, Q)d\nu_1(Q)d\nu_1(P)-2(r'-aq)\varepsilon$$

for any measures ν_1 of finite energy of $\mathfrak{M}(1, F, 0)$ and for positive numbers ε, p, q, r and r' such that

$$p\leqq k(P, Q)\,, \quad |n(P, Q)|\leqq q \quad \text{and} \quad r'\leqq g(P)\leqq r$$

for any points P and Q of F. That first hand is greater than a constant added to $G(\mu)$ whatever ε may be, and that last hand is smaller than $G(\mu)$ on account of $aq<r'$ when ε is sufficiently small. So, we have $-\infty<G^*<\inf G(\mu)$ for measures μ of $\mathfrak{M}(a, F, 0)$. Take any sequence $\{\alpha_n\}$ of measures of $\mathfrak{M}(a, F)$ such that $G(\alpha_n)\downarrow G^*$, and put $\alpha_n=\mu_n+i\nu_n$. Then, we may suppose that the total mass of each ν_n is not greater than

$$\frac{2(r+aq)}{p},$$

therefore, we may consider both $\{\mu_n\}$ and $\{\nu_n\}$ as vaguely convergent sequences. Let μ and ν be their limiting measures respectively. The measure $\alpha=\mu+i\nu$ is naturally of $\mathfrak{M}(a, F)$. By the inequality

$$G^*\leqq G(\alpha)\leqq \varliminf_{n\to\infty} G(\alpha_n) = G^*\,,$$

we have $G^*=G(\alpha)$. Therefore, we can assert that $\nu\not\equiv 0$. Put

$$B(P) = \int k(P, Q)d\nu(Q)+\int n(P, Q)d\mu(Q)-g(P)$$

and suppose that, for any positive number δ, the set

$$S = \{P;\, P\in F,\, B(P)<-\delta\}$$

is of positive k-transfinite diameter. Then, there exists a positive measure σ of finite energy supported by S, and there holds the inequality $G(\alpha)\leqq G(\alpha+i\varepsilon\sigma)$ for any positive number ε. The kernel k being symmetric, this induces the inequality

$$0 \leqq 2\varepsilon \int B(P)d\sigma(P)+\varepsilon^2 \iint k(P, Q)d\sigma(Q)d\sigma(P)$$
$$<-2\delta\cdot\varepsilon+\varepsilon^2 \iint k(P, Q)d\sigma(Q)d\sigma(P)\,,$$

which is a contradiction if ε is sufficiently small. Accordingly, the set S is of k-transfinite diameter zero. Furthermore, suppose that the set

$$T=\{P;\, P\in F,\, B(P)>\delta\}$$

has any positive mass for ν. Denoting by ν' the restricted measure of ν to T, there holds the inequality

$$G(\alpha)\leqq G(\alpha-i\varepsilon\nu')$$

for any positive number ε (<1). This induces the inequality

$$\begin{aligned}0 &\leqq 2\varepsilon\int B(P)d\nu'(P)+\varepsilon^2\iint k(P,Q)d\nu'(Q)d\nu'(P)\\ &<-2\delta\cdot\nu(T)\cdot\varepsilon+\varepsilon^2\iint k(P,Q)d\nu'(Q)d\nu'(P)\,,\end{aligned}$$

which is a contradiction if ε is sufficiently small. Accordingly, the set T has no mass for ν. Making $\delta\to 0$, we have the properties (3) and (4). Next, in order to obtain the properties (1) and (2), take any real-valued measure σ supported by F such that $\alpha+\sigma\in\mathfrak{M}(a, F)$. Naturally, σ is a measure with total mass 0 and $\mu+\sigma$ is a positive measure with total mass a supported by F. As the measure $\alpha+\varepsilon\sigma$ is of $\mathfrak{M}(a, F)$ for any positive number $\varepsilon(<1)$, we have the inequality

$$G(\alpha)\leqq G(\alpha+\varepsilon\sigma)\,.$$

The kernel k being symmetric, this induces the inequality

$$0\leqq 2\varepsilon\int A(P)d\sigma(P)+\varepsilon^2\iint k(P,Q)d\sigma(Q)d\sigma(P)\,,$$

where

$$A(P)=\int k(P,Q)\,d\mu(Q)-\int n(P,Q)d\nu(Q)-f(P)\,.$$

Putting $\gamma=\int A(P)d\mu(P)$, we obtain in the same way as Theorem 1 that $A(P)\geqq\gamma/a$ on F with a possible exception of a set of k-transfinite diameter zero and $A(P)\leqq\gamma/a$ on the support of μ. Thus, the measure $\alpha=\mu+i\nu$ and the real number γ are what the theorem needs. The analogous arguments will give us the properties (1′), (2′), (3′) and (4′). Q.E.D.

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