Sugano, K.
Osaka J. Math.
8 (1971), 465-469

ON CENTRALIZERS IN SEPARABLE EXTENSIONS II

Kozo SUGANO

(Received February 9, 1971)

0. The aim of this paper is to improve and generalize some results of the author's previous paper [8]. Therefore, all notations and terminologies are same as those in [7] and [8]. In [8] the author studied some commutor theory of H-separable extension $\Lambda|\Gamma$ in the case where $\Lambda\simeq\Gamma\otimes_C\Delta$ with $\Delta(=V_\Lambda(\Gamma))$ central separable over C and $C=$the center of $\Lambda=$the center of Γ,and in the case where Λ is left or right Γ-$f.g.$ (finitely generated) projective and $\Lambda|\Gamma$ satisfies the following condition (*)

(*) 1) Λ is an H-separable extension of Γ such that ${}_\Gamma\Gamma_\Gamma\langle\oplus{}_\Gamma\Lambda_\Gamma$.
 2) $V_\Lambda(\Gamma)=C'$, where C' is the center of Γ.

(See Theorem 1.2, Corollary 1.4 and Theorem 1.3 [8]). In case $\Lambda|\Gamma$ satisfies the condition (*) 1), Λ is left Γ-$f.g.$ projective if and only if Λ is right Γ-$f.g.$ projective by Corollary 2 [9], hence we shall simply say that Λ is Γ-$f.g.$ projective in this case. We note also that the condition (*) implies that $V_\Lambda(C')=\Gamma$ by Proposition 1.2 [7]. In this paper, we shall consider the case where Λ is left or right Γ-$f.g.$ projective and Λ is an H-separable extension of Γ, and shall prove that there exists a one to one correspondence between the class of subrings B of Λ which is separable extensions of Γ and ${}_BB_B\langle\oplus{}_B\Lambda_B$ and the class of separable C-subalgebras of Δ (Theorem 1). From this theorem, Corollary 1.4 and a more beautiful result than Thoerem 1.3 [8] follows.

1. To obtain our main results we need the next lemma which appears in [6].

Lemma 1 (Corollary 1.2 [6]). *Let A be a ring, M a left A-module, $\Omega=End({}_AM)$ and $E=End(M_\Omega)$. Then if M is A-$f.g.$ projective, $E\otimes_AM\simeq M$ as E-Ω-module by the map: $e\otimes m\rightarrow em$ for $e\in E$ and $m\in M$.*

Proof. Since M is A-$f.g.$ projective, we have natural isomorphisms

$$E\otimes_AM=\mathrm{Hom}(M_\Omega,M_\Omega)\otimes_AM\simeq\mathrm{Hom}(\mathrm{Hom}({}_AM,{}_AM)_\Omega,M_\Omega)$$
$$=\mathrm{Hom}(\Omega_\Omega,M_\Omega)\simeq M$$

as E–Ω–module. The composition of the above isomorphisms is the required one.

For rings $\Gamma\subset B\subset\Lambda$, we shall say that $B\otimes_\Gamma\Lambda\to\Lambda$ splits if the map of $B\otimes_\Gamma\Lambda$ to Λ such that $b\otimes x\to bx$ for $b\in B$ and $x\in\Lambda$ splits as B–Λ–map. We also need Proposition 2.3 [8]. This proposition can be improved as follows

Proposition 1. *Let Λ be an H–separable extension of Γ. Then for any intermediate ring B between Γ and Λ such that ${}_BB_\Gamma<\oplus{}_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits, ${}_DD<\oplus{}_D\Delta$ and $D\otimes_C\Delta\to\Delta$ splits, where $D=V_\Lambda(B)$. Conversely for any C-subalgebra D of Δ such that ${}_DD<\oplus{}_D\Delta$ and $D\otimes_C\Delta\to\Delta$ splits, ${}_BB_\Gamma<\oplus{}_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits, where $B=V_\Lambda(D)$.*

Proof. The first part of this proposition have been proved in Proposition 2.3 [8]. Hence we need to prove only the second part without assuming that B is right Γ–*f. g.* projective. Suppose that D is a C–subalgebra of Δ such that $D\otimes_C\Delta\to\Delta$ splits. Then $B=V_\Lambda(D)\cong\mathrm{Hom}({}_D\Delta_\Delta,{}_D\Lambda_\Delta)<\oplus\mathrm{Hom}({}_DD\otimes_C\Delta_\Delta,{}_D\Lambda_\Delta)$ $\cong V_\Lambda(C)$ as B–$V_\Lambda(\Delta)$–module. Hence ${}_BB_\Gamma<\oplus{}_B\Lambda_\Gamma$. Then, since Λ is H–separable over Γ and ${}_BB_\Gamma<\oplus{}_B\Lambda_\Gamma$, we have a B–Λ–isomorphism η of $B\otimes_\Gamma\Lambda$ to $\mathrm{Hom}({}_D\Delta,{}_D\Lambda)$ such that $\eta(b\otimes x)(d)=bdx$ for $b\in B$, $d\in D$ and $x\in\Lambda$ by Proposition 1.3 [7]. Hence, we have a commutative diagram of B–Λ–maps

$$\begin{array}{ccc} B\otimes_\Gamma\Lambda & \xrightarrow{\ \eta\ } & \mathrm{Hom}({}_D\Delta,{}_D\Lambda) \\ \downarrow & & \downarrow i_* \\ \Lambda & \xrightarrow[\ j\ ]{} & \mathrm{Hom}({}_DD,{}_D\Lambda) \end{array}$$

where j is the natural isomorphism and i_* is the one induced by the inclusion map i: $D\subset\Delta$. Then if ${}_DD<\oplus{}_D\Delta$, i_* B–Λ–splits and $B\otimes_\Gamma\Lambda\to\Lambda$ splits.

Let Λ be a semisimple R–algebra in the sense of A. Hattori [2], that is, Λ is a weakly semisimple extension of $R\cdot 1$ in the sense of [3]. Then every finitely generated Λ–module which is R–projective is Λ–projective, and by Proposition 4.1 [1] if Σ is a finitely generated projective R–algebra which contains Λ, ${}_\Lambda\Lambda<\oplus{}_\Lambda\Sigma$ and $\Lambda_\Lambda<\oplus\Sigma_\Lambda$. It is also well known that a separable algebra is a semisimple algebra.

Proposition 2. *Let Λ be an H–separable extension of Γ. If (1) D is a separable C–subalgebra of Δ, or if (2) Δ is a separable C–algebra (e. g., if ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$) and D is a semisimple C–subalgebra of Δ, then $V_\Lambda(V_\Lambda(D))=D$, ${}_BB_\Gamma<\oplus{}_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits, where $B=V_\Lambda(D)$.*

Proof. Suppose (1). Then since $D\otimes_CD\to D$ splits as D–D–map, $D\otimes_C\Delta$ $\to\Delta$ splits as D–Δ–map. Suppose (2). Then $D\otimes_C\Delta\to\Delta$ splits as D–C–map, since D is C–semisimple. Then $D\otimes_C\Delta\to\Delta$ splits as D–Δ–map, since Δ is C–separable. Thus in both cases, $D\otimes_C\Delta\to\Delta$ splits and ${}_DD<\oplus{}_D\Delta$. The latter follows from Proposition 4.1 [1], since D is C-semisimple and Δ is C–$f.g.$ projective. Then $B\otimes_\Gamma\Lambda\to\Lambda$ splits and ${}_BB_\Gamma\langle\oplus_B\Lambda_\Gamma$ by Proposition 1. Hence Λ is H-separable over B by Proposition 2.2 [8]. Let $D'=V_\Lambda(B)$. Then there exists a ring isomorphism η: $D'\otimes_C\Lambda^0\to\mathrm{End}({}_B\Lambda)$ such that $\eta(d\otimes x^0)(y)=dyx$ for $x, y\in\Lambda$, $d\in D'$ (see Proposition 3.3 [5]). Then $D'\otimes_C\Lambda^0$ is the double centralizer of a left $D\otimes_C\Lambda^0$-module Λ, since $B=\mathrm{End}({}_D\Lambda_\Lambda)$. While $D\otimes_C\Lambda\to\Lambda$ splits, since $D\otimes_C\Delta\to\Delta$ splits. This implies that Λ is left $D\otimes_C\Lambda^0$–$f.g.$ projective. Then by lemma 1, $(D'\otimes_C\Lambda^0)\otimes_{D\otimes_C\Lambda^0}\Lambda\cong\Lambda$, hence $D'\otimes_D\Lambda^0\cong\Lambda$. This isomorphism is given by $d\otimes x\to dx$ for $d\in D'$, $x\in\Lambda$. Then for every $d\in D'$, $d\otimes 1=1\otimes d$ in $D'\otimes_D\Lambda$, since both are mapped to d by this isomorphism. On the other hand, since Λ is H-separable over B, D' is C–$f.g.$ projective, and D' is right D–$f.g.$ projective, since D is C-semisimple. Hence $D'\otimes_D D'\subset D'\otimes_D\Lambda$, and $d\otimes 1=1\otimes d$ in $D'\otimes_D D'$ for every $d\in D'$. Since ${}_DD\langle\oplus_D D'$, $D'=D\oplus A$ for some left D-submodule A of D' and $D'\otimes_D D'=D'\otimes_D D\oplus D'\otimes_D A$. Let x be an arbitrary element of D' and $x=d+a$ for $d\in D$, and $a\in A$. Then $D'\otimes_D D'\ni x\otimes 1=1\otimes x$ $=1\otimes d+1\otimes a$, and $1\otimes a=0$, $x\otimes 1=1\otimes d$. Thus $x=d\in D$. Thus $D'=D$. Thus $D=V_\Lambda(V_\Lambda(D))$.

The next proposition is a generalization of Proposition 1.5 [8].

Proposition 3. *Let Λ be an arbitrary R-algebra which is R–$f.g.$ projective. Then for any separable R-subalgebra Γ of Λ, Γ is a Γ-Γ-direct summand of Λ.*

Proof. Since Γ is R-separable, there exists $\Sigma r_i\otimes s_i\in(\Gamma\otimes_C\Gamma)^\Gamma$ such that $\Sigma r_is_i=1$. While, since Γ is R-semisimple, ${}_\Gamma\Gamma\langle\oplus_\Gamma\Lambda$. Let p be the left Γ-projedtion of Λ to Γ. Then the map p^* of Λ to Γ such that $p^*(x)=\Sigma p(xr_i)s_i$ for $x\in\Lambda$ is a Γ–Γ-map, and $p^*(r)=\Sigma p(rr_i)s_i=\Sigma rr_is_i=r$ for every $r\in\Gamma$. Thus ${}_\Gamma\Gamma_\Gamma\langle\oplus_\Gamma\Lambda_\Gamma$.

Now we are ready to get our main theorem.

Theorem 1. *Let Λ be an H-separable extension of Γ. Then if Λ is left or right Γ–$f.g.$ projective, there exists a one to one correspondence $V: A \rightsquigarrow V_\Lambda(A)$ such that V^2=identity between the class of separable extensions B of Γ such that ${}_BB_B\langle\oplus_B\Lambda_B$ and the class of C-separable subalgebras of Δ.*

Proof. Let D be an arbitrary separable C-subalgebra of Δ and $B=V_\Lambda(D)$. Then ${}_BB_\Gamma\langle\oplus_B\Lambda_\Gamma$ and $V_\Lambda(B)=D$. This and Corollary 1.3 [8] imply that B is separable over Γ, since B is left or right Γ–$f.g.$ projective and ${}_DD_D\langle\oplus_D\Delta_D$. ${}_BB_B\langle\oplus_B\Lambda_B$ follows from ${}_BB_\Gamma\langle\oplus_B\Lambda_\Gamma$ and the separability of B over Γ. On the other hand, if B is a separable extension of Γ such that ${}_BB_B\langle\oplus_B\Lambda_B$, then $D=V_\Lambda(B)$ is a separable C-algebra and $V_\Lambda(V_\Lambda(B))=B$ by Proposition 1.4 [8].

Corollary 1. *Let Λ be an H-separable extension of Γ with the condition (*) of §0. Then if Λ is Γ–$f.g.$ projective, there exists a one to one correspondence*

V: $A \rightsquigarrow V_\Lambda(A)$ such that V^2=identity between the class of subrings of Λ which are H-separable extensions of Γ and the class of separable C-subalgebras of C'. In this case V corresponds each H-separable extension of Γ to its center.

Proof. Let B be any ring with $\Gamma\subset B\subset\Lambda$. Then $V_\Lambda(B)\subset V_\Lambda(\Gamma)=C'\subset B$, hence the center of $B=V_B(B)=B\cap V_\Lambda(B)=V_\Lambda(B)$. On the other hand, by Propositions 1.8 and 1.9 B is H-separable over Γ, if and only if ${}_BB_B\langle\oplus{}_B\Lambda_B$. Thus the assertion follows from Theorem 1.

REMARK. Ring extension $\Lambda|\Gamma$ which satisfy the condition (*) and such that Λ is Γ-*f. g.* projective really exists. Let Λ be a central separable C-algebra and Γ a C-separable subalgebra with its center $C'\neq C$. Then Λ is H-separable over Γ, ${}_\Gamma\Gamma_\Gamma\langle\oplus{}_\Gamma\Lambda_\Gamma$ and Λ is Γ-*f. g.* projective. Let $\Lambda'=V_\Lambda(C')$. Then $\Lambda|\Lambda'$ satisfy the condition (*) by Proposition 1.3 [8].

In [10] we considered ring extension $\Lambda|\Gamma$ which satisfy the following condition (#).

(#) (1) Λ is a separable extension of Γ such that $V_\Lambda(\Gamma)=C$.
(2) Λ is Γ-centrally projective (i.e., ${}_\Gamma\Lambda_\Gamma\langle\oplus{}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$).

And we proved that if $\Lambda|\Gamma$ satisfy the condition (#), there exist one to one correspondences U and V between the class $\mathfrak{A}$ of separable extensions B of Γ such that ${}_BB_B\langle\oplus{}_B\Lambda_B$ and the class $\mathfrak{B}$ of separable C'-subalgebras of C, defined by V: $B\rightsquigarrow B\cap C$ and U: $R\rightsquigarrow R\Gamma$ for $B\in\mathfrak{A}$ and $R\in\mathfrak{B}$, with $UV=1_\mathfrak{A}$ and $VU=1_\mathfrak{B}$ (Theorem 8 [10]).

Let a ring extension $\Lambda|\Gamma$ satisfy the condition (*) of §0. Then $\Omega=[\mathrm{End}({}_\Gamma\Lambda)]^0$ $=C'\otimes_C\Lambda$ and C' is a commutative C-separable algebra and C-*f. g.* projective. Then clearly, the center of $\Omega=C'=V_\Omega(\Lambda)$, and $\Omega|\Lambda$ satisfies the condition (#). Let $\mathfrak{A}$ be the class of separable extensions Σ of Λ such that ${}_\Sigma\Sigma_\Sigma\langle\oplus{}_\Sigma\Omega_\Sigma$, $\mathfrak{B}$ the class of separable C-subalgebras of C', and let U and V be such that $U(R)=R\Lambda$ for $R\in\mathfrak{B}$ and $V(\Sigma)=\Sigma\cap C'$ for $\Sigma\in\mathfrak{A}$. Then by Theorem 8 [10], U and V provide one to one correspondences between $\mathfrak{A}$ and $\mathfrak{B}$ with $UV=1_\mathfrak{A}$ and $VU=1_\mathfrak{B}$. Furthermore, let $\mathfrak{C}$ be the class of subrings of Λ which are H-separable extensions of Γ. Then by Corollary 1 we have.

Proposition 4. *Let a ring extension $\Lambda|\Gamma$ satisfy the condition (*) and Λ be Γ-f. g. projective. Then if we define $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ as above, the correspondences W: $\mathfrak{A}\rightsquigarrow\mathfrak{C}$ such that $W(\Sigma)=V_\Lambda(\Sigma\cap(C'\otimes 1))$ for $\Sigma\in\mathfrak{A}$ and T: $\mathfrak{C}\rightsquigarrow\mathfrak{A}$ such that $T(B)=End({}_B\Lambda)$ for $B\in\mathfrak{C}$ are one to one with $WT=1_\mathfrak{C}$ and $TW=1_\mathfrak{A}$.*

Proof. For $B\in\mathfrak{C}$, $V_\Lambda(B)\otimes_C\Lambda=\mathrm{End}({}_B\Lambda)$. Then by Corollary 1 and Theorem 8[10], $TW=1_\mathfrak{A}$ and $WT=1_\mathfrak{C}$.

HOKKAIDO UNIVERSITY

References

[1] S. Endo and Y. Watanabe: *The centers of semisimple algebras over a commutative ring*, Nagoya Math. J. **30** (1967), 285–293.

[2] A. Hattori: *Semisimple algebras over a commutative ring*, J. Math. Soc. Japan **15** (1963), 404–419.

[3] K. Hirata and K. Sugano: *On semisimple extensions and separable extensions over non commutative rings*, J. Math. Soc. Japan, **18** (1966), 360–373.

[4] K. Hirata: *Some types of separable extensions of rings*, Nagoya Math. J. **33** (1968), 107–116.

[5] K. Hirata: *Separable extensions and centralizers of rings*, Nagoya Math. J. **35** (1969), 31–45.

[6] K. Morita: *Localizations in categories of modulues*, Math. Z. **114** (1970), 121–144.

[7] K. Sugano: *Note on semisimple extensions and separable extensions*, Osaka J. Math. **4** (1967), 265–270.

[8] K. Sugano: *On centralizers in separable extensions*, Osaka J. Math. **7** (1970), 29–40.

[9] K. Sugano: *Separable extensions and Frobenius extensions*, Osaka J. Math. **7** (1970), 291–299.

[10] K. Sugano: *Note on separability of endomorphism rings*, J. Fac. Sci. Hokkaido Univ. **21** (1971), 196–208.

[11] H. Tominaga and T. Nagahara: Galois Theory of Simple Rings, Okayama Math. Lectures, Okayama, 1970.