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DOUBLY TRANSITIVE PERMUTATION REPRESENTATIONS OF THE FINITE PROJECTIVE SPECIAL LINEAR GROUPS $PSL(n, q)$

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1. Introduction

In this note we will determine all doubly transitive permutation representations of the projective special linear groups $PSL(n, q)$ over the finite field $\boldsymbol{F}_q$. Our main result (Theorem 1) asserts that these are all well known ones, namely

Theorem 1. *If the group $G=PSL(n, q)$ is reperesented as a faithful doubly transitive permutation group on a set Ω, $|\Omega|=m$, then (G, Ω) is isomorphic with one of the members in the following list:*

I) *G acts on the set Ω of points of the $(n-1)$-dimensional projective space over $\boldsymbol{F}_q$: $\mathcal{P}(n-1, q)$, $m=(q^n-1)/(q-1)$, via the natural action.*

II) *G acts on the set Ω of hyperplanes of $\mathcal{P}(n-1, q)$ via the natural action, $m=(q^n-1)/(q-1)$.*

III) $G=PSL(2, 5)$ $(\cong A_5)$, $m=5$.

IV) $G=PSL(2, 7)$ $(\cong PSL(3, 2))$, $m=7$.

V) $G=PSL(2, 9)$ $(\cong A_6)$, $m=6$.

VI) $G=PSL(2, 11)$, $m=11$.

VII) $G=PSL(3, 2)$ $(\cong PSL(2, 7))$, $m=8$.

VIII) $G=PSL(4, 2)$ $(\cong A_8)$, $m=8$.

For $n=2$, Theorem 1 has been given by E. Galois, L. E. Dickson and others (*cf.* B. Huppert [4]). Furthermore, for $n=3$, or also for particular pairs of (n, q) provided n, q are small the result above might have been proved by making use of the classifications of the maximal subgroups due to H.H. Mitchell [7], R.E. Hartley [3] and others.

Recently N. Ito [5] classified all premutation representations of the group $PSL(n, q)$ whose degrees are prime numbers. On the other hand, T. Tsuzuku [10] has shown that, if a finite simple group of Lie type has a primitive permutation representation whose degree is relatively prime to the characteristic of the basic field, then the stabilizer of a point must be a maximal parablilc subgroup. (This was also obtained independently by J. Tits). Especially Tsuzuku has shown that, if $PSL(n, q)$ is repersented as a doubly transitive perumtation group whose degree is relatively prime to q, then this permutation group must be either the case (I) or (II) in Theorem 1.

Nevertheless, it seems to the author that Theorem 1 has not yet been given in such a general form as was stated above as Theorem 1.

The outline of the proof of Theorem 1 is as follows: to begin with, it is shown that if $n \geqslant 4$ and $q^{n-2} \nmid m$, then the case (I) or (II) must hold. The proof depends heavily on a theorem of F.C. Piper [8 and 9] which characterizes the group $PSL(n, q)$ from a geometric view point.

Next we show that $m-1$ is bounded by a fixed value depending only on q and n, say $(q^n-1)(q^{n-1}-1)/(q-1)$. Then we determine irreducible characters φ of $G=PSL(n, q)$ which satisfy the conditions

1) $\varphi(1) \leqslant (q^n-1)(q^{n-1}-1)/(q-1)$,
2) $q^{n-2} \mid (\varphi(1)+1)$,

$\varphi(1)$ being the degree of the character φ. There, we are deeply indebted to the well-known construction of irreducible characters of the group $GL(n, q)$ by J.A. Green [2].

Suppose now that $n \geqslant 4$ and $q^{n-2} \mid m$. Since G is doubly transitive, G must have an irreducible character φ satisfying the above conditions (1) and (2). However we can easily show that, there exists no such irreducible character φ for $n \geqslant 5$, and so there exists no such doubly transitive permutation representation of G. Finally we will make some further observations for $m \leqslant 4$, and complete the proof of Theorem 1.

Our method is rather unrefined, because of its heavy dependence on other papers (especially on [2], [8] and [9]). Thus it is far from self-containedness. Therefore it is desirable to give a simple proof of Theorem 1 without using the character theory of $GL(n, q)$.

We use the following notation: let G be a permutation group on a set Ω, and let, $\Delta \subset \Omega$ then G_Δ (resp. $G_{(\Delta)}$) denotes the pointwise (resp.setwise) stabilizer of Δ. Moreover let Δ be invariant by G, then G^Δ denotes the constituent of G on Δ. Moreover let us set $G^{(\Delta)}=(G_{(\Delta)})^\Delta$.

2. A review of a theorem of Piper. Proof of Theorem 1 for the case $n \geqslant 4$ and $q^{n-2} \nmid m$

A projective space is defined as a system of points and lines (i.e., subsets of points) connected by axioms of incidence in the usual way (see, for example, O. Veblen and J.W. Young [11]).

We dentote by $\mathcal{P}(d, q)$ the d-dimensional projective space defined over a finite field $\boldsymbol{F}_q$ with q elements, and denote by $\boldsymbol{P}$ (resp. $\boldsymbol{L}$) the set of points (resp. lines) in $\mathcal{P}(d, q)$.

A system S of points $\boldsymbol{P}'$ and lines $\boldsymbol{L}'$ is said to be a subspace of $\mathcal{P}(d, q)$, if $\boldsymbol{P}'\subset\boldsymbol{P}$ and any line $l'\in\boldsymbol{L}'$ is contained in some line $l\in\boldsymbol{L}$, and if $\boldsymbol{P}'$ and $\boldsymbol{L}'$ themselves form a projective space. A subspace S is said to be complete, if $l\in\boldsymbol{L}'$ implies $l\in\boldsymbol{L}$. Note that every complete subspace is a subspace of $\mathcal{P}(d, q)$ naturally induced from a linear subspace of the $(d+1)$-dimensional vector space over $\boldsymbol{F}_q$ defining $\mathcal{P}(d, q)$, and vice versa.

A collineation of $\mathcal{P}(d, q)$ is a permutation of the points which transforms every three collinear points onto three collinear points, and this is equivalent to say that a collineation is a permutation of the complete subspaces preserving their dimension and incidence.

A collineation σ of $\mathcal{P}(d, q)$ is said to be an elation, if it fixes every point on a fixed hyperplane (called an axis of σ) and every hyperplane through a fixed point (called center of σ) lying on the hyperplane and fixes no other points or hpyerplanes. Let π be a collineation group of $\mathcal{P}(d, q)$, and let there exist two elations in π which have same axis and distinct centers, then the line joining the two centers is called an axis line for π.

In [8, 9] F.C. Piper proved the following theorem.

Theorem of Piper. *Let π be a collineation group of $\mathcal{P}(d, q)$ such that* (i) *π fixes no subspace of $\mathcal{P}(d, q)$,* (ii) *some hyperplane is the axis of elations in π for more than one centers. Then either π contains the little projective group $PSL(d+1, q)$, or $(d, q)=(2, 4)$ and $\pi\cong A_6$ or S_6.*

We will prove the following lemma which is a slight extension of Theorem of Piper.

Lemma 1. *Let a proper subgroup π of $PSL(d+1, q)$ $(d\geqslant 3)$, regarded as a collineation group of $\mathcal{P}(d, q)$, fix no complete subspace of $\mathcal{P}(d, q)$ $(d\geqslant 3)$, and let some axis has more than one center, then π fixes the subspace S consisting of all the elation centers and the axis lines for π. Moreover, S is a desarguesian projective space of dimension d defined over $F_{q'}$ with $(q')^j=q$ for some $j\geqslant 2$.*

Proof. By examining the proof of the theorem of Piper in [8 and 9], we can easily see that π fixes the subspace S consisting of all the elation centers and the axis lines for π. Therefore we have only to prove the latter assertion that $S\cong\mathcal{P}(d, q')$ with $(q')^j=q$ for some $j\geqslant 2$. Since π fixes no complete subspace, the complete subspace generated by S in $\mathcal{P}(d, q)$ is $\mathcal{P}(d, q)$ itself. So we have $\dim S\geqslant d$, because there exist $d+1$ points of S which are in general position in $\mathcal{P}(d, q)$ and these $d+1$ points are of course in general position in S. Thus S is desarugesian, since $\dim S\geqslant d\geqslant 3$. Next we will show that $\dim S\leqslant d$. Let $H^{(1)}$ be an axis for π. Then $S\cap H^{(1)}$ is clearly a subspace of S, and moreover is a complete subspace, since every line in S meets the complete subspace according to Lemma 3 in [8] and Remark 4 in [9]. (Note that the conclusion of Lemma 3 in [8] and Remark 4 in [9] are both valid under the assumption of our Lemma 1.) Thus we have $\dim S \leqslant \dim (S \cap H^{(1)})+1$. Now there exists an axis $H^{(2)}$ for π such that $H^{(1)} \supsetneqq H^{(1)} \cap H^{(2)}$, according to an extension of Lemma 5 in [8]. (Note that the conclusion of Lemma 5 in [8] is valid for π under the assumption of this lemma. Especially this is valid even if q is even.) Thus $S \cap H^{(1)} \cap H^{(2)}$ is a complete subspace of $S \cap H^{(1)}$, and we have $\dim (S \cap H^{(1)}) \leqslant \dim (S \cap H^{(1)} \cap H^{(2)})+1$ by Lemma 3 in [8] and Remark 4 in [9], since every line in $S \cap H^{(1)}$ meets the complete subspace $S \cap H^{(1)} \cap H^{(2)}$. Thus, there exists inductively for $i=3, 4, \cdots, d-1$ an axis $H^{(i)}$ for π such that $S \cap H^{(1)} \cap \cdots \cap H^{(i)}$ is a complete subspace of $S \cap H^{(1)} \cap \cdots \cap H^{(i-1)}$ by Lemma 5 in [8], and we have

$$\dim (S \cap H^{(1)} \cap \cdots \cap H^{(i-1)}) \leqslant \dim (S \cap H^{(1)} \cap \cdots \cap H^{(i)})+1$$

by Lemma 3 in [8] and Remark 4 in [9]. Clearly $\dim(S \cap H^{(1)} \cap \cdots \cap H^{(d-1)}) \leqslant 1$. Hence, we have $\dim S \leqslant d$, and so we have $\dim S=d$. Let $S \simeq \mathcal{P}(d, q')$. We have obviously from the existence of an elation, $q' | q$ ($q' \lneqq q$). Now we can assume that q' is not a prime. Let $l \in L$ be an axis line. Then $PSL(d+1, q)^{(l)}$ [1] is a subgroup of $PGL(2, q)$, the group of projective collineations of the projective line l, and so $\pi^{(l)}$ is a subgroup of $PGL(2, q)$. While $\pi^{(l \cap S)}$ is a subgroup of $PGL(2, q')$. By Result 1 in [8] together with Lemma 5 in [8] $\pi^{(l \cap S)}$ is transitive on $S \cap l$, and the classification of subgroups of $PGL(2, q')$ shows that either $\pi^{(l \cap S)} \supseteqq PSL(2, q')$ or $q=$even and $\pi^{(l \cap S)}$ is the dihedral group of order $2(q'+1)$[2]. Since $|\pi^{(l \cap S)}|$ must divide $|PGL(2, q)|$, we have $(q')^j=q$ for some j, owing to the classification of subgroups of $PGL(2, q)$. Hence we completed the proof of Lemma 1.

Lemma 2. *Let H be a subgroup of index m of $G=PSL(n, q)$ with $n \geqslant 4$, and let $q^{n-2} \nmid m$. Then H fixes some complete subspace of $\mathcal{P}(n-1, q)$.*

(This is a generalization of the result concerning $PSL(n, q)$ in [11]. The result of this lemma may have an independent interest.)

Proof. Let $x=\begin{pmatrix} 1 & a_2 \cdots a_n \\ & 1 \quad 0 \\ & \ddots \\ & 0 \quad 1 \end{pmatrix} \in GL(n, q)$ with some $a_i \neq 0$, then the collineation $\bar{x}$ of $\mathcal{P}(n-1, q)$ is an elation with the axis $H_{n-1}=\overline{\{(x_1, \cdots, x_n)}; x_i \in F_q, x_1=0\}$, and the center $\overline{(0, a_2, \cdots, a_n)}$. And the Sylow's theorem shows that H contains two elations with the same axis and distinct centers. (Note that a Sylow p-subgroup of some conjugate of H is contained in the group of upper triangular unipotent matrices (i.e., a Sylow p-subgroup of G) and the index of the Sylow

1) See the notation at the end of Section 1.

2) Cf. D.G. Higman and J.E. McLaughlin, Rank 3 subgroups of finite symplectic and unitary groups, Lemma 1, page 179.

p-subgroup of the conjugate subgroup of H in the upper triangular unipotent matrices is not divisible by q^{n-2}, and that the Sylow p-subgroup of the conjugate subgroup of H (hence the conjugate subgroup of H) contains two such elations with the axis H_{n-1}.) Let us assume that H fixes no complete subspace of $\mathcal{P}(d, q)$. Then, by Lemma 1, H fixes the subspace S, and we have $|H|=|H_S|\cdot|H^S|$. But H_S is not divisible by p, because the set of the fixed points by an element of order p of $PSL(n, q)$ is contained in some hyperplane and S is not contained in any hyperplane. While, since every element of $PSL(n, q)$ which fixes the subspace S induces a collineation of S, (because, since S is a subspace, any three collinear points in S is transformed onto three collinear points) H^S is regared as a subgroup of the full collineation group $P\Gamma L(n, q')$ of S. But clearly $|P\Gamma L(n, q')|$ is not divisible by $q'\cdot q'^{(n/2)(n-1)}$. Therefore index m is divisible by $q^{n(n-1)/2}/(q'\cdot q'^{(n/2)(n-1)})\geqslant q^{n-2}$, but this is a contradication and the lemma is proved.

Proof of Theorem 1 *for the case* $n\geqslant 4$ *and* $q^{n-2}\not| m$. Let $n\geqslant 4$ and $q^{n-2}\not| m$. Then by Lemma 2, the stabilizer H of a point of Ω, must fix some complete subspace of $\mathcal{P}(n-1, q)$. Since H is maximal in G, H is the subgroup consisting of all elements of G which fix an r-dimentional complete subspace of $\mathcal{P}(n-1, q)$, and it is well known that the number of orbits of H on Ω (i.e., the rank of the permutation group (G, Ω)) is equal to $\min\{2+r, n+1-r\}$. Especially this is equal to 2 if and only if $r=0$ or $r=n-1$, hence the assertion is proved.

3. A bound of the degree m

Lemma 3. *Let a finite group G be doubly transitive on a set Ω, $|\Omega|=m$, then for each non-identity element of G, there exist at least $m-1$ elements of G which are conjugate to the element.*

(This in the Lemma 1 in Ed. Maillet [6], However we repeat the proof for completeness.)

Proof. Let a non-identity element x of G be expressed as a cyclic permutation on the set Ω as follows:

$$x=(a, b, \cdots)\cdots, \qquad a, b\in\Omega,$$

where the cycle containing a is of length greater than 1. Since G is doubly transitive on Ω, G_a, the stabilizer of a point $a\in\Omega$, is transitive on the set $\Omega-\{a\}$, hence for every $b_i\in\Omega-\{a\}$ $(i=s, \cdots, m-1)$ there exists an element $y_i\in G_a$ such that $b^{y_i}=b_i$. But $y_i^{-1}xy_i$ $(i=1, \cdots, m-1)$ are all distinct from each other, and the assertion is proved.

Lemma 4. *Under the assumption of Theorem* 1, *we have* $m-1\leqslant(q^n-1)\cdot(q^{n-1}-1)/(q-1)$.

Proof. The number of elements of $PSL(n, q)$ which are conjugate to a fixed elation is $\leq (q^n-1)(q^{n-1}-1)/(q-1)$, hence we have the sasertion by Lemma 3.

4. Characters of the group $GL(n, q)$. Proof of Theorem 1 for the case $n \geqslant 5$ and $q^{n-2} | m$

Let $G=PSL(n, q)$ be doubly transitive on a set Ω, $|\Omega|=m$, and let us assume that $n \geqslant 4$ and $q^{n-2}|m$. Then G has the irreducible character φ_1 such that $\varphi_1(x)=I(x)-1$ $(x \in G)$ where I denotes the permutation character of (G, Ω).

Now we will determine which irreducible character φ of G satisfy the following two conditions (1) and (2).

1) $\varphi(1) \leqslant (q^n-1)(q^{n-1}-1)/(q-1)$,
2) $q^{n-2} | (\varphi(1)+1)$.

Clearly, from our assumption and Lemma 4, the irreducible character φ_1 must satisfy the conditions (1) and (2).

As is obvious from the theorem of Clifford, for any irreducible character φ of $G=PSL(n, q)$, there is associated some irreducible character χ of $GL(n, q)$ such that

$$\varphi(1) = \frac{1}{\alpha}\chi(1),$$

where $\alpha | (n, q-1)$.

(Note that $PGL(n, q)$ is a factor group of $GL(n, q)$ and that $PSL(n, q)$ is a normal subgroup of $PGL(n, q)$ such that the factor group $PGL(n, q)/PSL(n, q)$ is a cyclic group of order $(n, q-1)$).

As the first step of the determination of irreducible characters of G satisfying the conditions (1) and (2), we will determine which irreducible character χ of $GL(n, q)$ with $n \geqslant 4$ satisfy the following two conditions,

1') $\chi(1) \leqslant (q^n-1)(q^{n-1}-1)$,
2) $\chi(1)$ is prime to q.

Clearly, if χ is an irreducible character of $GL(n, q)$ associated to an irreducible character of G satisfying the conditions (1) and (2), then χ satisfies the conditions (1') and (2').

Owing to J.A. Green [2], we have the following lemma.

Lemma 5. *Let χ be an irredubcile character of $GL(n, q)$ whose degree $\chi(1)$ is prime to q, then there exists a partition of n, $n_1+n_2+\cdots+n_r=n$, positive integers s_i and v_i such that $s_i v_i = n_i$ $(i=1, \cdots, r)$ and s_i-simplexes $k^{(i)}(i=1, \cdots, r)$, and we have*

$$\chi = I_{s_1}^{k^{(1)}}[v_1] \circ \cdots \circ I_{s_r}^{k^{(r)}}[v_r]\,.^{3)}$$

3) In this notation we understand that if the right hand side is a negative character then χ is (-1) multiple of the negative character.

Moreover,

$$\chi(1)=\frac{\psi_n(q)}{\psi_{n_1}(q)\cdots\psi_{n_r}(q)}\frac{\psi_{n_1}(q)}{\psi_{v_1}(q^{s_1})}\cdots\frac{\psi_{n_r}(q)}{\psi_{v_r}(q^{s_r})},$$

where $\psi_l(q)=(q^l-1)(q^{l-1}-1)\cdots(q-1)$.

(For the notation and the proof of the lemma, see [2], especially Lemma 2.7, Lemma 7.4 and Theorem 13 in [2].)

Using Lemma 5, we can get the following lemma. Since the proof is straightforward and easy, we omit it.

Lemma 6. *If an irredicuble character* χ *of* $GL(n, q)$ *with* $n\geqslant 4$ *satisfies the conditions* (1′) *and* (2′), *then one of the following cases occurs.*

(Here we may assume that $n_1\leqslant n_2\leqslant\cdots\leqslant n_n$, and that $s_i\leqslant s_j$, if $n_i=n_j$ and $i\leqslant j$. Here we omit the parameter $k^{(i)}$ of $I_{s_i}^{k^{(i)}}[v_i]$. The s_i-simplexes $k^{(i)}$ must be suitably chosen. Especially, if $q=2$, then the cases 1°) 2°), 4°), 13°) and 16°) do not occur, because there exists only one 1-simplex if $q=2$, see [2].)

1°) $\chi=I_1[1]\circ I_1[n-1]$, $\chi(1)=(q^{n-1}+\cdots+q+1)$.
2°) $\chi=I_1[2]\circ I_1[n-2]$, $\chi(1)=(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)/(q+1)$.
3°) $\chi=I_2[1]\circ I_1[n-2]$. $\chi(1)=(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)(q-1)/(q+1)$.
4°) $\chi=I_1[1]\circ I_1[1]\circ I_1[n-2]$, $\chi(1)=(q^{n-1}+\cdots+q+1)(q^{n-2}+\cdots+q+1)$.
5°) $n=4$, $\chi=I_4[1]$, $\chi(1)=(q^3-1)(q^2-1)(q-1)$
6°) $n=4$, $\chi=I_2[2]$, $\chi(1)=(q^3-1)(q-1)$
7°) $n=4$, $\chi=I_1[1]\circ I_3[1]$, $\chi(1)=(q^3+q^2+q^2+1)(q^2-1)(q-1)$.
8°) $n=4$, $\chi=I_2[1]\circ I_2[1]$, $\chi(1)=(q^3+q^2+q+1)(q^2+q+1)(q-1)^2/(q+1)$.
9°) $n=4$, $\chi=I_1[2]\circ I_2[1]$, $\chi(1)=(q^3+q^2+q+1)(q^2+q+1)(q-1)/(q+1)$
10°) $n=5$, and $q=2$, $\chi=I_5[1]$, $\chi(1)=(q^4-1)(q^3-1)(q^2-1)/(q-1)$.
11°) $n=5$, $\chi=I_1[1]\circ I_2[2]$, $\chi(1)=(q^4+q^3+q^2+q+1)(q^3-1)(q-1)$.
12°) $n=5$, $\chi=I_1[2]\circ I_3[1]$, $\chi(1)=(q^4+q^3+q^2+q+1)(q^3+q^2+q+1)(q-1)^2$.
13°) $n=5$, $\chi=I_1[1]\circ I_1[2]\circ I_1[2]$,
$\chi(1)=(q^4+q^3+q^2+q+1)(q^3+q^2+q+1)(q^2+q+1)/(q+1)$.
14°) $n=6$, $\chi=I_2[3]$, $\chi(1)=(q^5-1)(q^3-1)(q-1)$.
15°) $n=6$, $\chi=I_3[2]$, $\chi(1)=(q^5-1)(q^4-1)(q^2-1)(q-1)$.
16°) $n=6$, $\chi=I_1[3]\circ I_1[3]$,
$\chi(1)=(q^5+\cdots+q+1)(q^4+\cdots+q+1)(q^3+q^2+q+1)/(q^2+q+1)(q+1)$.
17°) $n=8$ and $q=2$, $\chi=I_2[4]$, $\chi(1)=(q^7-1)(q^5-1)(q^3-1)(q-1)$.

Using Lemma 6 together with the following easily verified Remark, we have the next Lemma 7.

REMARK. Let $f(x)$ be a polynomial with integral coefficients such that $f(0)=1$ (resp. $f(0)=-1$). If $\frac{1}{\alpha}f(q)+1$, where $\alpha\mid(q-1)$, is an integer and is divisible by q, then $\alpha=q-1$ (resp. $\alpha=1$).

Lemma 7. *If φ is an irreducible character of $G=PSL(n, q)$ satisfying the conditions* (1) *and* (2), *then one of the following cases occurs.*

i) $n=4$, $\varphi(1)=(q^2+1)(q^2+q+1)(q-1)$, *the associated character* χ *of* φ *is* $I_1[2]\circ I_2[1]$ *and* $\alpha=1$

ii) $n=4$, $\varphi(1)=q^3-1$, *the associated character* χ *of* φ *is* $I_2[2]$ *and* $\alpha=q-1$.

iii) $n=4$, $\varphi(1)=(q^2+1)(q^2+q+1)(q-1)$, *the associated character* χ *of* φ *is* $I_2[1]\circ I_2[1]$ *and* $\alpha=q-1$.

Proof. Let χ be an irreducible character of $GL(n, q)$ associated to φ. Then, χ is one of the characters (1°)~(17°) in Lemma 6. Let us assume that for χ the case (1°) or (2°) holds. Then $\alpha=q-1$ by the above Remark, and $q^{n-2}\nmid\left(\frac{1}{\alpha}\chi(1)+1\right)$, since $n\geqslant 4$ and $q\neq 2$. But this contradicts the assumption that φ satisfies the condition (2). Let us assume that for χ the case (3°) of Lemma 6 holds. Then $\alpha=1$, and $q^{n-2}\left|\left(\frac{1}{\alpha}\chi(1)+1\right)\right.$ if and and only if $n=4$, hence the case (i) holds. By the similar argument we can easily show that only the cases (ii) and (iii) hold, if one of the cases (4°)~(17°) of Lemma 6 holds for χ.

Proof of Theorem 1 *for the case* $n\geqslant 5$ *and* $q^{n-2}|m$. This case does not occur, because by Lemma 7, there exists no irreducible character φ of G satisfying the conditions (1) and (2).

5. Proof of Theorem 1 for the case $n\leqslant 4$

The case $n=4$. Let $n=4$, then we may assume that $q^2|m$. By Lemma 7, we have either $m=q^2(q^3+q-1)$ (q being arbitrary) or $m=q^3$ ($q=2$, 3 or 5). The first case is impossible, since it is easily verified that m does not divide the order of G.

(1) Let $q=2$ ($m=8$). Then this case does occur because $PSL(4, 2)\cong A_8$ and doubly transitive on 8 points. Therefore the case (VIII) in Theorem 1 holds. Uniqueness of doubly transitive permutation representation of $PSL(4, 2)$ on 8 points is clear.

(2) Let $q=3$ ($m=27$). Let H be the stabilizer of a point in Ω. Obviously, G contains a subgroup K which is isomorphic to $PSp(4, 3)$. But according to L.E. Dickson [1], $PSp(4, 3)$ is represented as a permutation group on 27 points, and is not represented on less than 27 points if the action is nontrivial. Moreover we have that the minimal degree (=class) of $PSp(4, 3)$ on the 27 points is 12. But the result of W.A. Manning (cf. [12], page 43) on permutation groups of small minimal degrees shows that $PSL(4, 3)$ is not represented as a doubly transitive permutation group on 27 points. Hence, this case does not occur.

(3) Let $q=5$ ($m=125$). Let H be the stabilizer of a point in Ω. Obviously, G contains a subgroup K which is isomorphic to $PSp(4, 5)$. We have $|K\colon K\cap H|=|KH|/|H|\leqslant|H\colon G|=125$. But according to L.E. Dickson [1],

$PSp(4, 5)$ contains no proper subgroup whose index is not greater than 125 (this also due to C. Jordan). Hence $H \supseteqq K$, but this is a contracdition, since $125 \not| \, |G : K|$.

The case $n=3$. Let $n=3$. If m is prime to q, then owing to the Theorem of F.C. Piper [8 and 9], the case (I) or (II) of Theorem 1 hold. Let us assume that m is not prime to q. If $q=2$, $G=PSL(3, 2) \cong PSL(2, 7)$ and it has a doubly transitive permutation representation on 8 points, and has no other doubly transitive permutation representation of even degree. Uniqueness of doubly transitive representations on 8 points is clear. Let $q \neq 2$, then the same methods as previous section shows that the degree of φ_1 must be either 7 (for $q=4$), 28 (for $q=4$), q^3-1 (q being arbitrary), 15 (for $q=4$). But according to H.H. Mitchell [7] and R.W. Hartley [3], there exist no subgroups of index 8 (for $q=4$), 29 (for $q=4$), 16 (for $q=4$) and q^3 (for arbitrarily q). This is a contradiction.

The assertion of Theorem 1 for $n=2$ is well known, and we omit the proof.

Thus, we completed the proof of Theorem 1.

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