Fujiwara, T.
Osaka J. Math.
8 (1971), 425-436

ON THE CONSTRUCTION OF THE LEAST UNIVERSAL HORN CLASS CONTAINING A GIVEN CLASS

Dedicated to Professor Keizo Asano on his 60th birthday

TSUYOSHI FUJIWARA

(Received November 7, 1970)

Many mathematicians have studied universal Horn classes of structures or quasi-primitive (or implicational) classes of algebras. As is well known, these classes have many important algebraic properties which are satisfied in primitive (or equational) classes, for example, to be closed under the formation of direct products, to possess free structures or free algebras if these classes are non-empty. But they are not generally closed under the formation of homomorphic images. Among the theorems with respect to primitive classes, there is the well known most fundamental theorem concerning the construction of the least primitive class containing a given class of algebras. That is, it is obtained by making direct products, subalgebras, and homomorphic images. The main purpose of this paper is to show the analogical theorem which gives the construction of the least universal Horn class containing a given class of structures.

In §1, we simply explain the basic concept and notation with respect to structures for a first order language, and state some well known results which are used in the succeeding sections. In §2, we shall show the definition of a free structure satisfying defining relations, and study the relation between the class K with free structures satisfying defining relations and the formation of substructures and direct products of structures in K. In §3, we shall give the definition of a natural limit structure as a direct limit of a special direct family, and shall show the fact that, a structure $\mathfrak{A}$ is in a universal Horn class K if and only if $\mathfrak{A}$ can be represented as a natural limit structure with respect to K. In §4, applying the results in the above sections, we shall prove the theorem: All isomorphic copies of direct limits of substructures of direct products of structures in a class K form the least universal Horn class containing the class K. Finally, §5 gives a characterization of universal Horn classes by using natural limit structures.

1. Terminologies, notation, and some lemmas

A first order structure or simply structure $\mathfrak{A}$ means a non-empty set A on which finitary operations and finitary relations are defined. As is well known, there is the one to one correspondence between the similarity types of structures and the first order languages (with the equality symbol$=$). A structure of the similarity type corresponding to a first order language L is simply called a structure for L. Let F be the set of all operation symbols of a first order language L, and R the set of all relation symbols of L. The operation and the relation on the base set A of a structure $\mathfrak{A}$ for L which correspond to an operation symbol f and a relation symbol r are called interpretations of f and r on A, and denoted by $(f)_A$ and $(r)_A$ respectively. Then the structure $\mathfrak{A}$ is denoted by $\langle A, F_A, R_A\rangle$, where $F_A=\{(f)_A \mid f\in F\}$ and $R_A=\{(r)_A \mid r\in R\}$. Sometimes the base set A of $\mathfrak{A}$ is denoted by $|\mathfrak{A}|$.

Let $\mathfrak{A}=\langle A, F_A, R_A\rangle$ and $\mathfrak{B}=\langle B, F_B, R_B\rangle$ be structures for L. A mapping Φ of A into (or onto) B is called an L-homomorphism of $\mathfrak{A}$ into (or onto) $\mathfrak{B}$ if the following two conditions are satisfied for any finite number of elements $a_1, \cdots, a_n\in A$:

(H_1) $\Phi((f)_A(a_1, \cdots, a_n))=(f)_B(\Phi(a_1), \cdots, \Phi(a_n))$ for any n-ary operation symbol $f\in F$.

(H_2) For any n-ary relation symbol $r\in R$, if $(r)_A(a_1, \cdots, a_n)$ holds in $\mathfrak{A}$ then $(r)_B(\Phi(a_1), \cdots, \Phi(a_n))$ holds in $\mathfrak{B}$.

The L-homomorphism Φ of $\mathfrak{A}$ into $\mathfrak{B}$ may be defined as a mapping Φ of A into B such that for any atomic formula† $\theta(x_1,\cdots, x_n)$ of L and for any elements $a_1, \cdots, a_n\in A$, if $a_1,\cdots, a_n$ satisfy $\theta(x_1,\cdots, x_n)$ in $\mathfrak{A}$ then $\Phi(a_1), \cdots, \Phi(a_n)$ satisfy $\theta(x_1,\cdots, x_n)$ in $\mathfrak{B}$. An L-homomorphism Φ of $\mathfrak{A}$ onto $\mathfrak{B}$ is called an L-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$ if the mapping Φ is one to one and the inverse mapping Φ^{-1} is also an L-homomorphism. If there exists an L-isomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, we say that $\mathfrak{A}$ is L-isomorphic to $\mathfrak{B}$ or that $\mathfrak{A}$ and $\mathfrak{B}$ are L-isomorphic. A structure $\mathfrak{B}=\langle B, F_B, R_B\rangle$ is called a substructure of a structure $\mathfrak{A}=\langle A, F_A, R_A\rangle$, if B is a subset of A, and if each operation $(f)_B$ is the restriction of $(f)_A$ to B and each relation $(r)_B$ is the restriction of $(r)_A$ to B.

Let Σ be a set of sentences of L. A structure $\mathfrak{A}$ for L is called a model of Σ if every sentence in Σ is valid in $\mathfrak{A}$. The class of all models of Σ is denoted by Σ^*. If Σ is a set of universal sentences, then the class Σ^* is called a universal class. A formula of L is called a basic Horn formula (or a basic s-Horn formula), if it is of the form $\theta_1 \vee \cdots \vee \theta_n$, where θ_i is an atomic formula or the negation of an atomic formula, and at most (or exactly) one of them is an atomic formula. A universal sentence is called a universal Horn sentence (or a universal s-Horn sentence), if its matrix is a conjunction of basic Horn formulas (or basic s-Horn

---

†) We use an atomic formula in the usual meaning, not in the meaning in Grätzer [2]. That is, an atomic formula of L means a formula of the form $t_1=t_2$ or $r(t_1, \cdots, t_m)$, where r is an m-ary relation symbol of L and $t_1, \cdots, t_m$ are terms of L. Moreover an atomic formula $\theta(x_1, \cdots, x_n)$ denotes an atomic formula with free variables among $x_1, \cdots, x_n$.

formulas). If Σ is a set of universal Horn sentences (or universal s-Horn sentences), then the class Σ^* is called a universal Horn class (or a universal s-Horn class).

Let L be a first order language. $\boldsymbol{X}$ is called an operator if for every class K of structures for L, $\boldsymbol{X}(K)$ is a class of structures for L. If $\boldsymbol{X}$ and $\boldsymbol{Y}$ are operators, so is $\boldsymbol{XY}$ defined by $\boldsymbol{XY}(K)=\boldsymbol{X}(\boldsymbol{Y}(K))$. The operators used in this paper are $\boldsymbol{I}$, $\boldsymbol{S}$, $\boldsymbol{P}$, $\boldsymbol{P}^*$, $\boldsymbol{L}$ defined as follows:

$\boldsymbol{I}(K)$: all L-isomorphic copies of structures in K;
$\boldsymbol{S}(K)$: all substructures of structures in K;
$\boldsymbol{P}(K)$: all direct products of non-empty families of structures in K;
$\boldsymbol{P}^*(K)$: all direct products of empty or non-empty families of structures in K, where the direct product of the empty family of structures for L means the one-element structure $\mathfrak{E}=\langle\{\phi\}, F_{\{\phi\}}, R_{\{\phi\}}\rangle$† for L in which every atomic formula of L is valid;
$\boldsymbol{L}(K)$: all direct limits of direct families of structures in K.

A class K of structures for L is said to be abstract if K is closed under L-isomorphisms, i.e. $\boldsymbol{I}(K)\subseteq K$. The following results are well known (cf. [2; Chapter 7]):

Lemma 1. *Let K be a universal class. Then K is closed under the formation of substructures, i.e. $\boldsymbol{S}(K)\subseteq K$.*

Lemma 2. *Let K be a universal Horn class. Then K is closed under the formation of direct products of non-empty families of structures, i.e. $\boldsymbol{P}(K)\subseteq K$.*

Any universal s-Horn sentence of L is valid in the one-element structure $\mathfrak{E}=\langle\{\phi\}, F_{\{\phi\}}, R_{\{\phi\}}\rangle$. Hence the following lemma can be immediately obtained from Lemma 2:

Lemma 2^s. *Let K be a universal s-Horn class. Then K is closed under the formation of direct products of empty or non-empty families of structures, i.e. $\boldsymbol{P}^*(K)\subseteq K$.*

Let $\langle\{\mathfrak{A}_\mu \mid \mu\in M\}, \{\Phi_\mu^\nu \mid \mu, \nu\in M \text{ and } \mu\leqq\nu\}\rangle$ be a direct family of structures for L over a directed partially ordered set M, i.e. a family of structures $\mathfrak{A}_\mu$, $\mu\in M$, and of L-homomorphisms Φ_μ^ν of $\mathfrak{A}_\mu$ into $\mathfrak{A}_\nu$ for all $\mu\leqq\nu$ such that Φ_μ^μ is the identity mapping for all μ, and $\Phi_\mu^\nu\Phi_\lambda^\mu=\Phi_\lambda^\nu$ if $\lambda\leqq\mu\leqq\nu$. Now let $\hat{\mathfrak{A}}$ be the direct limit of the above direct family, and Φ_μ the canonical L-homomorphism of $\mathfrak{A}_\mu$ into $\hat{\mathfrak{A}}$. Then, the following result is well known (cf. [1; Lemma 1], [2; Lemma 8, p. 138])

Lemma 3. *Let $\theta(x_1, \cdots, x_n)$ be any atomic formula of L, and let $\hat{a}_1, \cdots, \hat{a}_n$*

---

†) ϕ denotes the empty set.

be any elements in $|\hat{\mathfrak{A}}|$. *Then,* $\hat{a}_1, \cdots, \hat{a}_n$ *satisfy* $\theta(x_1, \cdots, x_n)$ *in* $\hat{\mathfrak{A}}$ *if and only if there exist a structure* $\mathfrak{A}_\mu$ *and elements* $a_1, \cdots, a_n$ *in* $|\mathfrak{A}_\mu|$ *such that* $a_1, \cdots, a_n$ *satisfy* $\theta(x_1, \cdots, x_n)$ *in* $\mathfrak{A}_\mu$ *and* $\Phi_\mu(a_i) = \hat{a}_i$, $i=1, \cdots, n$.

Using the above lemma, we can easily verify the

Lemma 4. *Let K be a universal class. Then K is closed under the formation of direct limits, i.e.* $\boldsymbol{L}(K) \subseteq K$.

2. Free structures

Let E be a set of nullary operation symbols not belonging to a first order language L. Then, a new first order language can be obtained from L by adding all the nullary operation symbols $e \in E$, which is denoted by $L(E)$. Let $\mathfrak{A} = \langle A, F_A, R_A \rangle$ be a structure for L. Then $\mathfrak{A}$ can be considered as a structure for $L(E)$ by adding interpretations $(e)_A$ to A, i.e. by determining a mapping Ψ of E into A and putting $(e)_A = \Psi(e)$ for all $e \in E$. The resultant structure $\langle A, F_A \cup \{(e)_A \mid e \in E\}, R_A \rangle$ for $L(E)$ is denoted by $\mathfrak{A}(\Psi)$. Let K be a class of structures for L, and let Ω be a set of atomic sentences of $L(E)$, i.e. atomic formulas of $L(E)$ in which no free variable occurs. Then, a structure $\mathfrak{A} = \langle A, F_A, R_A \rangle$ for L is called a *K-free structure* satisfying Ω with a set E of generator symbols, and is denoted by $\mathfrak{F}(E, \Omega; K)$, if there exists a mapping Ψ of E into A such that the following three conditions are satisfied by putting $(e)_A = \Psi(e)$ for all $e \in E$:

(F_1) $\mathfrak{A}$ is generated by the set $\{(e)_A \mid e \in E\}$.

(F_2) $\mathfrak{A}$ is in K and $\mathfrak{A}(\Psi)$ is a model of Ω. We simply say that $\mathfrak{A}$ is a K-model of Ω under the interpretations $(e)_A$, $e \in E$.

(F_3) If $\mathfrak{B} = \langle B, F_B, R_B \rangle$ is a K-model of Ω under some interpretations $(e)_B$, $e \in E$, then there exists an $L(E)$-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$, i.e. an L-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ which maps $(e)_A$ to $(e)_B$ for each $e \in E$.

Note that if $\mathfrak{F}(E, \Omega; K)$ exists, then it is unique up to L-isomorphism. If both E and Ω are finite sets, then the K-free structure $\mathfrak{F}(E, \Omega; K)$ is said to be *finitely defined*.

First we shall state the

Theorem 1. *Let K be an arbitrary class of structures for a first order language L. Then any structure* $\mathfrak{A} = \langle A, F_A, R_A \rangle$ *in K can be represented as a K-free structure* $\mathfrak{F}(E, \Omega; K)$.

Proof. Let $E = \{e_a \mid a \in A\}$ be a set of nullary operation symbols not belonging to L, and let Ω be the set of all atomic sentences of $L(E)$ which are valid in $\mathfrak{A}$ under the interpretations $(e_a)_A = a$, $e_a \in E$. Then it is easy to see that $\mathfrak{A}$ is a K-free structure satisfying Ω with the set E of generator symbols.

Let E be an empty or non-empty set of nullary operation symbols not belonging to a first order language L. The set E is said to be *L-generative*, if $L(E)$ contains at least one nullary operation symbol. Now we have the

Theorem 2. *Let K be an abstract class of structures for a first order language L. Then the following two conditions are equivalent:*

(1) *For any L-generative set E of nullary operation symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, there exists a K-free structure $\mathfrak{F}(E, \Omega; K)$ whenever a K-model of Ω exists.*

(2) $\boldsymbol{S}(K) \subseteq K$ *and* $\boldsymbol{P}(K) \subseteq K$.

Proof of (1)⇒(2). First we shall prove that $\boldsymbol{S}(K) \subseteq K$. Let $\mathfrak{A} = \langle A, F_A, R_A \rangle$ be any structure in K, and let $\mathfrak{B} = \langle B, F_B, R_B \rangle$ be any substructure of $\mathfrak{A}$. Now let $E = \{e_b \mid b \in B\}$ be a set of nullary operation symbols not belonging to L, and Ω the set of all atomic sentences of $L(E)$ which are valid in $\mathfrak{B}$ under the interpretations $(e_b)_B = b$, $e_b \in E$. Then clearly $\mathfrak{A}$ is a K-model of Ω under the interpretations $(e_b)_A = b$, $e_b \in E$. Hence there exists a K-free structure $\mathfrak{F}(E, \Omega; K)$, and hence there exists an $L(E)$-homomorphism of $\mathfrak{F}(E, \Omega; K)$ into $\mathfrak{A}$, i.e. an L-homomorphism of $\mathfrak{F}(E, \Omega; K)$ into $\mathfrak{A}$ which maps $(e_b)_{|\mathfrak{F}(E,\Omega;K)|}$ to $(e_b)_A$ for each $e_b \in E$. Therefore there exists an L-homomorphism Φ of $\mathfrak{F}(E, \Omega; K)$ onto $\mathfrak{B}$ which maps $(e_b)_{|\mathfrak{F}(E,\Omega;K)|}$ to $(e_b)_B$ for each $e_b \in E$. On the other hand, any atomic sentence of $L(E)$ which is valid in $\mathfrak{B}$ under the interpretations $(e_b)_B$, $e_b \in E$, is also valid in $\mathfrak{F}(E, \Omega; K)$ under the interpretations $(e_b)_{|\mathfrak{F}(E,\Omega;K)|}$, $e_b \in E$, because any atomic sentence of $L(E)$ which is valid in $\mathfrak{B}$ is contained in Ω and $\mathfrak{F}(E, \Omega; K)$ is a K-model of Ω. Hence Φ is an L-isomorphism, and hence $\mathfrak{B}$ is in K. Therefore we have $\boldsymbol{S}(K) \subseteq K$.

Hereafter we shall prove that $\boldsymbol{P}(K) \subseteq K$. Let $\mathfrak{A}_i = \langle A_i, F_{A_i}, R_{A_i} \rangle$, $i \in I$, be any structures in K, where I is a non-empty set. And let $\mathfrak{A} = \langle A, F_A, R_A \rangle$ be the direct product of all $\mathfrak{A}_i$, $i \in I$. Now let $E = \{e_a \mid a \in A\}$ be a set of nullary operation symbols not belonging to L, and Ω the set of all atomic sentences of $L(E)$ which are valid in $\mathfrak{A}$ under the interpretations $(e_a)_A = a$, $e_a \in E$. Then $\mathfrak{A}_i$ is clearly a K-model of Ω under the interpretations $(e_a)_{A_i} = \Pi_i(a)$, $e_a \in E$, where Π_i is the projection of A onto A_i. Hence there exists a K-free structure $\mathfrak{F}(E, \Omega; K)$. Moreover there exists an L-homomorphism Φ of $\mathfrak{A}$ onto $\mathfrak{F}(E, \Omega; K)$ which maps $(e_a)_A$ to $(e_a)_{|\mathfrak{F}(E,\Omega;K)|}$ for each $e_a \in E$, because any atomic sentence of $L(E)$ which is valid in $\mathfrak{A}$ under the interpretations $(e_a)_A$, $e_a \in E$, is contained in Ω, and $\mathfrak{F}(E, \Omega; K)$ is a K-model of Ω under the interpretations $(e_a)_{|\mathfrak{F}(E,\Omega;K)|}$ and is generated by $\{(e_a)_{|\mathfrak{F}(E,\Omega;K)|} \mid e_a \in E\}$. On the other hand, for each $i \in I$ there exists an L-homomorphism Φ_i of $\mathfrak{F}(E, \Omega; K)$ onto $\mathfrak{A}_i$ such that $\Phi_i((e_a)_{|\mathfrak{F}(E,\Omega;K)|}) = \Pi_i(a)$ for all $e_a \in E$, because $\mathfrak{A}_i$ is a K-model of Ω under the interpretations $(e_a)_{A_i} = \Pi_i(a)$, $e_a \in E$. Hence any atomic sentence θ of $L(E)$ which is valid in

$\mathfrak{F}(E, \Omega; K)$ under the interpretations $(e_a)_{|\mathfrak{F}(E,\Omega;K)|}$, $e_a \in E$, is valid in every $\mathfrak{A}_i$ under the interpretations $(e_a)_{A_i} = \Pi_i(a)$, $e_a \in E$, and hence the θ is valid in $\mathfrak{A}$ under the interpretations $(e_a)_A = a$, $e_a \in E$. Therefore the L-homomorphism Φ of $\mathfrak{A}$ onto $\mathfrak{F}(E, \Omega; K)$ is an L-isomorphism. Hence $\mathfrak{A}$ is in K. Therefore we have $\boldsymbol{P}(K) \subseteq K$.

Proof of (2)$\Rightarrow$(1). Assume that $\boldsymbol{S}(K) \subseteq K$ and $\boldsymbol{P}(K) \subseteq K$. Let E be any L-generative set of nullary operation symbols not belonging to L, and let Ω be any set of atomic sentences of $L(E)$ such that a K-model of Ω exists. In the following, we shall prove that a K-free structure $\mathfrak{F}(E, \Omega; K)$ exists. It is easily seen that there exists a non-empty set $\{\mathfrak{A}_i = \langle A_i, F_{A_i}, R_{A_i} \rangle \mid i \in I\}$ of K-models of Ω such that, if $\mathfrak{B} = \langle B, F_B, R_B \rangle$ is a K-model of Ω under interpretations $(e)_B$, $e \in E$, then for some $i \in I$ the substructure $\mathfrak{A}_i'$ of $\mathfrak{A}_i$ generated by the set $\{(e)_{A_i} \mid e \in E\}$ and the substructure $\mathfrak{B}'$ of $\mathfrak{B}$ generated by the set $\{(e)_B \mid e \in E\}$ are $L(E)$-isomorphic. The direct product $\mathfrak{A} = \langle A, F_A, R_A \rangle$ of all $\mathfrak{A}_i$, $i \in I$, is contained in K, because $\boldsymbol{P}(K) \subseteq K$. For every $e \in E$, the interpretation $(e)_A$ is defined as the element $a \in A$ that satisfies $\Pi_i(a) = (e)_{A_i}$ for all $i \in I$, where Π_i are the projections of A onto A_i. Then it is clear that $\mathfrak{A}$ is a K-model of Ω, and the substructure $\mathfrak{A}'$ of $\mathfrak{A}$ generated by the set $\{(e)_A \mid e \in E\}$ is also a K-model of Ω, because $\boldsymbol{S}(K) \subseteq K$. Next let $\mathfrak{B}$ be any K-model of Ω. Then for some $i \in I$, there exists an $L(E)$-isomorphism Φ_i of the substructure $\mathfrak{A}_i'$ of $\mathfrak{A}_i$ generated by the set $\{(e)_{A_i} \mid e \in E\}$ into $\mathfrak{B}$. On the other hand, let Π_i' be the restriction of Π_i to $|\mathfrak{A}'|$. Then Π_i' is an $L(E)$-homomorphism of $\mathfrak{A}'$ onto $\mathfrak{A}_i'$. Hence the product $\Phi_i \Pi_i'$ is an $L(E)$-homomorphism of $\mathfrak{A}'$ into $\mathfrak{B}$. Therefore $\mathfrak{A}'$ is a K-free structure satisfying Ω with the set E of generator symbols. This completes the proof.

The next theorem can be easily obtained from the above theorem:

Theorem 2^s. *Let K be an abstract class of structures for a first order language L. Then the following two conditions are equivalent:*

(1) *For any L-generative set E of nullary operation symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, there always exists a K-free structure $\mathfrak{F}(E, \Omega; K)$.*

(2) $\boldsymbol{S}(K) \subseteq K$ *and* $\boldsymbol{P}^*(K) \subseteq K$.

Proof. Suppose that there always exists $\mathfrak{F}(E, \Omega; K)$. Then $\boldsymbol{S}(K) \subseteq K$ and $\boldsymbol{P}(K) \subseteq K$ follow from Theorem 2. Moreover, it is easy to see that K contains a one-element structure for L in which every atomic formula of L is valid. Hence K contains the one-element structure $\mathfrak{E} = \langle \{\phi\}, F_{\{\phi\}}, R_{\{\phi\}} \rangle$ for L in which every atomic formula of L is valid, because K is an abstract class. Hence we have $\boldsymbol{P}^*(K) \subseteq K$.

Conversely, assume that $\boldsymbol{S}(K) \subseteq K$ and $\boldsymbol{P}^*(K) \subseteq K$. Then K contains the one-element structure $\mathfrak{E}=\langle\{\phi\}, F_{\{\phi\}}, R_{\{\phi\}}\rangle$ for L in which every atomic formula of L is valid. Now it is easy to see that, for any L-generative set E of nullary operation symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, $\mathfrak{E}$ is a K-model of Ω. Hence by Theorem 2, $\mathfrak{F}(E, \Omega; K)$ always exists.

3. Natural limit structures

Let K be a class of structures for a first order language L. Let E be an L-generative set of nullary operation symbols not belonging to L, and let Ω be a set of atomic sentences of $L(E)$. Now we define $M(E, \Omega)$ as the set of all pairs (X, Γ) such that X is a finite L-generative subset of E and Γ is a finite set of atomic sentences of $L(X)$ which belong to Ω. Then, $M(E, \Omega)$ forms a directed partially ordered set under the order relation defined as follows: $(X, \Gamma) \leqq (Y, \Delta)$ if and only if both $X \subseteq Y$ and $\Gamma \subseteq \Delta$. Now we assume that a finitely defined K-free structure $\mathfrak{F}(X, \Gamma; K)$ exists for each $(X, \Gamma) \in M(E, \Omega)$. Then it is easy to see that, if $(X, \Gamma) \leqq (Y, \Delta)$ then there exists an L-homomorphism $\Phi^{(Y,\Delta)}_{(X,\Gamma)}$ of $\mathfrak{F}(X, \Gamma; K)$ into $\mathfrak{F}(Y, \Delta; K)$ which maps $(e)_{|\mathfrak{F}(X,\Gamma;K)|}$ to $(e)_{\mathfrak{F}|(Y,\Delta;K)|}$ for each $e \in X$. Thus the pair of sets $\langle\{\mathfrak{F}(X, \Gamma; K) \mid (X, \Gamma) \in M(E, \Omega)\}, \{\Phi^{(Y,\Delta)}_{(X,\Gamma)} \mid (X, \Gamma), (Y, \Delta) \in M(E, \Omega)$ and $(X, \Gamma) \leqq (Y, \Delta)\}\rangle$ forms a direct family, which is called a direct family naturally defined by $(E, \Omega; K)$. A direct limit of a direct family naturally defined by $(E, \Omega; K)$ is called a *natural limit structure* with respect to $(E, \Omega; K)$, and denoted by $\mathfrak{L}(E, \Omega; K)$. It is easy to see that $\mathfrak{L}(E, \Omega; K)$ is unique up to L-isomorphism. Now the following lemma is known (cf. [1; Theorem 1]):

Lemma 5. *Let K be a class of structures for a first order language L. Let E be an L-generative set of nullary operation symbols not belonging to L, and let Ω be a set of atomic sentences of $L(E)$. If there exists a natural limit structure $\mathfrak{L}(E, \Omega; K)$, and it is in K, then there exists a K-free structure $\mathfrak{F}(E, \Omega; K)$, which is L-isomorphic to $\mathfrak{L}(E, \Omega; K)$.*

Using the above lemma, we can obtain the

Lemma 6. *Let Σ be a set of universal Horn sentences of a first order language L. Let E be an L-generative set of nullary operation symbols not belonging to L, and let Ω be a set of atomic sentences of $L(E)$. If there exists a Σ^*-free structure $\mathfrak{F}(E, \Omega; \Sigma^*)$, then there exists a natural limit structure $\mathfrak{L}(E, \Omega; \Sigma^*)$, which is L-isomorphic to $\mathfrak{F}(E, \Omega; \Sigma^*)$.*

Proof. Let (X, Γ) be any member of $M(E, \Omega)$. Then it is easy to see that $\mathfrak{F}(E, \Omega; \Sigma^*)$ is a Σ^*-model of Γ under the interpretations $(e)_{|\mathfrak{F}(E,\Omega;\Sigma^*)|}$, $e \in X$. On the other hand, $\boldsymbol{S}(\Sigma^*) \subseteq \Sigma^*$ and $\boldsymbol{P}(\Sigma^*) \subseteq \Sigma^*$ follow from Lemmas 1 and 2 respectively. Hence by Theorem 2, there exists a Σ^*-free structure $\mathfrak{F}(X, \Gamma; \Sigma^*)$, and hence there exists a direct family naturally defined by $(E, \Omega; \Sigma^*)$. Therefore there exists a natural limit structure $\mathfrak{L}(E, \Omega; \Sigma^*)$, and by Lemma 4, it is in Σ^*. Hence by Lemma 5, $\mathfrak{F}(E, \Omega; \Sigma^*)$ and $\mathfrak{L}(E, \Omega; \Sigma^*)$ are L-isomorphic.

Now we can see the frame of a universal Horn class by the following

Theorem 3. *Let K be a universal Horn class. Then, a structure $\mathfrak{A}$ is in K if and only if $\mathfrak{A}$ is L-isomorphic to a natural limit structure $\mathfrak{L}(E, \Omega; K)$.*

Proof. Assume that $\mathfrak{A}$ is in K. Then by Theorem 1, $\mathfrak{A}$ can be represented as a K-free structure $\mathfrak{F}(E, \Omega; K)$. Hence by Lemma 6, $\mathfrak{A}$ is L-isomorphic to a natural limit structure $\mathfrak{L}(E, \Omega; K)$. Conversely, assume that $\mathfrak{A}$ is L-isomorphic to a natural limit structure $\mathfrak{L}(E, \Omega; K)$. Then it is obvious from Lemma 4 that $\mathfrak{A}$ is in K.

4. Constructions of universal Horn classes

First we shall prove the

Theorem 4. *Let K be a class of structures for a first order language L such that, for any finite L-generative set X of nullary operation symbols not belonging to L and for any finite set Γ of atomic sentences of $L(X)$, if a K-model of Γ exists then a finitely defined K-free structure $\mathfrak{F}(X, \Gamma; K)$ exists. Then the following three properties are satisfied:*

(A) *$\boldsymbol{IL}(K)$ is the least universal Horn class that contains the class K.*

(B) *Let E be a finite L-generative set of nullary operation symbols not belonging to L, and Ω a finite set of atomic sentences of $L(E)$. Then, there exists a finitely defined $\boldsymbol{IL}(K)$-free structure $\mathfrak{F}(E, \Omega; \boldsymbol{IL}(K))$ if and only if there exists a finitely defined K-free structure $\mathfrak{F}(E, \Omega; K)$.*

(C) *Let E and Ω be the same sets as in* (B). *If there exists a finitely defined K-free structure $\mathfrak{F}(E, \Omega; K)$, then $\mathfrak{F}(E, \Omega; K)$ is L-isomorphic to $\mathfrak{F}(E, \Omega; \boldsymbol{IL}(K))$.*

Proof. Let Σ be the set of all universal Horn sentences which are valid in every structure in K. Then it is clear that Σ^* is the least universal Horn class containing the class K. Now let $E=\{e_1, \cdots, e_n\}$ be any finite L-generative set of nullary operation symbols not belonging to L, and let $\Omega=\{\theta_i(e_1, \cdots, e_n) \mid i=1, \cdots, m\}$ be any finite set of atomic sentences of $L(E)$.

First we shall prove the following:

(a) If there exists no finitely defined K-free structure $\mathfrak{F}(E, \Omega; K)$, then there exists no finitely defined Σ^*-free structure $\mathfrak{F}(E, \Omega; \Sigma^*)$.

Since $\mathfrak{F}(E, \Omega; K)$ does not exist, it is obvious by the assumption of this theorem that there exists no K-model of Ω. Hence all $\theta_i(e_1, \cdots, e_n)$ in Ω are not valid in any structure $\mathfrak{A}$ in K under any interpretations $(e_j)_{|\mathfrak{A}|}, j=1, \cdots, n$. That is, the sentence

$$\exists x_1 \cdots \exists x_n(\theta_1(x_1, \cdots, x_n) \wedge \cdots \wedge \theta_m(x_1, \cdots, x_n))$$

is not valid in any structure $\mathfrak{A}$ in K. Hence the universal Horn sentence

$$\forall x_1 \cdots \forall x_n(\neg\theta_1(x_1, \cdots, x_n) \vee \cdots \vee \neg\theta_m(x_1, \cdots, x_n))$$

is valid in every structure in K. Therefore this universal Horn sentence belongs to Σ. Hence there exists no Σ^*-model of Ω, and hence there exists no finitely defined Σ^*-free structure $\mathfrak{F}(E, \Omega; \Sigma^*)$.

Next we shall prove the following:

(b) If there exists a finitely defined K-free structure $\mathfrak{F}(E, \Omega; K)$, then there exists a finitely defined Σ^*-free structure $\mathfrak{F}(E, \Omega; \Sigma^*)$, moreover, in this case, $\mathfrak{F}(E, \Omega; K)$ and $\mathfrak{F}(E, \Omega; \Sigma^*)$ are L-isomorphic.

The K-free structure $\mathfrak{F}(E, \Omega; K)$ is a Σ^*-model of Ω, because $K \subseteq \Sigma^*$. Hence by Lemmas 1 and 2, and Theorem 2, we know that there exists a finitely defined Σ^*-free structure $\mathfrak{F}(E, \Omega; \Sigma^*)$. Next we shall prove that $\mathfrak{F}(E, \Omega; K)$ and $\mathfrak{F}(E, \Omega; \Sigma^*)$ are L-isomorphic. Since $\mathfrak{F}(E, \Omega; K)$ is a Σ^*-model of Ω and is generated by the set $\{(e_j)_{|\mathfrak{F}(E,\Omega;K)|} \mid e_j \in E\}$, there exists an L-homomorphism Φ of $\mathfrak{F}(E, \Omega; \Sigma^*)$ onto $\mathfrak{F}(E, \Omega; K)$ which maps $(e_j)_{|\mathfrak{F}(E,\Omega;\Sigma^*)|}$ to $(e_j)_{|\mathfrak{F}(E,\Omega;K)|}$ for each $e_j \in E$. Now let $\theta(e_1, \cdots, e_n)$ be any atomic sentence which is valid in $\mathfrak{F}(E, \Omega; K)$ under the interpretations $(e_j)_{|\mathfrak{F}(E,\Omega;K)|}$, $e_j \in E$. Then the universal s-Horn sentence

$$\forall x_1 \cdots \forall x_n((\theta_1(x_1, \cdots, x_n) \wedge \cdots \wedge \theta_m(x_1, \cdots, x_n)) \rightarrow \theta(x_1, \cdots, x_n))$$

is valid in every structure in K, because for any K-model $\mathfrak{B}$ of Ω under some interpretations $(e_j)_{|\mathfrak{B}|}$, there exists an L-homomorphism of $\mathfrak{F}(E, \Omega; K)$ into $\mathfrak{B}$ which maps $(e_j)_{|\mathfrak{F}(E,\Omega;K)|}$ to $(e_j)_{|\mathfrak{B}|}$ for each $e_j \in E$. Hence this universal s-Horn sentence belongs to Σ. Therefore $\theta(e_1, \cdots, e_n)$ is valid in $\mathfrak{F}(E, \Omega; \Sigma^*)$ under the interpretations $(e_j)_{|\mathfrak{F}(E,\Omega;\Sigma^*)|}$, $e_j \in E$, because $\mathfrak{F}(E, \Omega; \Sigma^*)$ is a model of Σ and $\theta_1(e_1, \cdots, e_n) \wedge \cdots \wedge \theta_m(e_1, \cdots, e_r)$ is valid in $\mathfrak{F}(E, \Omega; \Sigma^*)$ under the interpretations $(e_j)_{|\mathfrak{F}(E,\Omega;\Sigma^*)|}$, $e_j \in E$. Hence the L-homomorphism Φ is an L-isomorphism.

In order to prove the property (A), it is sufficient to show $\boldsymbol{IL}(K) = \Sigma^*$. By Theorem 3, any structure in the universal Horn class Σ^* is L-isomorphic to a natural limit structure, which is, of course, a direct limit of a direct family of finitely defined Σ^*-free structures. Moreover by (a) and (b), every finitely defined Σ^*-free structure is L-isomorphic to a finitely defined K-free structure. Hence we have $\boldsymbol{IL}(K) \supseteq \Sigma^*$. The converse inclusion $\boldsymbol{IL}(K) \subseteq \Sigma^*$ follows from $K \subseteq \Sigma^*$, by Lemma 4. Hence we have $\boldsymbol{IL}(K) = \Sigma^*$, as desired. Finally, the properties (B) and (C) can be immediately obtained from (a) and (b), because $\boldsymbol{IL}(K) = \Sigma^*$.

Next we shall prove the

Theorem 4^s. *Let K be a class of structures for a first order language L such that, for any finite L-generative set X of nullary operation symbols not belonging to L and for any finite set Γ of atomic sentences of $L(X)$, there exists a finitely defined K-free structure $\mathfrak{F}(X, \Gamma; K)$. Then, $\boldsymbol{IL}(K)$ is the least universal s-Horn class that contains the class K.*

Proof. By the assumption, K contains a one-element structure $\mathfrak{A}$ in which every atomic formula of L is valid. Now, it is easy to see that, if a universal Horn sentence σ is not a universal s-Horn sentence then σ is not valid in $\mathfrak{A}$. Hence the set of all universal Horn sentences which are valid in every structure in K consists of only universal s-Horn sentences. Hence the least universal Horn class containing the class K is the least universal s-Horn class containing the class K. Therefore this theorem follows from Theorem 4.

The following theorem gives the construction of the least universal Horn class containing a given class:

Theorem 5. *Let K be a class of structures for a first order language L. Then, the class $\boldsymbol{ILSP}(K)$ is the least universal Horn class that contains the class K.*

Proof. Let $\bar{K}$ be the least universal Horn class containing the class K. Then by Lemmas 1 and 2, $\bar{K}$ is closed with respect to the operators $\boldsymbol{S}$ and $\boldsymbol{P}$, and by Lemma 4, $\bar{K}$ is closed with respect to the operator $\boldsymbol{L}$. Hence we have that $\boldsymbol{ILSP}(K)\subseteq\bar{K}$. Therefore, in order to prove this theorem, it suffices to show the property:

(♯) $\boldsymbol{ILSP}(K)$ is a universal Horn class containing the class K.

Now it is clear that $\boldsymbol{SI}(C)=\boldsymbol{IS}(C)=\boldsymbol{ISS}(C)$, $\boldsymbol{PI}(C)=\boldsymbol{IP}(C)=\boldsymbol{IPP}(C)$, and $\boldsymbol{PS}(C)\subseteq\boldsymbol{SP}(C)$ for any class C of structures for L. Hence it is easy to see that $\boldsymbol{SISP}(K)\subseteq\boldsymbol{ISP}(K)$ and $\boldsymbol{PISP}(K)\subseteq\boldsymbol{ISP}(K)$. Therefore by Theorem 2, for any L-generative set E of nullary operation symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, if there exists an $\boldsymbol{ISP}(K)$-model of Ω then there exists an $\boldsymbol{ISP}(K)$-free structure $\mathfrak{F}(E, \Omega; \boldsymbol{ISP}(K))$. Hence by Theorem 4, $\boldsymbol{ILISP}(K)$ is a universal Horn class containing the class $\boldsymbol{ISP}(K)$. Therefore we have the property (♯), because $\boldsymbol{ILSP}(K)=\boldsymbol{ILISP}(K)$ and $\boldsymbol{ISP}(K)\supseteq K$.

The following theorem can be obtained in the similar way as in the proof of the above theorem:

Theorem 5^s. *Let K be a class of structures for a first order language L. Then, the class $\boldsymbol{ILSP}^*(K)$ is the least universal s-Horn class that contains the class K.*

5. Characterizations of universal Horn classes

As an immediate consequence of Theorems 5 and 5^s, we can obtain the

Theorem 6. *Let K be an abstract class of structures of the same type. Then,*

(1) *K is a universal Horn class if and only if* $\boldsymbol{S}(K)\subseteq K$, $\boldsymbol{P}(K)\subseteq K$, *and* $\boldsymbol{L}(K)\subseteq K$;

(2) *K is a universal s-Horn class if and only if* $\boldsymbol{S}(K)\subseteq K$, $\boldsymbol{P}^*(K)\subseteq K$, *and* $\boldsymbol{L}(K)\subseteq K$.

We now give another characterization of universal Horn classes, by using natural limit structures.

Theorem 7. *Let K be an abstract class of structures for a first order language L. Then, in order that K is a universal Horn class, it is necessary and sufficient that, for any L-generative set E of nullary operation symbols not belonging to L and for any set* Ω *of atomic sentences of* $L(E)$, *if a K-model of* Γ *exists for each* $(X, \Gamma)\in M(E, \Omega)$ *then a natural limit structure* $\mathfrak{L}(E, \Omega; K)$ *exists in the class K.*

Proof. Let K be a universal Horn class. Then $\boldsymbol{S}(K)\subseteq K$ and $\boldsymbol{P}(K)\subseteq K$ follow from Lemmas 1 and 2 respectively. Now suppose that a K-model of Γ exists for each $(X, \Gamma)\in M(E, \Omega)$. Then by Theorem 2, a K-free structure $\mathfrak{F}(X, \Gamma; K)$ exists for each $(X, \Gamma)\in M(E, \Omega)$. Hence by the definition of a natural limit structure, $\mathfrak{L}(E, \Omega; K)$ exists, and by Lemma 4, $\mathfrak{L}(E, \Omega; K)$ is in K.

Conversely, let us assume that, for any L-generative set E of nullary operation symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, if a K-model of Γ exists for each $(X, \Gamma)\in M(E, \Omega)$ then $\mathfrak{L}(E, \Omega; K)$ exists in the class K. First we shall prove that, for any L-generative set E' of nullary operation symbols not belonging to L and for any set Ω' of atomic sentences of $L(E')$, if a K-model of Ω' exists then a K-free structure $\mathfrak{F}(E', \Omega'; K)$ exists. Now suppose that a K-model of Ω' exists. Then a natural limit structure $\mathfrak{L}(E', \Omega'; K)$ exists in the class K, because a K-model of Ω' is a K-model of Γ' for each $(X', \Gamma')\in M(E', \Omega')$. Therefore by Lemma 5, there exists a K-free structure $\mathfrak{F}(E', \Omega'; K)$, as desired. Hence by Theorem 4, $\boldsymbol{IL}(K)$ is the least universal Horn class that contains the class K, moreover for any finite L-generative set X of nullary operation symbols not belonging to L and for any finite set Γ of atomic sentences of $L(X)$, a finitely defined $\boldsymbol{IL}(K)$-free structure $\mathfrak{F}(X, \Gamma; \boldsymbol{IL}(K))$ exists if and only if a finitely defined K-free structure $\mathfrak{F}(X, \Gamma; K)$ exists, and they are L-isomorphic if both exist. Hence by Theorem 3, any structure in $\boldsymbol{IL}(K)$ is L-isomorphic to some natural limit structure $\mathfrak{L}(E, \Omega; \boldsymbol{IL}(K))$, and hence it is L-isomorphic to $\mathfrak{L}(E, \Omega; K)$. Moreover we have that $\mathfrak{L}(E, \Omega; K)$ is in K, because each $\mathfrak{F}(X, \Gamma; K)$ is a K-model of Γ, where $(X, \Gamma)\in M(E, \Omega)$. Hence any structure in $\boldsymbol{IL}(K)$ is in K, i.e. $\boldsymbol{IL}(K)\subseteq K$, and hence $\boldsymbol{IL}(K)=K$. Therefore K is a universal Horn class.

Theorem 7^s. *Let K be an abstract class of structures for a first order language L. Then, in order that K is a universal s-Horn class, it is necessary and sufficient that for any L-generative set E of nullary operation symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, a natural limit structure $\mathfrak{L}(E, \Omega; K)$ exists in the class K.*

Proof. Let K be a universal s-Horn class. Then K contains a one-element structure $\mathfrak{A}$ in which every atomic formula of L is valid. Now let E be an L-generative set of nullary operation symbols not belonging to L, and Ω a set of atomic sentences of $L(E)$. Then $\mathfrak{A}$ is a K-model of Γ for each $(X, \Gamma)\in M(E, \Omega)$. Hence by Theorem 7, a natural limit structure $\mathfrak{L}(E, \Omega; K)$ exists in the class K.

Conversely, assume that for any L-generative set E of nullary operation symbols not belonging to L and for any set Ω of atomic sentences of $L(E)$, a natural limit structure $\mathfrak{L}(E, \Omega; K)$ exists in the class K. Then by Theorem 7, K is a universal Horn class, and clearly K contains a one-element structure $\mathfrak{A}$ in which every atomic formula of L is valid. Now let Σ be a set of universal Horn sentences which defines the class K. Then every universal Horn sentence in Σ is a universal s-Horn sentence, because it must be valid in the one-element structure $\mathfrak{A}$. Hence K is a universal s-Horn class.

UNIVERSITY OF OSAKA PREFECTURE

References

[1] T. Fujiwara and Y. Nakano: *On the relation between free structures and direct limits*, Math. Japon. **15** (1970), 19–23.
[2] G. Grätzer: Universal Algebra, Van Nostrand, 1968.