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BORDISM AND MAPS OF ODD PRIME PERIOD

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1. Introduction

Let X be a topological space with $A\subset X$ a subspace, and let τ: $(X, A)\rightarrow(X, A)$ be a continuous map of period p, p an odd prime. We define oriented equivariant bordism groups with maps of preiod p which are analogues of the equivariant bordism groups of involutions given by Stong [6]. As a special case we obtain Z_p-bordism groups defined by Conner and Floyd [2].

Our aim is to compute such bordism groups and to catch a clearer view of their structures.

The main results of this paper are as follows.

In §2 we define (free) oriented equivariant bordism groups $\mathcal{O}_*(X, A, \tau)$ $(\Omega_*(X, A, \tau))$ and another bordism group $\mathcal{M}_*(X, A, \tau)$, a generalization of the bordism groups $\mathcal{M}_*=\sum\mathfrak{N}_m(BO(*-m))$ of involutions given by Conner and Floyd in [2,28.1]. And we obtain

Theorem 1. *The sequence*
$\cdots\rightarrow\Omega_n(X, A, \tau)\xrightarrow{i_*}\mathcal{O}_n(X, A, \tau)\xrightarrow{\nu}\mathcal{M}_n(X, A, \tau)\xrightarrow{\partial}\Omega_{n-1}(X, A, \tau)\rightarrow\cdots$ *is exact, where i_* forgets freeness, ν is defined by taking the normal disk bundle of the fixed point sets and ∂ is defined by taking boundary.*

As a special case we obtain an exact sequence

$$0\rightarrow\Omega_*\xrightarrow{i_*}\mathcal{O}_*(Z_p)\xrightarrow{\nu}\mathcal{M}_*(Z_p)\xrightarrow{\partial}\tilde{\Omega}_*(Z_p)\rightarrow 0\,.$$

The Ω-modules $\mathcal{M}_*(Z_p)$ and $\mathcal{O}_*(Z_p)$ may be given ring structure, and in this sequence we see that $\mathcal{J}=\operatorname{im} i_*$ is an ideal of $\mathcal{O}_*(Z_p)$. We then have

Corollary 1.2. *Let $\hat{\mathcal{O}}_*(Z_p)=\mathcal{O}_*(Z_p)/\mathcal{J}$. Then the sequence*

$$0\rightarrow\hat{\mathcal{O}}_*(Z_p)\xrightarrow{\nu}\mathcal{M}_*(Z_p)\xrightarrow{\partial}\tilde{\Omega}_*(Z_p)\rightarrow 0$$

is exact.

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In §3 we define the Smith homomorphism and obtain

Theorem 2. *The sequence*

$$\cdots \to \tilde{\Omega}_n(X \times S^1, \tau \times \rho) \xrightarrow{\pi} \tilde{\Omega}_n(X, \tau) \xrightarrow{\Delta} \Omega_{n-2}(X, \tau) \xrightarrow{\mathcal{P}} \tilde{\Omega}_{n-1}(X \times S^1, \tau \times \rho)$$

$\to \cdots$ *is exact, where* π *is defined by taking projection and* $\mathcal{P}$ *is defined by taking product,* $\rho = exp\ (2\pi i/p)$.

As a special case we have

Theorem 3. *The sequence*

$$0 \to \Omega_{n-1} \xrightarrow{\mathcal{P}'} \Omega_{n-1} \xrightarrow{\pi'} \tilde{\Omega}_n(Z_p) \xrightarrow{\Delta} \tilde{\Omega}_{n-2}(Z_p) \to 0$$

is exact, where $\mathcal{P}'$ *is defined by sending* $[M]$ *into* $p[M]$ *and* π' *is defined by sending* $[N]$ *into* $[N] \cdot [S^1, \rho]$.

This theorem gives immediate corollaries of well-known results discussed in [2].

In §4 we define weakly complex bordism groups and get some analogous results obtained in §3 which we list in Theorem 4.

In §5 we determine the Ω-module structures of $\hat{\mathcal{O}}_*(Z_3)$ and $\mathcal{O}_*(Z_3)$, and obtain

Theorem 5. *As free* Ω*-module,* $\hat{\mathcal{O}}_*(Z_3) \approx \sum_{k \geqslant 1} \Omega \cdot \hat{\beta}_k \oplus \mathcal{O}_*(S^1)$, *where* $\sum_{k \geqslant 1} \Omega \cdot \hat{\beta}_k$ *is a free* Ω*-module generated by* $\hat{\beta}_k = 3\theta_0^k + [M^4]\theta_0^{k-2} + [M^8]\theta_0^{k-4} + \cdots$, *with* M^{4k}, $k = 1, 2, \cdots$, *closed oriented manifolds such that for each* $k \geqslant 1$, $3\alpha_{2k-1} + [M^4]\alpha_{2k-5} + [M^8]\alpha_{2k-9} + \cdots = 0$ *in* $\tilde{\Omega}_*(Z_3)$ *where* $\alpha_{2k-j} = [S^{2k-j}, \rho]$ *with* $\rho = exp\ (2\pi i/3)$, *and with* $\theta_0 = [\xi^2 \to *]$ *the trivial* 2*-plane bundle over a point* $*$; *and* $\mathcal{O}_*(S^1)$ *is the bordism group of semi-free* S^1*-action formed from* $\mathcal{O}_*(Z_3)$ *just by replacing* Z_3*-action by semi-free* S^1*-action in* $\mathcal{O}_*(Z_3)$. *The* Ω*-module structure of* $\mathcal{O}_*(S^1)$ *has been determined by Uchida* [7], *Shimada and the author* [5].

We also obtain

Theorem 6. *As free* Ω*-module,* $\mathcal{O}_*(Z_3) \approx \sum_{k \geqslant 0} \Omega \cdot \mu_k \oplus \mathcal{O}_*(S^1)$, *where* $\mu_0 = [Z_3, \sigma]$, σ *the map of period* 3 *which interchanges elements of* Z_3, *and* μ_k *is taken to be such an element of* $\mathcal{O}_*(Z_3)$ *that* $\nu(\mu_k) = \hat{\beta}_k$ *for each* $k \geqslant 1$.

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2. Bordism groups with maps of odd prime period

In this section we study the oriented equivariant bordism groups with maps of period p, p an odd prime, which are analogues of the equivariant bordism groups of involutions provided by Stong [6], and as a special case we obtain Z_p-bordism groups given by Conner and Floyd in [2].

Let X be a topological space with $A\subset X$ a subspace, and let $\tau\colon (X, A)\to (X, A)$ be a continuous map of period p. A (free) oriented equivariant bordism class of (X, A, τ) is an equivalence class of triples (M^n, T, f) with M^n a compact oriented differentiable manifold with boundary, $T\colon M^n\to M^n$ a (fixed point free) orientation preserving diffeomorphism of period p, and $f\colon (M^n, \partial M^n)\to(X, A)$ a continuous equivariant map sending ∂M^n into A. Two triples (M_0^n, T_0, f_0) and (M_1^n, T_1, f_1) are bordant if there is a 4-tuple (W^{n+1}, V^n, T, f) such that W^{n+1} and V^n are compact oriented differentiable manifolds with boundary; $\partial V^n= \partial M_0^n\cup -\partial M_1^n$ and $\partial W^{n+1}=M_0^n\cup -V^n\cup -M_1^n/\partial M_1^n\cup -\partial M_1^n\equiv\partial V^n$; M_0^n, M_1^n and V^n are regular submanifolds of ∂W whose orientation are induced by that of W; T; $(W, V)\to(W, V)$ is a (fixed point free) orientation preserving diffeormorphism of period p restricting to T_0 on M_0 and T_1 on M_1; and $f\colon (W, V)\to(X, A)$ is a continuous equivariant map restricting to f_0 on M_0 and f_1 on M_1. Denote the equivariant bordism class of (M, T, f) by $[M, T, f]$, and the collection of all such classes by $\mathcal{O}_n(X, A, \tau)$ in which $T\colon M\to M$ is not necessarily free. $\mathcal{O}_n(X, A, \tau)$ is called the group of n-dimensional oriented equivariant bordism classes of (X, A, τ). The group of n-dimensional free oriented equivariant bordism classes of (X, A, τ) is denoted by $\Omega_n(X, A, \tau)$ in which every $T\colon M\to M$ is fixed point free. An abelian group structure is imposed on $\mathcal{O}_n(X, A, \tau)$ $(\Omega_n(X, A, \tau))$ via disjoint union. The weak direct sum $\mathcal{O}_*(X, A, \tau)= \sum\mathcal{O}_n(X, A, \tau)$ is a graded Ω-module. From element $[M^n, T, f]\in\mathcal{O}_n(X, A, \tau)$ (or $\Omega_n(X, A, \tau)$) and a closed manifold $V^m\in\Omega_m$ we give an element $[M^n\times V^m, T\times 1, f\circ\pi_1]\in\mathcal{O}_{n+m}(X, A, \tau)$ (or $\Omega_{n+m}(X, A, \tau)$).

Notice that if X is a point and τ is the identity map, then $\mathcal{O}_*(pt, 1)=\mathcal{O}_*(Z_p)$ and $\Omega_*(pt, 1)=\Omega_*(Z_p)$ where $\mathcal{O}_*(Z_p)$ is the unrestricted Z_p-bordism group in which the action is not necessarily free and $\Omega_*(Z_p)$ is the free (i.e., fixed point free) Z_p-bordism group. We also notice that an action of Z_p is equivalent to a map $T\colon M\to M$ of period p, so (M, Z_p) is replaced by (M, T) to denote a Z_p-manifold in these cases.

Given an equivariant map $\varphi\colon (X, A, \tau)\to(X', A', \tau')$ there is associated a natural homomorphism $\varphi_*\colon \mathcal{O}_n(X, A, \tau)\to\mathcal{O}_n(X', A', \tau')$ given by $\varphi_*[M^n, T, f] =[M^n, T, \varphi f]$. There is also a homomorphism $\partial\colon \mathcal{O}_n(X, A, \tau)\to\mathcal{O}_{n-1}(A, \tau)$ given by $\partial[M^n, T, f]=[\partial M^n, T\,|\,\partial M, f\,|\,\partial M]$. Let $\mathcal{C}$ denote the category of pairs with map of period p, (X, A, τ), and equivariant maps of pairs. We then have

Proposition 1. *On the category $\mathcal{C}$ of pairs with map of period p and* *equivariant maps of pairs the oriented equivariant bordism functor* $\{\mathcal{O}_*(X, A, \tau), \varphi_*, \partial\}$ *satisfies*

(1) *If* φ_0, φ_1 *are equivariantly homotopic maps, then* $\varphi_{0*}=\varphi_{1*}$.

(2) *If* U *is an invariant open set with* $\bar{U}\subset \mathrm{Int}\, A$, A *closed, then the inclusion* i: $(X-U, A-U)\rightarrow(X, A)$ *induces an isomorphism*

$$i_*: \mathcal{O}_n(X-U, A-U) \rightarrow \mathcal{O}_n(X, A).$$

(3) *The sequence*

$\cdots\rightarrow\mathcal{O}_n(A, \tau) \xrightarrow{i_*} \mathcal{O}_n(X, \tau) \xrightarrow{j_*} \mathcal{O}_n(X, A, \tau) \xrightarrow{\partial} \mathcal{O}_{n-1}(A, \tau)\rightarrow\cdots$ *with* (A, ϕ, τ) $\xrightarrow{i} (X, \phi, \tau) \xrightarrow{j} (X, A, \tau)$ *the inclusions, is exact.*

Note. The same is true for the free oriented equivariant bordism functor $\{\Omega_*(X, A, \tau), \varphi_*, \partial\}$. And these oriented equivariant bordisms are equivariant generalized homology theories on the category of pairs with map of odd prime period.

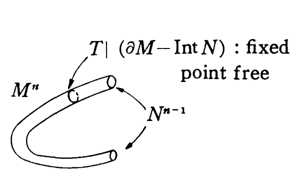
We also have

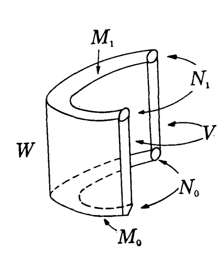
Proposition 2. $\Omega_n(X, A, \tau)\approx\Omega_n(X\times S^\infty/\tau\times\rho, A\times S^\infty/\tau\times\rho)$ *where* $\rho=$ *exp* $(2\pi i/p)$ *is the action on the infinite sphere* $S^\infty\subset C^\infty$.

The proofs of Proposition 1 and Proposition 2 are entirely analogous to those given by Stong in [6], replacing involutions and unorientedness by maps of period p and the requirments of orientability or orientedness, so we omit the proofs here.

We next define an oriented equivariant bordism group $\mathcal{M}_n(X, A, \tau)$ as follows, where X is a topological space with $A\subset X$ a subspace, and τ: $(X, A)\rightarrow(X, A)$ is a map of odd prime period p. An oriented equivariant bordism class of (X, A, τ) is an equivalence class of 4-tuple (M^n, N^{n-1}, T, f) with M^n and N^{n-1} compact oriented differentiable manifolds with boundary, N^{n-1} a regular submanifold of ∂M^n; T; $(M, N)\rightarrow(M, N)$ an orientation preserving diffeomorphism of period p with $T\,|\,(\partial M^n-\mathrm{Int}\, N^{n-1})$ fixed point free; and f: $(M, N)\rightarrow(X, A)$ a continuous equivariant map sending N^{n-1} into A. Two 4-tuples $(M_0^n, N_0^{n-1}, T_0, f_0)$ and $(M_1^n, N_1^{n-1}, T_1, f_1)$ are bordant if there is a 4-tuple (W^{n+1}, V^n, T, f) such that

i) W^{n+1} and V^n are compact oriented differentiable manifolds with boundary;





$M_0^n \cup -V^n \cup -M_1^n$ is contained in ∂W as regular submanifolds whose orientation are induced by that of W; $N_0 \cup -N_1$ is contained in ∂V as regular submanifolds whose orientation are induced by that of V, with $M_0 \cap V = N_0$ and $M_1 \cap V = N_1$;

ii) $T: (W, V) \to (W, V)$ is an orientation preserving diffeomorphism of period p restricting T_0 on M_0 and T_1 on M_1 with $T \mid \partial W - \operatorname{Int}(M_0 \cup V \cup M)$ fixed point free; and

iii) $f: (W, V) \to (X, A)$ is a continuous equivariant map restricting to f_0 on M_0 and f_1 on M_1.

Denote the equivariant bordism class of (M^n, N^{n-1}, T, f) by $[M^n, N^{n-1}, T, f]$, and the collection of all such classes by $\mathcal{M}_n(X, A, \tau)$.

If $A = \phi$, then $N = \phi$ and $[M, N, T, f] = [M, T, f] \in \mathcal{M}_n(X, \tau)$. Therefore (M, T, f) consists of a compact oriented differentiable manifold with boundary, $T: M \to M$ an orientation preserving diffeomorphism of period p with $T \mid \partial M$ fixed point free, and $f: M \to X$ a continuous equivariant map. And so the situation is simpler.

Suppose that $\xi: E \to X$ is an $0(2k)$ bundle with fibre R^{2k} over a connected, locally connected, paracompact base, and that $T: E \to E$ is a map of odd prime period p which carries each fibre orthogonally onto itself leaving only the zero vector fixed. There are then linear subbundles $\xi_j: E_j \to X$ of ξ, $j = 1, 2, \cdots, (p-1)/2$ with $\xi = \xi_1 + \cdots + \xi_{(p-1)/2}$ and there exists a complex linear structure on E_j such that $T(E_j) \subset E_j$ and $T(v) = \rho^j v$ for $v \in E_j$ where $\rho = \exp(2\pi i/p)$, [2, 38.3]. Here the centralizer $C(Z_p) = U(k_1) \times \cdots \times U(k_{(p-1)/2})$ in $0(2k)$ where $k_1 + \cdots + k_{(p-1)/2} = k$ and we may as well suppose that the structural group of ξ is reduced to $U(k_1) \times \cdots \times U(k_{(p-1)/2})$, [2, 38.2]. It follows that if $T: M \to M$ is a differentiable map of odd prime period on an oriented n-manifold, then the structure group of the normal bundle to the fixed point set F can be reduced on each component of F to the unitary group.

For given $[M^n, T, f] \in \mathcal{O}_n(X, A, \tau)$, let F_T be the fixed point set of T, and let F_T^{n-2k} be the union of the $(n-2k)$-dimensional components of F_T which is orientable. And consider the normal disc bundle $\pi: D(\nu_k) \to F_T^{n-2k}$ which is identified with a closed tubular neighborhood of F_T^{n-2k}, and whose orientation is given as follows. We orient F_T^{n-2k} so that the orientation of the fibre followed by that of F_T^{n-2k} yields the orientation of $D(\nu_k)$. Let $k = k_1 + \cdots + k_{(p-1)/2}$ and let $T: D(\nu_k) \to D(\nu_k)$ be the map $T(v) = \rho^j v$ for $v \in E_j$ as in [2, 38.3] which coincides with the restriction of T on the tubular neighborhood of F_T^{n-2k}. We now consider the 4-tuple $(D(\nu_k), D(\nu_k \mid \partial F_T^{n-2k}), T, f)$ where $D(\nu_k \mid \partial F_T^{n-2k}) = D(\nu_k) \cap \partial M$. Obviously, $T \mid \partial D(\nu_k) - \operatorname{Int} D(\nu_k \mid \partial F_T^{n-2k})$ is fixed point free, so $\sum [D(\nu_k), D(\nu_k \mid \partial F_T^{n-2k}), T, f]$ is an element of $\mathcal{M}_n(X, A, \tau)$. We may then define a homomorphism $\nu: \mathcal{O}_n(X, A, \tau) \to \mathcal{M}_n(X, A, \tau)$ by $\nu[M^n, T, f] = \sum [D(\nu_k), D(\nu_k \mid \partial F_T^{n-2k}), T, f]$. Notice that for the absolute case, $\nu: \mathcal{O}_n(X, \tau) \to \mathcal{M}_n(X, \tau)$ is the forgetting homomorphism.

We next let $[M^n, N^{n-1}, T, f]\in\mathcal{M}_n(X, A, \tau)$ and let $\tilde{M}^{n-1}=\partial M^n-\operatorname{Int} N^{n-1}$. The triple $(\tilde{M}^{n-1}, T\,|\,\tilde{M}^{n-1}, f\,|\,\tilde{M}^{n-1})$ is then fixed point free and thus represents an element of $\Omega_{n-1}(X, A, \tau)$. We may then define a homomorphism ∂: $\mathcal{M}_n(X, A, \tau)\to\Omega_{n-1}(X, A, \tau)$ by $\partial\,[M, N, T, f]=[\tilde{M}, T\,|\,\tilde{M}, f\,|\,\tilde{M}]$. Letting i_*: $\Omega_n(X, A, \tau)\to\mathcal{O}_n(X, A, \tau)$ be the homomorphism induced by forgetting the free condition, we thus obtain the following.

Theorem 1. *The sequence*

$$\cdots\to\Omega_n(X, A, \tau)\xrightarrow{i_*}\mathcal{O}_n(X, A, \tau)\xrightarrow{\nu}\mathcal{M}_n(X, A, \tau)\xrightarrow{\partial}\Omega_{n-1}(X, A, \tau)\to\cdots$$ *is exact.*

Proof. It is easy to see that $\partial\nu=0$ and $i_*\,\partial=0$. If $[M^n, T, f]\in\Omega_r(X, A, \tau)$, F_T is empty so $\nu i_*[M, T, f]=0$ in $\mathcal{M}_n(X, A, \tau)$.

im $i_*\supset\ker\nu$. Let $[M^n, T, f]\in\mathcal{O}_n(X, A, \tau)$ with $\sum[D(\nu_k), D(\nu_k\,|\,\partial F_T^{n-2k}), T, f]=0$ in $\mathcal{M}_n(X, A, \tau)$. For simplicity, put $[M', N', T, f]=\sum[D(\nu_k), D(\nu_k\,|\,\partial F_T^{n-2k}), T, f]$. Then there is $(W^{n+1}, V^n, \hat{T}, \hat{f})$ such that $\partial W\supset M'\cup -V$, $\partial V\supset N'$, $M'\cap V=N'$; $\hat{T}$: $(W, V)\to(W, V)$ is an orientation preserving diffeomorphism of period p with $\hat{T}|M'=T$, and with $\hat{T}|\partial W-\operatorname{Int}(M'\cup V)$ fixed point free; and $\hat{f}$: $(W, V)\to(X, A)$ is a continuous equivariant map with $\hat{f}\,|M'=f$. Let U^{n+1} be formed from $M\times I\cup W$ by identifying $M'\times 1$ and M', and let $\tilde{T}$: $U\to U$ be given by $T\times 1\cup\hat{T}$. The continuous equivariant map $\tilde{f}$: $U\to X$ is given by $f\circ\pi_1$ on $M\times I$ and by $\hat{f}$ on W. Next let $B^n=\{(M\times 1)-\operatorname{Int} N'\}\cup\tilde{W}$ by identifying the two copies of N' where $\tilde{W}=\partial W-\operatorname{Int} V$. Let $T'=T\cup\hat{T}$ and let f': $B\to X$ be $f\cup\hat{f}$. Then (B^n, T', f') is fixed point free, so $[B^n, T', f']\in\Omega_n(X, A, \tau)$. But $(U^{n+1}, \partial M\times I\cup V, \tilde{T}, \tilde{f})$ is a bordism of (M, T, f) and (B, T', f'). Hence there is $[B^n, T', f']\in\Omega_n(X, A, \tau)$ such that $i_*[B^n, T', f']=[M^n, T, f]$ in $\mathcal{O}_n(X, A, \tau)$.

im $\nu\supset\ker\partial$. Let $[M^n, N^{n-1}, T, f]\in\mathcal{M}_n(X, A, \tau)$ with $[\tilde{M}^{n-1}, T\,|\,\tilde{M}, f\,|\,\tilde{M}]=0$ in $\Omega_{n-1}(X, A, \tau)$ where $\tilde{M}^{n-1}=\partial M-\operatorname{Int} N$. Then there exists (B^n, C^{n-1}, T', f') such that $\partial C=\partial\tilde{M}$, $\partial B=-\tilde{M}\cup C/\partial\tilde{M}\equiv\partial C$; T': $(B, C)\to(B, C)$ is an orientation preserving, fixed point free, diffeomorphism of period p with $T'\,|\,\tilde{M}=T\,|\,\tilde{M}$; and f': $(B, C)\to(X, A)$ is a continuous equivariant map with $f'\,|\,\tilde{M}=f\,|\,\tilde{M}$. Let $E^n=-M^n\cup B^n$, identifying the two copies of $\tilde{M}$, and let $\tilde{T}$ be given by $T\cup T'$ and $\tilde{f}$ be given by $f\cup f'$ on E. Here notice that $F_{\tilde{T}}=F_T$. We then have $[E^n, \tilde{T}, \tilde{f}]\in\mathcal{O}_n(X, A, \tau)$ such that $\nu[E^n, \tilde{T}, \tilde{f}]=\sum[D(\nu_k), D(\nu_k\,|\,\partial F_T^{n-2k}), T, f]$ which can be shown to be $[M, N, T, f]$ by the following observation. Let $W^{n+1}=(M\times[0, 1/2])\bigcup_k(\cup D(\nu_k)\times[1/2, 1])$ by identifying the two copies of $\bigcup_k D(\nu_k)\times 1/2$. Then $\partial W\supset -M\times 0\bigcup_k(\cup D(\nu_k)\times 1)\cup V$ where $V=N\times[0, 1/2]\cup(\bigcup_k D(\nu_k\,|\,\partial F_T^{n-2k})\times[1/2, 1]$ and $\partial V\supset -N\cup(\bigcup_k D(\nu_k\,|\,\partial F_T^{n-2k})$. Let $\hat{T}$: $(W, V)\to(W, V)$ be given by $T\times 1$ and $\hat{f}$: $(W, V)\to(X, A)$ be given by

$f\circ\pi_1$ on both $M\times[0, 1/2]$ and $(\bigcup_k D(\nu_k)\times[1/2, 1]$. Then $F_{\hat{T}}=F_T\times[0, 1]$ and $\hat{T}|\partial W-\text{Int}\,(M\cup\bigcup_k D(\nu_k)\cup V)$ is fixed point free. Thus (W, V, T, f) is a bordism of $(\bigcup_k D(\nu_k), \bigcup_k D(\nu_k|\partial F_T^{n-2k}), T, f)$ and (M, N, T, f).

$\text{im}\,\partial\supset\ker i_*$. Let $[M^n, T, f]\in\Omega_n(X, A, \tau)$ with $i_*[M^n, T, f]=0$ in $\mathcal{O}_n(X, A, \tau)$. Then there is (W^{n+1}, V^n, T', f') such that $\partial V=\partial M$, $\partial W=-M\cup V/\partial M\equiv\partial V$, $T'|M=T, f'|M=f$. Moreover $T'|\partial W-\text{Int}\,V=T'|M=T$ is fixed point free, so $[W, V, T', f']\in\mathcal{M}_{n+1}(X, A, \tau)$ and $\partial[W, V, T', f']=[M, T, f]$. The theorem thus follows.

Let F_τ be the fixed point set of τ, and let $k=k_1+\cdots+k_{(p-1)/2}$. We then have

Proposition 3. $\mathcal{M}_n(X, A, \tau)\approx\sum\Omega_{n-2k}(F_\tau\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})), (F_\tau\cap A)\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})))$

Proof. Let $[M^n, N^{n-1}, T, f]\in\mathcal{M}_n(X, A, \tau)$ and let F_T^{n-2k} be the union of the $(n-2k)$-dimensional components of F_T. The normal bundle of F_T^{n-2k}, ν_k, is then a $U(k_1)\times\cdots\times U(k_{(p-1)/2})$-bundle classified by a map ν_k: $F_T^{n-2k}\to B(U(k_1)\times\cdots\times U(k_{(p-1)/2}))$ where $k=k_1+\cdots+k_{(p-1)/2}$. For $x\in F_T$, $\tau f(x)=fT(x)=f(x)$, so $f(x)\in F_\tau$, inducing a map $f|F_T$: $(F_T, \partial F_T)\to(F_\tau, F_\tau\cap A)$ where $\partial F_T=F_T\cap\partial M$. We thus have a map $\bigcup_{k\geq 0}(f|F_T^{n-2k}\times\nu_k)$: $\bigcup_k F_T^{n-2k}\to\bigcup_k F_\tau\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2}))$ which defines a homomorphism φ: $\mathcal{M}_n(X, A, \tau)\to\sum\Omega_{n-2k}(F_\tau\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})), (F_\tau\cap A)\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})))$ by $\varphi[M^n, N^{n-1}, T, f]=\sum_{k\geq 0}[F_T^{n-2k}, f|F_T^{n-2k}\times\nu_k]$.

Next, for given $[V^{n-2k}, g_k]\in\Omega_{n-2k}(F_\tau\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})), (F_\tau\cap A)\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})))$, let ξ_k be the complex k vector bundle over V induced by $\pi_2\circ g_k$ from the universal bundle $\gamma_{k_1}\times\cdots\times\gamma_{k_{(p-1)/2}}$ over $B(U(k_1)\times\cdots\times U(k_{(p-1)/2}))$. We then have f_k: $D(\xi_k)\to F_\tau\subset X$ given by $\pi_1\circ g_k\circ\pi$ with π the projection of the disc bundle $D(\xi_k)$ of ξ_k. Since there is the natural action of $U(k_1)\times\cdots\times U(k_{(p-1)/2})$ on the complex linear space $C^k=C^{k_1}\times\cdots\times C^{k_{(p-1)/2}}$ and ξ_k: $E(\xi_k)\to V^{n-2k}$ is the bundle with fibre C^k, with T: $C^k\to C^k$ defined by $T(v)=\rho^j v$ for $v\in C^{k_j}$, then T is in the center of $U(k_1)\times\cdots\times U(k_{(p-1)/2})$. Hence there is induced a T: $E\to E$. E is oriented by the usual way. Then T: $D(\xi_k)\to D(\xi_k)$ is a differentiable map of period p, preserving the orientation, we may thus define a homomorphism ψ: $\sum\Omega_{n-2k}(F_\tau\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})), (F_\tau\cap A)\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})))\to\mathcal{M}_n(X, A, \tau)$ by $\psi(\sum[V^{n-2k}, g_k])=\sum[D(\xi_k)D(\xi_k|\partial V), T, f_k]$. Then it is easy to see that $\varphi\circ\psi=1$, here we use the fact that $F_T=V\subset D(\xi_k)$. Next for any element $[M, N, T, f]\in\mathcal{M}_n(X, A, \tau)$, $\psi\circ\varphi[M, N, T, f]=\psi(\sum[F_T^{n-2k}, f|F_T^{n-2k}\times\nu_k])=\sum[D(\nu_k), D(\nu_k|\partial F_T^{n-2k}), T, f\circ\pi]=\sum[D(\nu_k), D(\nu_k|\partial F_T^{n-2k}), T, f]$. However, we may show that $\sum[D(\nu_k), D(\nu_k|\partial F_T^{n-2k}), T, f]=[M, N, T, f]$ as follows. Form W^{n+1} from $M^n\times[0, 1/2]\cup(\bigcup_k D(\nu_k)\times[1/2, 1])$ by identifying the two copies of $D(\nu_k)\times 1/2$, with $\tilde{T}$: $(W, V)\to(W, V)$ given by

$T\times 1$ and $\tilde{f}$: $(W, V)\to(X, A)$ given by $f\circ\pi_1$ on both of $M^n\times[0, 1/2]$ and $\bigcup_k D(\nu_k)\times[1/2, 1]$, where $V=N\times[0, 1/2]\cup(\bigcup_k D(\nu_k|\partial F_T^{n-2k})\times[1/2, 1])$. Then $(W, V, \tilde{T}, \tilde{f})$ is a bordism of (M, N, T, f) and $(\bigcup_k D(\nu_k), D(\nu_k|\partial F_T^{n-2k}), T, f)$. The assertion follows.

From the previous arguments we see immediately that the exact sequence of Theorem 1 is equivalent to the following exact sequence: $\cdots\to\Omega_n(X, A, \tau)\xrightarrow{i_*}\mathcal{O}_n(X, A, \tau)\xrightarrow{\nu}\sum\Omega_{n-2k}(F_\tau\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})), (F_\tau\cap A)\times B(U(k_1)\times\cdots\times U(k_{(p-1)/2})))\xrightarrow{\partial}\Omega_{n-1}(X, A, \tau)\to\cdots$.

For X a point and τ the identity map, this exact sequence becomes

$$\cdots\to\Omega_n(Z_p)\xrightarrow{i_*}\mathcal{O}_n(Z_p)\xrightarrow{\nu}\mathcal{M}_n(Z_p)\xrightarrow{\partial}\Omega_{n-1}(Z_p)\xrightarrow{i_*}\cdots$$

where $\mathcal{M}_n(Z_p)=\sum\Omega_{n-2k}(B(U(k_1)\times\cdots\times U(k_{(p-1)/2})))$.

Furthermore, we may reduce this exact sequence to a more compact form and obtain a corollary to Theorem 1 as follows.

Corollary 1.1. *The sequence*

$$0\to\Omega_n\xrightarrow{i_*}\mathcal{O}_n(Z_p)\xrightarrow{\nu}\mathcal{M}_n(Z_p)\xrightarrow{\partial}\tilde{\Omega}_{n-1}(Z_p)\to 0$$

is exact. Here i_*: $\Omega_n\to\mathcal{O}_n(Z_p)$ is defined by $i_*[M^n]=[M^n\times Z_p, 1\times\sigma]$ where σ is the map of period p which interchanges elements of Z_p.

Proof. From the exact sequence $\cdots\to\Omega_n(Z_p)\xrightarrow{i_*}\mathcal{O}_n(Z_p)\xrightarrow{\nu}\mathcal{M}_n(Z_p)\xrightarrow{\partial}\Omega_{n-1}(Z_p)\xrightarrow{i_*}\cdots$, it sufficies to show that ∂: $\mathcal{M}_n(Z_p)\to\tilde{\Omega}_{n-1}(Z_p)$ is an epimorphism.

(1) We first show that ∂ is a homomorphism of $\mathcal{M}_n(Z_p)$ into the reduced bordism group $\tilde{\Omega}_{n-1}(Z_p)$. For ∂: $\mathcal{M}_r(Z_p)\to\Omega_{n-1}\oplus\tilde{\Omega}_{n-1}(Z_p)$, we shall prove that for any element $\sum[D(\xi_k), T]\in\mathcal{M}_n(Z_p)$ its image $\partial(\sum[D(\xi_k), T])=\sum[S(\xi_k), T]$ is in $\tilde{\Omega}_{n-1}(Z_p)$. If ε_*: $\Omega_{n-1}(Z_p)\to\Omega_{n-1}$ is the augmentation defined by $\varepsilon_*[M, T]=[M/T]$, $\varepsilon_*(\sum[S(\xi_k), T])=\sum\varepsilon_*[S(\xi_k), T]=\sum[S(\xi_k)/T]$ which clearely vanishes in Ω_{n-1}. For Ω_* has no element of odd order, and $0=[S(\xi_k)]=p[S(\xi_k)/T]$, [2, 19.4], $[S(\xi_k)/T]=0$. Hence ∂ is the homomorphsim $\mathcal{M}_n(Z_p)\to\tilde{\Omega}_{n-1}(Z_p)$.

(2) We go on to show that ∂ is an epimorphism. Since $\{[S^{2i-1}, \rho]\}$ generates the Ω-module $\tilde{\Omega}_*(Z_p)$, [2, 34.3], any element $[M^{n-1}, T]\in\tilde{\Omega}_{n-1}(Z_p)$ can be written in the form $[M^{n-1}, T]=\sum_i[S^{2i-1}, \rho]\cdot[V^{n-2i}]=\sum_i[S^{2i-1}\times V^{n-2i}, \rho\times 1]$ where $V^{n-2i}\in\Omega_{n-2i}$. Consider now the trivial complex i vector bundle ε_i: $C^i\times V^{n-2i}\to V^{n-2i}$ where C^i is the i-dimensional complex vector space which is given the action ρ. Then there is $\sum[D(\varepsilon_i), \rho]\in\mathcal{M}_n(Z_p)$ such that $\partial(\sum[D(\varepsilon_i), \rho])=\sum[S(\varepsilon_i), \rho]=\sum[S^{2i-1}\times V, \rho\times 1]=[M^{n-1}, T]\in\tilde{\Omega}_{n-1}(Z_p)$. Hence ∂ is an

epimorphism. The assertion thus follows.

The Ω-modules $\mathcal{O}_*(Z_p)$ and $\mathcal{M}_*(Z_p)$ are graded ring with multiplication induced by cartesian porduct: $[M_0, T_0]\cdot[M_1, T_1]=[M_0\times M_1, T_0\times T_1]$. And in the exact sequence

$$0\rightarrow\Omega_*\xrightarrow{i_*}\mathcal{O}_*(Z_p)\xrightarrow{\nu}\mathcal{M}_*(Z_p)\xrightarrow{\partial}\tilde{\Omega}_*(Z_p)\rightarrow 0$$

if we let $\mathcal{J}=\operatorname{im} i_*$, $\mathcal{J}$ is an ideal of $\mathcal{O}_*(Z_p)$ since ν is a ring homomorphism. Therefore if we let $\hat{\mathcal{O}}_*(Z_p)=\mathcal{O}_*(Z_p)/\mathcal{J}$, then $\hat{\mathcal{O}}_*(Z_p)$ is also a ring and we obtain the following.

Corollary 1.2. *The sequence*

$$0\rightarrow\hat{\mathcal{O}}_*(Z_p)\xrightarrow{\nu}\mathcal{M}_*(Z_p)\xrightarrow{\partial}\tilde{\Omega}_*(Z_p)\rightarrow 0$$

is exact.

This short exact sequence is an analogue of the exact sequence

$$0\rightarrow I_*(Z_2)\xrightarrow{\nu}\mathcal{M}_*\xrightarrow{\partial}\mathfrak{N}_*(Z_2)\rightarrow 0$$

where $\mathcal{M}_*=\sum\mathfrak{N}_m(BO(*-m))$ and $I_*(Z_2)$ is the unrestricted Z_2-bordism group which was provided by Conner and Floyd in [2, 28.1].

3. The Smith homomorphism

Let τ: $X\rightarrow X$ be an action of Z_p, p an odd prime, on a space X, and let T: $M\rightarrow M$ be a free action of Z_p on a closed oriented manifold M. Given $[M^n, T, f]\in\Omega_n(X, \tau)$ and $2m+1>n$, there exists an equivariant differentiable map φ: $(M^n, T)\rightarrow(S^{2m+1}, \rho)$ which is transverse regular on $S^{2m-1}\subset S^{2m+1}$ where $\rho=\exp(2\pi i/p)$. Let $N^{n-2}=\varphi^{-1}(S^{2m-1})$. Then N is a closed oriented submanifold of M. The Smith homomorphism Δ: $\Omega_n(X, \tau)\rightarrow\Omega_{n-2}(X, \tau)$ is defined by $\Delta[M^n, T, f]=[N^{n-2}, T\,|\,N, f\,|\,N]$.

Letting ι_*: $\mathcal{M}_n(X, \tau)\rightarrow\mathcal{M}_{n+2}(X, \tau)$ be defined by sending $[M^n, T, f]$ into $[M^n\times D^2, T\times\rho, f\circ\pi_1]$ where D^2 is a disk whose boundary is a unit sphere S^1, we then have

Proposition 4. *The diagram*

$$\begin{array}{ccc}\mathcal{M}_n(X,\tau) & \xrightarrow{\partial} & \Omega_{n-1}(X,\tau)\\ \downarrow{\scriptstyle\iota_*} & & \uparrow{\scriptstyle\Delta}\\ \mathcal{M}_{n+2}(X,\tau) & \xrightarrow{\partial} & \Omega_{n+1}(X,\tau)\end{array}$$

commutes.

Proof. If $[M^n, T, f] \in \mathcal{M}_n(X, \tau)$, then $\partial[M, T, f] = [\partial M, T \mid \partial M, f \mid \partial M]$ and $\partial\iota_*[M, T, f] = \partial[M \times D^2, T \times \rho, f \circ \pi_1] = [\partial(M \times D^2), T \times \rho \mid \partial(M \times D^2), f \circ \pi_1 \mid \partial(M \times D^2)]$. We shall show that $\Delta[\partial(M \times D^2), T \times \rho \mid \partial(M \times D^2), f \circ \pi_1 \mid \partial(M \times D^2)] = [\partial M, T \mid \partial M, f \mid \partial M]$. To do this, we use the equivariant differentiable map φ: $(\partial M; T \mid \partial M) \to (S^{2m+1}, \rho)$ to obtain an equivariant differentiable map $\tilde{\varphi}$: $(\partial(M \times D^2), T \times \rho \mid \partial(M \times D^2)) \to (S^{2m+3}, \rho) = (S^{2m+1}, \rho) * (S^1, \rho)$ defined by

$$\tilde{\varphi}(x, tz) = \begin{cases} (1-t)\varphi(x) + tz \text{ if } x \in \partial M, \ |z| = 1, \quad 0 \leqslant t \leqslant 1, \\ 0 + z \text{ if } x \in M, \ |z| = 1, \quad t = 1, \end{cases}$$

where $(S^{2m+1}, \rho) * (S^1, \rho)$ denotes the join of (S^{2m+1}, ρ) and (S^1, ρ). We thus have $\tilde{\varphi}^{-1}(S^{2m+1}) = \partial M \times 0$, so $\Delta[\partial(M \times D^2), T \times \rho \mid \partial(M \times D^2), f \circ \pi_1 \mid \partial(M \times D^2)] = [\partial M, T \mid \partial M, f \mid \partial M]$.

If X is a point and τ is the identity map, we obtain the following

Corollary. *The diagram*

$$\begin{array}{ccc} \mathcal{M}_n(Z_p) & \xrightarrow{\partial} & \Omega_{n-1}(Z_p) \\ \downarrow{\scriptstyle \iota_*} & & \uparrow{\scriptstyle \Delta} \\ \mathcal{M}_{n+2}(Z_p) & \xrightarrow{\partial} & \Omega_{n+1}(Z_p) \end{array}$$

commutes.

We also have

Proposition 5. *For any element $[M^n, T, f]$ in $\mathcal{O}_n(X, \tau)$ we have*

$$\partial\iota_*\nu[M^n, T, f] = [M^n \times S^1, T \times \rho, f \circ \pi_1]$$

in $\Omega_{n+1}(X, \tau)$ where ν: $\mathcal{O}_n(X, \tau) \to \mathcal{M}_n(X, \tau)$ is the homomorphism defined in the preceeding section.

This is a generalization of the case, $X = pt.$, $\tau = 1$, given in [2, 38.6].

Proof. Considering the diagram

$$\begin{array}{ccccc} & & \mathcal{M}_{n+2}(X, \tau) & \xrightarrow{\partial} & \Omega_{n+1}(X, \tau) \\ & & {\scriptstyle \iota_*} \nearrow & & \\ \mathcal{O}_n(X, \tau) & \xrightarrow{\nu} & \mathcal{M}_n(X, \tau) & & \end{array}$$

and letting $[M^n, T, f] \in \mathcal{O}_n(X, \tau)$, we have $\nu[M^n, T, f] = [M^n, T, f]$ since ν is the forgetting homomorphism. We then see that $\partial\iota_*\nu[M, T, f] = \partial\nu_*[M, T, f] = \partial[M \times D^2, T \times \rho, f \circ \pi_1] = [M \times S^1, T \times \rho, f \circ \pi_1]$, this completes the proof.

We next define a homomorphism $\mathcal{P}$: $\Omega_n(X, \tau) \to \Omega_{n+1}(X \times S^1, \tau \times \rho)$ by $\mathcal{P}[M^n, T, f] = [M \times S^1, T \times \rho, f \times 1]$.

We also consider a homomorphism π: $\Omega_n(X\times S^1, \tau\times\rho)\to\Omega_n(X, \tau)$ defined by $\pi[M, T, f]=[M, T, \pi_1\circ f]$.

Some of our main results are then obtained in the following.

Theorem 2. *The sequence*

$$\cdots\to\Omega_n(X\times S^1, \tau\times\rho)\xrightarrow{\pi}\Omega_n(X, \tau)\xrightarrow{\Delta}\Omega_{n-2}(X, \tau)\xrightarrow{\mathcal{P}}\Omega_{n-1}(X\times S^1, \tau\times\rho)\to\cdots$$

is exact.

Proof. (1) $\Delta\pi=0$ and $\mathcal{P}\Delta=0$. Consider the diagram

$$\begin{array}{ccccc}
\Omega_n(X, \tau) & \xrightarrow{\mathcal{P}} & \Omega_{n+1}(X\times S^1, \tau\times\rho) & \xrightarrow{\pi} & \Omega_{n+1}(X, \tau) \\
\downarrow\Delta & & \downarrow\Delta & & \downarrow\Delta \\
\Omega_{n-2}(X, \tau) & \xrightarrow{\mathcal{P}} & \Omega_{n-1}(X\times S^1, \tau\times\rho) & \xrightarrow{\pi} & \Omega_{n-1}(X, \tau)
\end{array}$$

which commutes by definition. Here

$$\Delta\colon\ \Omega_{n-2}(X\times S^1, \tau\times\rho)\to\Omega_{n-1}(X\times S^1, \tau\times\rho)$$

is a zero map since for any element $[M, T, f]\in\Omega_{n+1}(X\times S^1, \tau\times\rho)$, the map $\varphi=i\circ\pi_2\circ f$: $M\to S^1\to S^{2m+1}$ and $S^1\sim(S^{2m+1}-S^{2m-1})$, so $N=\varphi^{-1}(S^{2m-1})=\phi$, and $\Delta[M, T, f]=0$. Hence $\Delta\pi=\pi\Delta=0$ and $\mathcal{P}\Delta=\Delta\mathcal{P}=0$.

(2) $\pi\mathcal{P}=0$. Let $[M^{n-2}, T, f]\in\Omega_{n-2}(X, \tau)$, then $\pi\mathcal{P}[M, T, f]=[M\times S^1, T\times\rho, \pi_X\circ(f\times 1)]$. Notice that $\pi_X\circ(f\times 1)=f\circ\pi_M$, and evidently $(M\times D^2, T\times\rho, f\circ\pi_M)$ has boundary $(M\times S^1, T\times\rho, f\circ\pi_M)$ so $\pi\mathcal{P}=0$.

(3) $\operatorname{im}\pi\supset\ker\Delta$. If $[M^n, T, f]\in\Omega_n(X, \tau)$ with $\Delta[M^n, T, f]=[N^{n-2}, T|N, f|N]=0$, there is (V^{n-1}, T', f') such that $\partial V=N$, $T'|N=T|N$ and $f'|N=f|N$. Let $\tilde{M}^n=(M^n-N\times\operatorname{Int}D^2)\cup V\times S^1$, identifying the two copies of $N\times S^1$, with $\tilde{T}$: $\tilde{M}^n\to\tilde{M}^n$ given by $T\cup T'\times\rho$, and with $\tilde{f}$: $\tilde{M}\to X$ given by $\tilde{f}=f\cup g$ where g is defined as follows. Let r_t: $N\times S^1\to N\times D^2$ be defined by $r_t(x, s)=(x, (1-t)s)$, then $f\circ r_0=f\circ 1=f|N\times S^1$: $N\times S^1\to X$ is equivariantly homotopic to $f\circ r_1=(f|N)\circ\pi_N=f'\circ\pi_N|N\times S^1$: $N\times S^1\xrightarrow{\pi_N}N\xrightarrow{f'}X$. The map g is given by $g_1=f'\circ\pi_V$ on $V\times S^1-N\times S^1\times[0,1)$ and g_2 on $N\times S^1\times[0,1]$ defined by $g_2(x, s, t)=f\circ r_t(x, s)$. Then $(\tilde{M}, \tilde{T}, \tilde{f})$ and (M, T, f) are bordant. For if we form $M\times I\cup V\times D^2$ by identifying $N\times D^2\times 1$ with $N\times D^2$, with action given by $T\times 1\cup T'\times\rho$, and with map $M\times I\cup V\times D^2\to X$ given by $f\circ\pi_M\cup f'\circ\pi_V$, then $\partial(M\times I\cup V\times D^2)=-\tilde{M}\cup M$, $(T\times 1\cup T'\times\rho)|\tilde{M}\cup M=\tilde{T}\cup T$, and $(f\circ\pi_M\cup f'\circ\pi_V)|\tilde{M}\cup M=\tilde{f}\cup f$. Now let $\tilde{\varphi}=\psi\cup\pi_{S'}$: $(M-N\times\operatorname{Int}D^2)\cup(V\times S^1)\to S^1$ where ψ is the map defined as follows. Let $B=M-N\times\operatorname{Int}D^2$. Then since φ: $M^n\to S^{2m+1}$ is equivariant and transverse regular on S^{2m-1} and $N=\varphi^{-1}(S^{2m-1})$, $\varphi|B$: $B\to(S^{2m+1}-S^{2m-1}\times\operatorname{Int}D^2)\approx D^{2m}\times S^1$ is equivariant. There is also an equivariant homotopy $h\times 1$: $D^{2m}\times S^1\to 0\times S^1\approx S^1$. We then define ψ: $B\to S^1$ by $(h\times 1)\circ(\varphi|B)$ which is equivariantly homotopic to $\varphi|B$.

We thus have $[\tilde{M}, \tilde{T}, \tilde{f}\times\tilde{\varphi}]\in\Omega_n(X\times S^1, \tau\times\rho)$ such that $\pi[\tilde{M}, \tilde{T}, \tilde{f}\times\tilde{\varphi}]=[\tilde{M}, \tilde{T}, \pi_X\circ(\tilde{f}\times\tilde{\varphi})]=[\tilde{M}, \tilde{T}, \tilde{f}]=[M, T, f]$.

(4) im $\Delta\supset$ ker $\mathscr{P}$, Let $[M^{n-2}, T, f]\in\Omega_{n-2}(X, \tau)$ such that $\mathscr{P}[M, T, f]=[M\times S^1, T\times\rho, f\times 1]=0$ in $\Omega_{n-1}(X\times S^1, \tau\times\rho)$. Then there is (W^n, T', f') with $\partial W=M\times S^1$, with T': $W\to W$ an extension of $T\times\rho$, and with an equivariant map f': $W\to X\times S^1$ extending f. Form $(W\cup M\times D^2, T'\cup T\times\rho, \pi_X\circ f'\cup f\circ\pi_M)$ by identifying the two copies of $M\times S^1$. Then $T'\cup T\times\rho|M=T$ and $\pi_X\circ f'\cup f\circ\pi_M|M=f$. We shall show that there is an equivariant differentiable map φ: $(W\cup M\times D^2, T'\cup T\times\rho)\to(S^{2m+1}, \rho)$ which is transverse regular on S^{2m-1} and $\varphi^{-1}(S^{2m-1})=M$. If so, we then have $[W\cup M\times D^2, T'\cup T\times\rho, \pi_X\circ f'\cup f\circ\pi_M]\in\Omega_n(X, \tau)$ with $\Delta[W\cup M\times D^2, T'\cup T\times\rho, \pi_X\circ f\cup f\circ\pi_M]=[M, T, f]$. In fact, the map φ can be obtained in the following way. Since $W\xrightarrow{f'}X\times S^1\xrightarrow{\pi_2}S^1\xrightarrow{i}S^{2m+1}$ and $S^{2m+1}-0\times S^1\approx S^{2m-1}\times\text{Int}\,D^2$, so if we consider $(W-\partial W\times I)\xrightarrow{\pi_2\circ f'}0\times S^1$ and equivariant maps μ_t: $M\times S^1_t\to S^{2m-1}_{1-t}\times S^1$, $0\leqslant t\leqslant 1$, then the maps μ_t define an equivariant map μ: $M\times D^2\to S^{2m-1}\times D^2$. This implies $M\times 0\to S^{2m-1}\times 0$. We thus define φ by $i\circ\pi_2\circ f'$ on W and μ on $M\times D^2$. Then φ: $(W\cup M\times D^2, T'\cup T\times\rho)\to(S^{2m+1}, \rho)$ is an equivariant differentiable map which is transverse regular on S^{2m-1} and $\varphi^{-1}(S^{2m-1})=M$.

(5) im $\mathscr{P}\supset$ker π. Let $[M^{n-1}, T, f]\in\Omega_{n-1}(X\times S^1, \tau\times\rho)$ such that $\pi[M^{n-1}, T, f]=[M, T, \pi_1\circ f]=0$ in $\Omega_{n-1}(X, \tau)$. There is then (W^n, T', f') with $\partial W=M$, $T'|M=T$, and $f'|M=\pi_1\circ f$. Extending $M\xrightarrow{\pi_2\circ f}S^1$, we have an equivariant differentiable map φ: $(W, T')\to(D^2, \rho)$ and a commutaive diagram

$$\begin{array}{ccc} W & \xrightarrow{\varphi} & D \\ \downarrow & & \downarrow\eta \\ W/T' & \xrightarrow{\bar{\varphi}} & D^2/\rho\,. \end{array}$$

For the regular value $\bar{y}\in D^2/\rho-(S^1/\rho\cup\{0\})$ of $\bar{\varphi}$, there are p regular values, y, $\rho(y)$, $\cdots$, $\rho^{p-1}(y)$, of φ corresponding to $\bar{y}$. Let $N^{n-2}=\varphi^{-1}\{y, \rho(y), \cdots, \rho^{p-1}(y)\}$. We then have $[N, T'|N, f'|N]\in\Omega_{n-2}(X, \tau)$ with $\mathscr{P}[N, T'|N, f'|N]=[N\times S^1, (T'|N)\times\rho, (f'|N)\times 1]=[M, T, f]$. The last equality follows by the fact that $(W-N\times\text{Int}\,D^2, T', f')$ has boundary the disjoint union of $(N\times S^1, (T'|N)\times\rho, (f'|N)\times 1)$ and (M, T, f). The theorem follows.

Notes. (1) The same arguments may be applied to the relative case and obtain the exact sequence:

$$\cdots\to\Omega_n(X\times S^1, A\times S^1, \tau\times\rho)\xrightarrow{\pi}\Omega_n(X, A, \tau)\xrightarrow{\Delta}\Omega_{n-2}(X, A, \tau)\xrightarrow{\mathscr{P}}$$
$$\Omega_{n-1}(X\times S^1, A\times S^1, \tau\times\rho)\to\cdots$$

(2) Let $\tilde{\Omega}_n(X, \tau)=\ker(\Omega_n(X, \tau)\to\Omega_n)$ where the augmentation $\varepsilon_*: \Omega_n(X, \tau)\to\Omega_n$ is defined by $\varepsilon_*[M^n, T, f]=[M^n/T]\in\Omega_n$. We then have a split exact sequence

$$0\to\tilde{\Omega}_n(X, \tau)\to\Omega_n(X, \tau)\to\Omega_n\to 0\,.$$

To see this, taking any element $[M^n]\in\Omega_n$ we form $(M\times Z_p, 1\times\sigma, f)$ where σ is the action of preiod p which interchanges elements of $Z_p=\{0, 1, \cdots, p-1\}$, and f is given by $f(M\times 0)=x\in X$ and $f(M\times k)=\tau^k(x)$ for $k\geqslant 1$. We then have an element $[M^n\times Z_p, 1\times\sigma, f]$ of $\Omega_n(X, \tau)$. The assignment $[M^n]\to[M^n\times Z_p, 1\times\sigma, f]$ induces a homomorphism $i_*: \Omega_n\to\Omega_n(X, \tau)$ such that $\varepsilon_*\circ i_*=1$.

We may reduce the preceding theorem in the following form.

Corollary 2.1. *The sequence*

$$\cdots\to\tilde{\Omega}_n(X\times S^1, \tau\times\rho)\xrightarrow{\pi}\tilde{\Omega}_n(X, \tau)\xrightarrow{\Delta}\Omega_{n-2}(X, \tau)\xrightarrow{\mathcal{P}}\tilde{\Omega}_{n-1}(X\times S^1, \tau\times\rho)\to\cdots$$

is exact.

Proof. (1) Let $[M^n, T, f]\in\tilde{\Omega}_n(X\times S^1, \tau\times\rho)$. Then $[M/T]=0$, and $\pi[M, T, f]=[M, T, \pi_1\circ f]\in\Omega_n(X, \tau)$ also satisfies $[M/T]=0$ in Ω_n.

(2) Next if $[M^{n-2}, T, f]\in\Omega_{n-2}(X, \tau)$, $\mathcal{P}[M, T, f]=[M\times S^1, T\times\rho, f\times 1]$ in $\Omega_{n-1}(X\times S^1, \tau\times\rho)$. But since $M\times S^1=\partial(M\times D^2)$ and Ω_* has no odd torsion, $0=[M\times S^1]=p[M\times S^1/T\times\rho]$ implies $[M\times S^1/T\times\rho]=0$ in Ω_{n-1}.

If X is a point and $\tau=1$, we have the following corollary.

Corollary 2.2. *The sequence*

$$\cdots\to\tilde{\Omega}_n(S^1, \rho)\xrightarrow{\pi}\tilde{\Omega}_n(Z_p)\xrightarrow{\Delta}\Omega_{n-2}(Z_p)\xrightarrow{\mathcal{P}}\tilde{\Omega}_{n-1}(S^1, \rho)\to\cdots$$ *is exact.*

We can now reduce the Corollary 2.2. to an exact sequence in which only free Z_p-bordism groups and the Thom groups are concerned, and from which some well-known properties of $\tilde{\Omega}_n(Z_p)$ are derivable.

Theorem 3. *The sequence*

$$0\to\Omega_{n-1}\xrightarrow{\mathcal{P}'}\Omega_{n-1}\xrightarrow{\pi'}\tilde{\Omega}_n(Z_p)\xrightarrow{\Delta}\tilde{\Omega}_{n-2}(Z_p)\to 0$$ *is exact, where $\mathcal{P}': \Omega_{n-1}\to\Omega_{n-1}$ is defined by $\mathcal{P}'[M^{n-1}]=p[M^{n-1}]$ and $\pi': \Omega_{n-1}\to\tilde{\Omega}_n(Z_p)$ is defined by $\pi'[N^{n-1}]=[N]\cdot[S^1, \rho]$.*

Proof. We first recall that $\tilde{\Omega}_n(S^1, \rho)\approx\tilde{\Omega}_n(S^1/\rho)$. This is induced by $\theta[M^n, T, f]=[M^n/T, \bar{f}]$ where $\bar{f}: M/T\to S^1/\rho$ is induced from f in the following diagram:

$$\begin{array}{ccc} M & \xrightarrow{f} & S^1 \\ \downarrow & & \downarrow \\ M/T & \xrightarrow{\bar{f}} & S^1/\rho . \end{array}$$

Also, $\tilde{\Omega}_n(S^1/\rho)\approx\tilde{\Omega}_n(S^1)\approx\Omega_{n-1}$. We now have $\tilde{\Omega}_n(S^1, \rho)\approx\Omega_{n-1}$, so consider next the following diagram

$$\begin{array}{ccccccccccc} \to \tilde{\Omega}_{n+1}(Z_p) & \xrightarrow{\Delta} & \Omega_{n-1}(Z_p) & \xrightarrow{\mathcal{P}} & \tilde{\Omega}_n(S^1, \rho) & \xrightarrow{\pi} & \tilde{\Omega}_n(Z_p) & \xrightarrow{\Delta} & \Omega_{n-2}(Z_p) \to \\ & \searrow^{\Delta} & \wr & & \wr & \nearrow^{\pi'} & & \searrow^{\Delta} & \\ 0 & & \tilde{\Omega}_{n-1}(Z_p) & & \Omega_{n-1} & & & & \tilde{\Omega}_{n-2}(Z_p) \\ & \searrow & + & \nearrow^{\mathcal{P}'} & & & & & \\ & & \Omega_{n-1} & & & & & & \end{array}$$

The theorem then follows by showing that the homomorphism Δ: $\tilde{\Omega}_*(Z_p)\to\tilde{\Omega}_*(Z_p)$ is an epimorphism, and the homomorphisms $\mathcal{P}'$ and π' are compatible with $\mathcal{P}$ and π respectively. The homomorphism Δ is surely an epimorphism [2, 34.9]. Consider next the diagram

$$\begin{array}{ccc} \Omega_{n-1}(Z_p) & \xrightarrow{\mathcal{P}} & \tilde{\Omega}_n(S^1, \rho) \\ i_*\downarrow & & \approx\downarrow\mu \\ \Omega_{n-1} & \xrightarrow{\mathcal{P}'} & \Omega_{n-1} \end{array}$$

where i_*: $\Omega_{n-1}\to\Omega_{n-1}(Z_p)$ is defined by $i_*[M]=[M\times Z_p, 1\times\sigma]$ and μ: $\tilde{\Omega}_n(S^1, \rho)\to\Omega_{n-1}$ is defined by $\lambda\circ\theta$: $\tilde{\Omega}_n(S^1, \rho)\xrightarrow{\theta}\tilde{\Omega}_n(S^1/\rho)\xrightarrow{\lambda}\Omega_{n-1}$ with λ the map defined by sending $[M^n, h]\in\tilde{\Omega}_n(S^1/\rho)$ to $[N^{n-1}=h^{-1}(*)]$, $*\in S^1/\rho$ being a regular value of h. Taking any element $[M]\in\Omega_{n-1}$, we have $\mathcal{P}\circ i_*[M]=[M\times Z_p\times S^1, 1\times\sigma\times\rho, \pi_{S^1}]$ which is equivariantly diffeomorphic to $[M\times Z_p\times S^1, 1\times\sigma\times 1, g]$ by an equivariant diffeomorphism φ defined by $\varphi(x, k, t)=(x, k, \rho^{-k}(t))$. The map g: $M\times Z_p\times S^1\to S^1$ is defined by $g(x, k, t)=\rho^k(t)$. We then have a commutative diagram

$$\begin{array}{ccc} M\times Z_p\times S^1 & \xrightarrow{g} & S^1 \\ \downarrow & & \downarrow\eta \\ M\times S^1 & \xrightarrow{\bar{g}} & S^1/\rho \end{array}$$

where $\bar{g}=\eta\circ\pi_2$. And so $\theta[M\times Z_p\times S^1, 1\times\sigma\times 1, g]=[M\times S^1, \bar{g}]\in\tilde{\Omega}_n(S^1/\rho)$. Moreover $\lambda[M\times S^1, \bar{g}]=[\bar{g}^{-1}(*)=M\times Z_p]=p[M]$. Hence $\mu\mathcal{P}i_*[M]=p[M]=\mathcal{P}'[M]$.

Finally, in the diagram

$$\begin{array}{ccc} \tilde{\Omega}_n(S^1, \rho) & \xrightarrow{\pi} & \tilde{\Omega}_n(Z_p) \\ \kappa \uparrow \approx & & = \downarrow \\ \Omega_{n-1} & \xrightarrow{\pi'} & \tilde{\Omega}_n(Z_p) \end{array}$$

where $\kappa\colon \Omega_{n-1} \to \tilde{\Omega}_n(S^1, \rho)$ is an isomorphism defined by $\kappa[N]=[N\times S^1, 1\times\rho, \pi_2]$, we have $\pi\kappa[N]=\pi[N\times S^1, 1\times\rho, \pi_2]=[M\times S^1, 1\times\rho]=[N]\cdot[S^1, \rho]=\pi'[N]$. Since $\kappa=\mu^{-1}$ the assertion follows.

This theorem yields immediate corollaries, which are well-known results shown by Conner and Floyd in [2] in different way.

Corollary 3.1. *For* $k\geqslant 0$, $\tilde{\Omega}_{2k}(Z_p)=0$.

Proof. Since Ω_{2k-1} consists of 2-torsion, it is seen that $\mathcal{P}'$ is an epimorphism and $\pi'=0$. We thus get $\tilde{\Omega}_{2k}(Z_p)\approx\tilde{\Omega}_{2k-2}(Z_p)\approx\cdots\approx\tilde{\Omega}_0(Z_p)$. But since $\Omega_0(Z_p)\approx\Omega_0(BZ_p)\approx\Omega_0(pt.)=\Omega_0$, we have $\tilde{\Omega}_0(Z_p)=0$. Hecne $\tilde{\Omega}_{2k}(Z_p)=0$ for all $k\geqslant 0$.

Corollary 3.2. *For* $k\geqslant 0$, $\tilde{\Omega}_{4k+3}(Z_p)\approx\tilde{\Omega}_{4k+1}(Z_p)$.

Proof. Since Ω_{4k+2} consists of 2-torsion, the result follows immediately.

Corollary 3.3. *The sequence*

$0\to\Omega_{4k}/p\Omega_{4k}\xrightarrow{\pi'}\tilde{\Omega}_{4k+1}(Z_p)\xrightarrow{\Delta}\tilde{\Omega}_{4k-1}(Z_p)\to 0$ *is exact.*

4. Weakly complex bordism groups

Being given a $2k$-plane bundle ξ over a space X, a complex structure for ξ is a homotopy class of maps J mapping each fiber of ξ linearly into itself and having $J^2=-1$. If X is a finite dimensional CW complex and if ξ is a real n-plane bundle over X, a weakly complex structure for ξ is complex structure for the Whitney sum, $\xi+\varepsilon^{2k-n}$ of ξ and the trivial $(2k-n)$-plane bundle, $2k-2\geqslant$ dim X; this is independent of k. A weakly complex oriented manifold is a pair consisting of a differentiable oriented manifold M and a weakly complex structure on the tangent bundle of M. Let G be a compact Lie group acting differentiably on M. If $\zeta\colon E\to M$ is the tangent bundle to M, then G acts on the Whitney sum $\zeta+\varepsilon^{2k-n}$ as a group of bundle maps, acting trivially on the trivial bundle. An invariant complex structure is a complex structure which commutes with the action of G. A wealky complex action of the compact Lie group G on the differentiable manifold M is a pair consisting of a differentiable action of G on M and an invariant weakly complex structure for the action. Consider a free weakly complex action of Z_p on a closed manifold M; denote the pair by (M, T) where $T\colon M\to M$ is a map of odd prime period. There is a natural equivariant Z_p-bordism group of such pairs, denoted by $\Omega^U_*(Z_p)$. The weakly complex bordism groups of the form $\Omega^U_*(X)$ and $\Omega^U_*(X, \tau)$ are also constructed in the same way. As in the case of Z_p-bordism groups, we have $\Omega_*^U(Z_p)\approx\Omega_*^U(BZ_p)$, and as an Ω^U-module, a generating set of $\Omega_*^U(Z_p)$ is given by $\{[S^{2k-1}, \rho]\}$, $\rho=\exp(2\pi i/p)$, [1, p. 63].

We can also introduce the Smith homomorphism $\Delta\colon \Omega_n^U(Z_p)\to\Omega_{n-2}^U(Z_p)$ as follows. Given $[M^n, T]\in\Omega_n^U(Z_p)$ and $2m+1>n$, there is a unique equivariant homotopy class of equivariant maps $\varphi\colon (M^n, T)\to(S^{2m+1}, \rho)$ which is transverse regular on the invariant $S^{2m-1}\subset S^{2m+1}$. Let $\varphi^{-1}(S^{2m-1})=N^{n-2}$. The closed invariant submanifold $N\subset M$ has a trivial complex normal bundle. An invariant weakly complex structure on N is uniquely determined by this normal bundle together with the weakly complex structure on M. The Smith homomorphism $\Delta\colon \Omega_n^U(X, \tau)\to\Omega_{n-2}^U(X, \tau)$ is defined by $\Delta[M, T, f]=[N, T|N, f|N]$.

We then obtain some results analogous to those of the preceding section, we now collect them in a theorem.

Theorem 4. *The following sequences* (1)–(5) *are exact.*

(1) $\cdots\to\Omega_n^U(X\times S^1, \tau\times\rho)\xrightarrow{\pi}\Omega_n^U(X, \tau)\xrightarrow{\Delta}\Omega_{n-2}^U(X, \tau)$
$\xrightarrow{\mathcal{P}}\Omega_{n-1}^U(X\times S^1, \tau\times\rho)\to\cdots$

(2) $\cdots\to\tilde{\Omega}_n^U(X\times S^1, \tau\times\rho)\xrightarrow{\pi}\tilde{\Omega}_n^U(X, \tau)\xrightarrow{\Delta}\Omega_{n-2}^U(X, \tau)$
$\xrightarrow{\mathcal{P}}\tilde{\Omega}_{n-2}^U(X\times S^1, \tau\times\rho)\to\cdots$

(3) $\cdots\to\tilde{\Omega}_n^U(S^1, \rho)\xrightarrow{\pi}\tilde{\Omega}_n^U(Z_p)\xrightarrow{\Delta}\Omega_{n-2}^U(Z_p)\xrightarrow{\mathcal{P}}\tilde{\Omega}_{n-1}^U(S^1, \rho)\to\cdots,$

(4) $0\to\Omega_{n-1}^U\xrightarrow{\mathcal{P}'}\Omega_{n-1}^U\xrightarrow{\pi'}\tilde{\Omega}_n^U(Z_p)\xrightarrow{\Delta}\tilde{\Omega}_{n-2}^U(Z_p)\to 0,$

(5) $0\to\Omega_{2k}^U/p\Omega_{2k}^U\xrightarrow{\pi'}\tilde{\Omega}_{2k+1}^U(Z_p)\xrightarrow{\Delta}\tilde{\Omega}_{2k-1}^U(Z_p)\to 0.$ *We also have*

(6) *For* $k\geqslant 0$, $\tilde{\Omega}_{2k}^U(Z_p)=0$.

Proof. The assertion (6) is proved by the fact that $\Omega_{2k-1}^U=0$, [4, Cor. to Th. 3], and (1)–(5) are verified in the same way given in the preceeding section.

5. The Ω-module structures of $\hat{\mathcal{O}}_*(Z_3)$ and $\mathcal{O}_*(Z_3)$

In this section we compute $\hat{\mathcal{O}}_*(Z_3)$ and $\mathcal{O}_*(Z_3)$, and determine their Ω-module structures.

We shall use several facts shown by Conner and Floyd in [2, 46.1–46.3].

Consider the generating set $[\alpha_{2k-1}\colon k=1, 2, \cdots\}$ for $\tilde{\Omega}_*(Z_3)$ where $\alpha_{2k-1}=[S^{2k-1}, \rho]$ and $\rho=\exp(2\pi i/3)$. There exist closed oriented manifolds M^{4k}, $k=1, 2, \cdots$ such that for each k,

$$\beta_{2k-1}=3\alpha_{2k-1}+[M^4]\alpha_{2k-5}+[M^8]\alpha_{2k-9}+\cdots=0$$

in $\tilde{\Omega}_*(Z_3)$. And $\tilde{\Omega}_*(Z_3)$ is isomorphic as an Ω-module to the quotient of the free Ω-module generated by $\alpha_1, \alpha_3, \cdots$ by the submodule generated by $\beta_1, \beta_3, \cdots$.

We shall need three bordism groups of S^1-actions, $\Omega_*(S^1)$, a bordism group of free S^1-action, $\mathcal{O}_*(S^1)$ and $\mathcal{M}_*(S^1)$, two bordism groups of semi-free S^1-actions which are entirely analogues of the bordism groups $\Omega_*(Z_3)$, $\mathcal{O}_*(Z_3)$ and $\mathcal{M}_*(Z_3)$ studied in §2. They are just formed from the latters by replacing Z_3-actions by S^1-actions. For such bordism groups we have an exact sequence

$$0 \to \mathcal{O}_*(S^1) \xrightarrow{\tilde{\nu}} \mathcal{M}_*(S^1) \xrightarrow{\tilde{\partial}} \Omega_*(S^1) \to 0$$

which is verified in the same way given in §2, (cf. [7]), where the homomorphisms $\tilde{\nu}$ and $\tilde{\partial}$ are entirely analogues of ν and ∂.

Meanwhile we have obtained in Corollary 1.2 the exact sequence

$$0 \to \hat{\mathcal{O}}_*(Z_3) \xrightarrow{\nu} \mathcal{M}_*(Z_3) \xrightarrow{\partial} \tilde{\Omega}_*(Z_3) \to 0$$

where $\hat{\mathcal{O}}_*(Z_3)=\mathcal{O}_*(Z_3)/\mathcal{J}$ and $\mathcal{J}$ is an ideal of $\mathcal{O}_*(Z_3)$ which is generated by $[Z_3, \sigma]$.

Consider now the following diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & B & & K=\ker\lambda & & \\
 & & & & \cap & & \downarrow & & \\
0 \to & \mathcal{O}_*(S^1) & \xrightarrow{\tilde{\nu}} & & \mathcal{M}_*(S^1) & \xrightarrow{\tilde{\partial}} & \Omega_*(S^1) & \to 0 \\
 & \downarrow\lambda & & & \downarrow\lambda & & \downarrow\lambda & \\
0 \to & \hat{\mathcal{O}}_*(Z_3) & \xrightarrow{\nu} & & \mathcal{M}_*(Z_3) & \xrightarrow{\partial} & \tilde{\Omega}_*(Z_3) & \to 0 \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

where λ is the homomorphism defined by sending an S^1-action $[M, \tau]$ to a Z_3-action $[M, T]$. B will be defined in the following.

We also need some results given in [5]. First, we have

$$\Omega_*(S^1) = \sum_{k\geqslant 1} \Omega\cdot\alpha_{2k-1},$$

a free Ω-module generated by $\alpha_{2k-1}=[S^{2k-1}, \tau_0]$ where τ_0 is the usual free S^1-action on S^{2k-1} given by $\tau_0(t,(z_0, z_1, \cdots, z_{2k-1}))=(tz_0, tz_1, \cdots, tz_{2k-1})$, $t\in S^1$.

Next, we need the fact that

$$\mathcal{M}_*(S^1) = \mathcal{M}_*(Z_3) = \sum \Omega_i(BU(k)) = \Omega[\theta_0, \theta_1, \theta_2, \cdots]$$

is a polynomial algebra in $\theta_0, \theta_1, \cdots$, where $\theta_0=[\varepsilon^2\to *]$, $\varepsilon^2\to *$ is the trivial 2-plane bundle over a point $*$, and $\theta_i=[\eta\to CP(i)]$, $\eta\to CP(i)$ is the complex line bundle over an i-dimensional complex projective space $CP(i)$ induced from the universal bundle over $BU(1)$ by the inclusion i: $CP(i)\to BU(1)$.

We then see immediately that

$$K=\ker\lambda=\sum_{k\geqslant 1}\Omega\cdot\bar{\beta}_k\,,$$

a free Ω-module, where $\bar{\beta}_k=3\bar{\alpha}_{2k-1}+[M^4]\bar{\alpha}_{2k-5}+[M^8]\bar{\alpha}_{2k-9}+\cdots$, and $\lambda(\bar{\beta}_k)=\beta_{2k-1}=0$ in $\tilde{\Omega}_*(Z_3)$.

Let $\hat{\beta}_k$ be defined by

$$\hat{\beta}_k=3\theta_0^k+[M^4]\theta_0^{k-2}+[M^8]\theta_0^{k-4}+\cdots\in\mathcal{M}_*(S^1)$$

and let

$$B=\sum_{k\geqslant 1}\Omega\cdot\hat{\beta}_k\,,$$

a free Ω-module, which is evidently a submodule of $\mathcal{M}_*(S^1)$.

We then have

Lemma 1. $\tilde{\nu}(\mathcal{O}_*(S^1))\cap B=\{0\}$.

Proof. Since $\tilde{\partial}(\theta_0^k)=\bar{\alpha}_{2k-1}$, $\tilde{\partial}(\hat{\beta}_k)=\bar{\beta}_k$ which implies $\tilde{\partial}\,|\,B\colon B\approx K$. Assume now that $\tilde{\nu}(\mathcal{O}_*(S^1))$ and B have a non-zero element, say a, in common. Then $a\in\tilde{\nu}(\mathcal{O}_*(S^1))$ implies $\tilde{\partial}(a)=0$. The same element $a\neq 0$ in B, which is isomorphic to K, implies $\tilde{\nu}(a)\neq 0$ in K. This is a contradiction. The assertion thus follows.

Lemma 2. $\nu(\hat{\mathcal{O}}_*(Z_3))=B\oplus\tilde{\nu}(\mathcal{O}_*(S^1))$.

Proof. The diagram

$$\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_*(S^1) & \xrightarrow{\tilde{\nu}} & \mathcal{M}_*(S^1) & \xrightarrow{\tilde{\partial}} & \Omega_*(S^1) & \to & 0\\
 & & & & =\big\downarrow\lambda & & \big\downarrow\lambda & & \\
0 & \to & \hat{\mathcal{O}}_*(Z_3) & \xrightarrow{\nu} & \mathcal{M}_*(Z_3) & \xrightarrow{\partial} & \tilde{\Omega}_*(Z_3) & \to & 0
\end{array}$$

commutes, and $\mathcal{M}_*(S^1)=\mathcal{M}_*(Z_3)$. Hence

$$\begin{aligned}\nu(\hat{\mathcal{O}}_*(Z_3))&=\ker(\lambda\circ\tilde{\partial})=\tilde{\partial}^{-1}(\ker\lambda)=\tilde{\partial}^{-1}(K)\\&=B+\ker\tilde{\partial}=B+\tilde{\nu}(\mathcal{O}_*(S^1))\,.\end{aligned}$$

But $B\cap\tilde{\nu}(\mathcal{O}_*(S^1))=\{0\}$ by Lemma 1, we thus have

$$\nu(\hat{\mathcal{O}}_*(Z_3))=B\oplus\tilde{\nu}(\mathcal{O}_*(S^1))\,.$$

We may now have the Ω-module structure of $\hat{\mathcal{O}}_*(Z_3)$ as follows.

Theorem 5. $\hat{\mathcal{O}}_*(Z_3)\approx\sum_{k\geqslant 1}\Omega\cdot\hat{\beta}_k\oplus\mathcal{O}_*(S^1)$ *as free Ω-module.*

We go on to study the Ω-module structure of $\mathcal{O}_*(Z_3)$. Let $\mu_0=[Z_3,\sigma]$, and let μ_k be an element of $\mathcal{O}_*(Z_3)$ such that $\nu(\mu_k)=\hat{\beta}_k$ for each $k\geqslant 1$.

We then obtain the following

Theorem 6. *$\mathcal{O}_*(Z_3)\approx \sum_{k\geq 0}\Omega\cdot\mu_k\oplus\mathcal{O}_*(S^1)$ as free Ω-module.*

Proof. We already have the exact sequence

$$0\to\Omega_*\xrightarrow{i_*}\mathcal{O}_*(Z_3)\xrightarrow{\nu}\mathcal{M}_*(Z_3)\xrightarrow{\partial}\tilde{\Omega}_*(Z_3)\to 0\,,$$

(Corollary 1.1). And from the construction of $\hat{\beta}_k$ and μ_k, it is evident that ν: $\sum_{k\geq 1}\Omega\cdot\mu_k\approx\sum_{k\geq 1}\Omega\cdot\hat{\beta}_k$. Recall that $\hat{\mathcal{O}}_*(Z_3)=\hat{\mathcal{O}}_*(Z_3)/\mathcal{I}$, $\mathcal{I}=\Omega\cdot[Z_3,\sigma]=\Omega\cdot\mu_0$, so $\mathcal{O}_*(Z_3)\approx\Omega\cdot\mu_0+(\sum_{k\geq 1}\Omega\cdot\mu_k\oplus\mathcal{O}_*(S^1))$. But $\Omega\cdot\mu_0=\ker\nu$, we thus have

$$\mathcal{O}_*(Z_3)\approx\Omega\cdot\mu_0\oplus(\sum_{k\geq 1}\Omega\cdot\mu_k\oplus\mathcal{O}_*(S^1))\,.$$

The theorem follows.

REMARKS (1) The Ω-module structure of $\mathcal{O}_*(S^1)$ is determined by Uchida in [7] and independently by us in [5]. The result is as follows. For any element $[M^n,\tau]\in\mathcal{O}_*(S^1)$, consider $(M\times D^2, 1\times\tau_0)$ and $(M\times D^2,\tau\times\tau_0)$ where τ_0 is the usual S^1-action on D^2. Then $\partial(M\times D^2, 1\times\tau_0)=(M\times S^1, 1\times\tau_0)$ and $\partial(M\times D^2, \tau\times\tau_0)=(M\times S^1,\tau\times\tau_0)$ are equivariantly diffeomorphic by an equivariant diffeomorphism φ: $M\times S^1\to M\times S^1$ defined by $\varphi(x,t)=(t(x),t)$, [2. P. 119]. And form (M^{n+2},τ') from $(M\times D^2, 1\times\tau_0)\cup(-M\times D^2,\tau\times\tau_0)$ by identifying $(M\times S^1, 1\times\tau_0)$ and $(M\times S^1,\tau\times\tau_0)$ via φ. We may then define an Ω-map Γ: $\mathcal{O}_n(S^1)\to\mathcal{O}_{n+2}(S^1)$ by $\Gamma[M^n,\tau]=[M^{n+2},\tau']$. Let $\sigma_i=[CP(i+1),\tau]$, $\tau(t,[z_0,z_1,\cdots,z_{i+1}])=[tz_0,z_1,\cdots,z_{i+1}]$, $t\in S^1$. Then $F_\tau=CP(i)\cup\{\text{a point}\}$ and $\tilde{\nu}(\sigma_i)=\theta_i-\theta_0^{i+1}$. And using such $[CP(i+1),\tau]\in\mathcal{O}_*(S^1)$ and Γ, we have

$$\mathcal{O}_*(S^1)\approx\sum_{l_0,\cdots,l_j\geq 0}\Omega\cdot\Gamma^{l_0}(\sigma_1^{l_1}\cdots\sigma_j^{l_j})$$

as a free Ω-module.

(2) The Theorem 6 gives a partial answer to the statement of Conner and Floyd in the last page of [3].

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