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ON REPRESENTATIONS OF DIRECT PRODUCTS OF FINITE SOLVABLE GROUPS

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Let K be a field and π a finite group. We denote by $G_0(K\pi)$ the Grothendieck ring of $K\pi$. Let π_i be a finite group and M_i be finitely generated $K\pi_i$-module, $i=1, 2$. Let us denote by $M_1 \# M_2$ the outer tensor product of M_1 and M_2. We can define the natural ring homomorphism φ: $G_0(K\pi_1)\otimes G_0(K\pi_2) \rightarrow G_0(K(\pi_1\times\pi_2))$ by putting $\varphi([M_1]\otimes[M_2])=[M_1\# M_2]$. In this paper we study the kernel and cokernel of φ.

1. Let π be a finite group, E a finite normal separable extension of K which is a splitting field of π, and $\mathcal{G}(E/K)$ the Galois group of E over K. Let N be an $E\pi$-module with character χ and $\sigma\in\mathcal{G}(E/K)$. Then we define an $E\pi$-module σN, the conjugate of N, as usual and denote it's character by $\sigma\chi$. We denote the Schur index of N over K by $m_K(N)$.

Now, let π be the direct product of finite groups π_1 and π_2, $\pi=\pi_1\times\pi_2$. Let M_i be an irreducible $K\pi_i$-module, $i=1, 2$, and denote an irreducible $E\pi_i$-component of $M_i^E=M_i\otimes_K E$ by N_i, the character of N_i by ψ_i and the Galois group E over $K(\psi_i)$ by $\mathcal{H}_i=\mathcal{G}(E/K(\psi_i))$. Then, the following results can be found in [3].

(1) If σ, $\tau\in\mathcal{G}(E/K)$, then $\sigma N_1\#\tau N_2$ is an irreducible $E[\pi_1\times\pi_2]$-module also and $m_K(N_1\# N_2)=m_K(\sigma N_1\#\tau N_2)$.

(2) $M_1\# M_2$ is completely reducible. $M_1\# M_2=k(T_1\oplus\cdots\oplus T_r)$, where the $\{T_i\}$ are nonisomorphic irreducible $K\pi$-modules and $k=m_K(N_1)m_K(N_2)/m_K(N_1\# N_2)$. The $\{T_i\}$ have common K-dimension s, where $s=m_K(N_1\# N_2)(K(\psi_1, \psi_2)\colon K)(N_1\# N_2\colon E)$.

(3) $M_1\# M_2$ is an irreducible $K\pi$-module if and only if the following conditions are satisfied:

(a) $m_K(N_1)m_K(N_2)=m_K(N_1\# N_2)$.

(b) $\mathcal{G}(E/K)=\mathcal{H}_1\mathcal{H}_2$.

(c) $(K(\psi_1)\colon K)(K(\psi_2)\colon K)=(K(\psi_1, \psi_2)\colon K)$.

(4) Let $\pi_1=\pi_2$, $\pi=\pi_1\times\pi_1$. Let M_1 be an irreducible $K\pi_1$-module. Then $M_1\# M_1$ is irreducible if and only if M_1 is an absolutely irreducible $K\pi_1$-module.

Since for any irreducible $K[\pi_1\times\pi_2]$-module M we can find a unique irreducible $K\pi_i$-module M_i, $i=1, 2$, satisfying $M_1\# M_2\oplus > M$, the following is an immediate corollary to (3).

(5) We denote the order of a group π by $|\pi|$. Let Q be the field of rational numbers. If $(|\pi_1|, |\pi_2|)=1$, then

$$\varphi\colon G_0(Q\pi_1)\otimes G_0(Q\pi_2) \xrightarrow{\sim} G_0(Q[\pi_1\times\pi_2]) .$$

One aim of this paper is to study the converse to (5).

2. Hereafter we assume char. $K=0$.

Lemma 1. *If π_1 and π_2 are finite abelian groups, then Ker $\varphi=0$ and Coker φ is torsion free.*

Proof. Since the Schur index of abelian groups is 1, then φ is a split map by (2). Q.E.D.
Let $j\colon \pi'\to\pi$ be a group homomorphism. Then we have the induction and restriction functors

$$\text{mod}-K\pi' \underset{j_*=\text{res}}{\overset{j^*=(\cdot\otimes_{K\pi'}K\pi)}{\rightleftarrows}} \text{mod}-K\pi ,$$

and these functors induce the additive homomorphisms of Grothendieck rings, $G_0(K\pi') \underset{j_*}{\overset{j^*}{\rightleftarrows}} G_0(K\pi)$. Let π_i' be a subgroup of π_i. Then the following diagram is commutative.

$$\begin{array}{ccccccc}
\text{Ker}\,\varphi & \longrightarrow & G_0(K\pi_1)\otimes G_0(K\pi_2) & \xrightarrow{\varphi} & G_0(K[\pi_1\times\pi_2]) & \longrightarrow & \text{Coker}\,\varphi \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
\text{Ker}\,\psi & \longrightarrow & G_0(K\pi_1')\otimes G_0(K\pi_2') & \xrightarrow{\psi} & G_0(K[\pi_1'\times\pi_2']) & \longrightarrow & \text{Coker}\,\psi
\end{array}$$

Proposition 2. *For any finite groups π_1, π_2, we have Ker $\varphi=0$.*

Proof. Since Ker $\psi=0$ for cyclic groups π_1' and π_2', by the commutativity of the above diagram and the Artin's induction theorem, Ker $\varphi=0$. Q.E.D.
(But we can prove this proposition without the induction theorem.)

Now let π_i' be a normal subgroup of π_i. Then we have the exact sequence $1 \longrightarrow \pi_i' \xrightarrow{j} \pi_i \xrightarrow{p} \pi_i'' \longrightarrow 1$, $i=1, 2$. From this we obtain the following commutative diagram.

$$\begin{array}{ccccc}
G_0(K\pi_1'')\otimes G_0(K\pi_2'') & \xrightarrow{\varphi_1} & G_0(K[\pi_1''\times\pi_2'']) & \xrightarrow{\varphi_1'} & \text{Coker}\,\varphi_1 \\
p^*\uparrow\downarrow p_* & & p^*\uparrow\downarrow p_* & & p^*\uparrow\downarrow p_* \\
G_0(K\pi_1)\otimes G_0(K\pi_2) & \xrightarrow{\varphi_2} & G_0(K[\pi_1\times\pi_2]) & \xrightarrow{\varphi_2'} & \text{Coker}\,\varphi_2 \\
j^*\uparrow\downarrow j_* & & j^*\uparrow\downarrow j_* & & j^*\uparrow\downarrow j^* \\
G_0(K\pi_1')\otimes G_0(K\pi_2') & \xrightarrow{\varphi_3} & G_0(K[\pi_1'\times\pi_2']) & \xrightarrow{\varphi_3'} & \text{Coker}\,\varphi_3 .
\end{array}$$

Let M be an irreducible $K[\pi_1'\times\pi_2']$-module, E a finite normal separable extension of K which is a splitting field of $\pi_1'\times\pi_2'$ and $N_1\# N_2$ an $E[\pi_1'\times\pi_2']$-irreducible component of M^E, where N_i is the $E\pi_i'$-irreducible module, $i=1, 2$. Denote the characters of M, N_i by χ, ψ_i respectively and put $m=|\pi_1''\times\pi_2''|$.

Lemma 3. (a) *If there exists an irreducible $K[\pi_1'\times\pi_2']$-module M such that $\varphi_3'([M])\neq 0$ and*

$$m_K(N_1)m_K(N_2)(K(\psi_1)\colon K)(K(\psi_2)\colon K)/m_K(N_1\# N_2)(K(\psi_1, \psi_2)\colon K)\nmid m\,,$$

then Coker $\varphi_2\neq 0$.
(b) *If there exists an irreducible $K[\pi_1'\times\pi_2']$-module M such that $\varphi_3'([M])\neq 0$ and the inertial group of χ, $I(\chi)=\{g\mid g\in\pi_1\times\pi_2\ \chi^g=\chi\}$, coincides with $\pi_1\times\pi_2$ and if Coker φ_3 is torsion free, then Coker $\varphi_2\neq 0$.*
(c) *Let $K=Q$. Let π_i' be an elementary abelian p-group and $|\pi_i'|=p^{n_i}$, $i=1, 2$, where p is an odd prime. Denoting by $\mathfrak{c}_i$ the centralizer of π_i' in π_i, then we can regard $\pi_i/\mathfrak{c}_i$ as a group of morphisms of the module π_i'. This identification induces the natural map*

$$\psi\colon\ \pi_1\times\pi_2\longrightarrow\pi_1/\mathfrak{c}_1\times\pi_2/\mathfrak{c}_2\longrightarrow PGL(n_1+n_2, p)\,.$$

Then $j_*\, j^*$ Coker $\varphi_3=0$ if and only if

$$\psi(\pi_1\times\pi_2)\ \text{contains}$$

$$\sigma=\begin{pmatrix} \overbrace{1 \quad\quad\quad}^{n_1} & & & \overbrace{\quad\quad}^{n_2} & \\ & \ddots & & & \\ & & 1 & & \\ & & & r & \\ & & & & \ddots \\ & & & & & r \end{pmatrix},$$

where r is a primitive root modulo p and the order of σ is $p-1$.
(d) *If Coker $\varphi_1\neq 0$, then Coker $\varphi_2\neq 0$.*

Proof. (a) Assume $j_*j^*[M]=[M\otimes_{K[\pi_1'\times\pi_2']}K[\pi_1\times\pi_2]]\in Im\ \varphi_3$. Then

$$M\otimes_{K[\pi_1'\times\pi_2']}K[\pi_1\times\pi_2]=M_{11}\# M_{21}\oplus M_{12}\# M_{22}\oplus\cdots\oplus M_{1s}\# M_{2s}$$

where each M_{ij} is a $K\pi_i'$-irreducible module, $i=1, 2, j=1, 2, \cdots, s$.

$$(*)\qquad M^E\otimes_{E[\pi_1'\times\pi_2']}E[\pi_1\times\pi_2]=M_{11}^E\# M_{21}^E\oplus M_{12}^E\# M_{22}^E\oplus\cdots\oplus M_{1s}^E\# M_{2s}^E\,.$$

Let N_{ij} be an $E\pi_i'$-irreducible component of M_{ij}^E, $i=1, 2, j=1, 2, \cdots, s$. Since $N_1\# N_2$ is an irreducible component of M^E, there exists an element g_{ij} of π_i and $\sigma_i\in\mathcal{G}(E/K)$ such that $N_{ij}=(\sigma_iN_i)g_{ij}$. Let ψ_{ij} be the character of N_{ij}. Then $m_K(N_{ij})=m_K((\sigma_iN_i)g_{ij})=m_K(N_i)$ and $K(\psi_{ij})=K(\psi_i)$. Comparing the E-dimensions of both sides in $(*)$, we obtain

$$m_K(N_1\# N_2)(K(\psi_1, \psi_2)\colon K)m(N_1\# N_2\colon E)$$
$$= s\cdot m_K(N_1)m_K(N_2)(K(\psi_1)\colon K)(K(\psi_2)\colon K)(N_1\# N_2\colon E)\,.$$

Hence

$$m = s\cdot m_K(N_1)m_K(N_2)(K(\psi_1)\colon K)(K(\psi_2)\colon K)/m_K(N_1\# N_2)(K(\psi_1, \psi_2)\colon K)\,.$$

This contradicts the assumption. Therefore Coker φ_2 is not zero.

(b) Since $I(\mathcal{X})=\pi_1\times\pi_2$, $M\otimes_{K[\pi_1'\times\pi_2']}K[\pi_1\times\pi_2]\cong M^m$ as $K[\pi_1'\times\pi_2']$-modules. Since Coker φ_3 is torsion free, we have Coker $\varphi_2\neq 0$.

(c) First, assume $j_* j^*$ Coker $\varphi_3=0$. We have $Q\pi_1'\cong Q[X_1, \cdots, X_{n_1}]/(X_1^p-1, \cdots, X_{n_1}^p-1)$ and $Q\pi_2'\cong Q[Y_1, \cdots, Y_{n_2}]/(Y_1^p-1, \cdots, Y_{n_2}^p-1)$. Let ζ be a primitive p-th root of unity and put $G=\mathcal{G}(Q(\zeta)/Q)$. Further put $M_1=Q[X_1, \cdots, X_{n_1}]/(X_1-\zeta, \cdots, X_{n_1}-\zeta)^G$ and $M_2=Q[Y_1, \cdots, Y_{n_2}]/(Y_1-\zeta, \cdots, Y_{n_2}-\zeta)^G$ where $(\ \ )^G$ is the set of all G-invariant elements of $(\ \ )$. Then each M_i is an irreducible $Q\pi_i'$-module.

$$\begin{aligned} M_1\# M_2 \cong\, & Q[X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}]/(X_1-\zeta, \cdots, X_{n_1}-\zeta, Y_1-\zeta, \cdots, Y_{n_2}-\zeta)^G \\ & \oplus Q[X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}]/(X_1-\zeta, \cdots, X_{n_1}-\zeta, Y_1-\zeta^2, \cdots, Y_{n_2}-\zeta^2)^G \\ & \oplus\cdots \\ & \oplus Q[X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}]/(X_1-\zeta, \cdots, X_{n_1}-\zeta, Y_1-\zeta^{p-1}, \cdots, Y_{n_2}-\zeta^{p-1})^G \end{aligned}$$

as $Q[\pi_1'\times\pi_2']$-modules. If we put

$$M = Q[X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}]/(X_1-\zeta, \cdots, X_{n_1}-\zeta, Y_1-\zeta, \cdots, Y_{n_2}-\zeta)^G\,,$$

we have $\varphi_3'([M])\neq 0$ and so, by the assumption, $j_* j^* M\oplus > M_1\# M_2$. Therefore we can find an element c of $\pi_1\times\pi_2$ such that

$$\begin{aligned} M\otimes c &= Q[X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}]/(X_1-\zeta, \cdots, X_{n_1}-\zeta, Y_1-\zeta, \cdots, Y_{n_2}-\zeta)^G\otimes c \\ &\cong Q[X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}]/(X_1-\zeta, \cdots, X_{n_1}-\zeta, Y_1-\zeta^r, \cdots, Y_{n_2}-\zeta^r)^G. \end{aligned}$$

Then we have $\psi(c)=\sigma$.

Conversely, assume $\psi(\pi_1\times\pi_2)\ni\sigma$. Let c be a representative of σ in $\pi_1\times\pi_2$, $\{g_i, g_ic, g_ic^2, g_ic^3, \cdots, g_ic^{p-2}\}$ representatives of $\pi_1''\times\pi_2''$ in $\pi_1\times\pi_2$ and M an irreducible $Q[\pi_1'\times\pi_2']$-module. (We can find representatives of above type.) Then $j_*j^*M=\sum_i^{\oplus}(M\otimes g_i\oplus M\otimes g_ic\oplus\cdots\oplus M\otimes g_ic^{p-2})$ and there exist integers $r_1, \cdots, r_{n_1}, t_1, \cdots, t_{n_2}$ such that $M\otimes g_i\cong Q[X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}]/(X_1-\zeta^{r_1}, \cdots, X_{n_1}-\zeta^{r_{n_1}}, Y_1-\zeta^{t_1}, \cdots, Y_{n_2}-\zeta^{t_{n_2}})^G$. By the assumption, $\sum_{j=0}^{p-2} M\otimes g_ic^j\cong \sum_{j=1}^{p-1}{}^{\oplus} Q[X_1, \cdots, X_{n_1}, Y_1, \cdots, Y_{n_2}]/(X_1-\zeta^{r_1}, \cdots, X_{n_1}-\zeta^{r_{n_1}}, Y_1-\zeta^{jt_1}, \cdots, Y_{n_2}-\zeta^{jt_{n_2}})^G$ $\cong [Q[X_1, \cdots, X_{n_1}]/(X_1-\zeta^{r_1}, \cdots, X_{n_1}-\zeta^{r_{n_1}})^G\# Q[Y_1, \cdots, Y_{n_2}]/(Y_1-\zeta^{t_1}, \cdots, Y_{n_2}-\zeta^{t_{n_2}})^G]^u$ where u is a positive integer. Therefore $[j_*j^*M]\in \operatorname{Im}\varphi_3$ and

$\varphi_3'(j_* j^*[M])=0$.
(d) Since $p^*p_*=1$, it is trivial. Q.E.D.
Denote by $e(\pi)$ the exponent of a group π and by ζ_n a primitive n-th root of unity for any integer n.

Lemma 4. *Let π_i be an abelian group, $i=1, 2$, and G.C.D.$(e(\pi_1), e(\pi_2)) =\prod p^{h_p}$. Let $s_p=\max\{s \mid \zeta_{p^s}\in K\}$ for each prime p. If there exists at least one prime p such that $h_p>s_p$, then φ: $G_0(K\pi_1)\otimes G_0(K\pi_2) \not\twoheadrightarrow G_0(K[\pi_1\times\pi_2])$.*

Proof. $K(\zeta_{p^{h_p}})$ is an irreducible $K\pi_i$-module. Let us consider the underlying abelian group of $K(\zeta_{p^{h_p}})\#K(\zeta_{p^{h_p}})$. There exists an integer n such that $K(\zeta_{p^{h_p}})\otimes_K K(\zeta_{p^{h_p}})\cong K(\zeta_{p^{h_p}})^n$. Since $(K(\zeta_{p^{h_p}}): K) \neq 1$, we have $n\neq 1$ and so Coker $\varphi\neq 0$. Q.E.D.

3. (I) We can determine Coker φ when π_1 and π_2 are abelian groups. Let π_1 be an abelian group with invariants $l_1, \cdots, l_n$ and π_2 an abelian group with invariants $l_{n+1}, \cdots, l_{n+m}$. Then

$$\begin{aligned}\text{rank}\quad \text{Coker}\ \varphi = \sum_{d_i|l_i} [\eta(d_1)\times\cdots\times\eta(d_{n+m})\times\{(K(\zeta_{\underset{1\leqslant i\leqslant n+m}{\text{L.C.M.}(d_i)}}): K)^{-1} \\ -(K(\zeta_{\underset{1\leqslant i\leqslant n}{\text{L.C.M.}(d_i)}}): K)^{-1}(K(\zeta_{\underset{1\leqslant j\leqslant m}{\text{L.C.M.}(d_{n+j})}}): K)^{-1}\}\end{aligned}$$

where η is the Euler's function.
(II) We denote the center of a group π by $Z(\pi)$.

Theoram 5. *Let $\underset{\pi_1'\triangleleft\pi_1}{\text{L.C.M.}}(e(Z(\pi_1/\pi_1')))=\prod p^{m_p}$, $\underset{\pi_2'\triangleleft\pi_2}{\text{L.C.M.}}(e(Z(\pi_2/\pi_2')))=\prod p^{n_p}$ and $s_p=\max\{s\mid\zeta_{p^s}\in K\}$. If there exists a prime p such that $\min(m_p, n_p)>s_p$, then $G_0(K\pi_1)\otimes G_0(K\pi_2) \not\twoheadrightarrow G_0(K[\pi_1\times\pi_2])$.*

Proof. By assumption, there exists a normal subgroup π_i' of π_i such that $p^{m_p}\mid e(Z(\pi_1/\pi_1'))$ and $p^{n_p}\mid e(Z(\pi_2/\pi_2'))$. Put $\pi_i''=\pi_i/\pi_i'$ and consider the following commutative diagram;

$$\begin{array}{ccccc}
G_0(K\pi_1)\otimes G_0(K\pi_2) & \xrightarrow{\varphi_1} & G_0(K[\pi_1\times\pi_2]) & \longrightarrow & \text{Coker}\ \varphi_1 \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
G_0(K\pi_1'')\otimes G_0(K\pi_2'') & \xrightarrow{\varphi_2} & G_0(K[\pi_1''\times\pi_2'']) & \longrightarrow & \text{Coker}\ \varphi_2 \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
G_0(K[Z(\pi_1'')])\otimes G_0(K[Z(\pi_2'')]) & \xrightarrow{\varphi_3} & G_0(K[Z(\pi_1''\times\pi_2'')]) & \longrightarrow & \text{Coker}\ \varphi_3\,.
\end{array}$$

Let G.C.D. $(e(Z(\pi_1'')), e(Z(\pi_2''))=\prod p^{h_p}$. Since $h_p>s_p$, Coker $\varphi_3\neq 0$ by Lemma 4 and since Coker φ_3 is torsion free by Lemma 1, then Coker $\varphi_2\neq 0$ by Lemma 3 (b), and terefore Coker $\varphi_1\neq 0$ by Lemma 3 (d). Q.E.D.

Corollary 6. *Let L.C.M.* $(e(Z(\pi/\pi')))=\prod_{\pi'\lhd\pi} p^{m_p}=h$. *Then any splitting field of π contains the primitive h-th root of unity.*

Proof. By (4) $G_0(K\pi)\otimes G_0(K\pi)\xrightarrow{\sim} G_0(K[\pi\times\pi])$ if and only if K is a splitting field of π. So this corollary is trivial. Q.E.D.

(III) **Theorem 7.** *Let π_i be a group of odd order. Assume that there exists an odd prime p such that $p|(|\pi_1|, |\pi_2|)$ and $2|(K(\zeta_p): K)$ where ζ_p is a primitive p-th root of unity. Then* φ; $G_0(K\pi_1)\otimes G_0(K\pi_2) \not\xrightarrow{\sim} G_0(K[\pi_1\times\pi_2])$.

Proof. Since π_i is a group of odd order, each π_i is solvable. We can consider a principal series $\pi_i=\pi_i^{(0)}\supset\pi_i^{(1)}\supset\cdots\supset\pi_i^{(n_i)}\supset\cdots\supset(1)$ and find integers n_i, r_i such that $|\pi_i^{(n_i)}:\pi_i^{(n_i+1)}|=p^{r_i}$, $r_i>0$, for each $i=1, 2$. And consider the following commutative diagram;

$$\begin{array}{ccc}
G_0(K\pi_1)\otimes G_0(K\pi_2) & \xrightarrow{\varphi_1} & \\
\uparrow\downarrow & & \\
G_0(K[\pi_1/\pi_1^{(n_1+1)}])\otimes G_0(K[\pi_2/\pi_2^{(n_2+1)}]) & \xrightarrow{\varphi_2} & \\
\uparrow\downarrow & & \\
G_0(K[\pi_1^{(n_1)}/\pi_1^{(n_1+1)}])\otimes G_0(K[\pi_2^{(n_2)}/\pi_2^{(n_2+1)}]) & \xrightarrow{\varphi_3} &
\end{array}$$

$$\begin{array}{ccc}
G_0(K[\pi_1\times\pi_2]) & \longrightarrow & \mathrm{Coker}\,\varphi_1 \\
\uparrow\downarrow & & \uparrow\downarrow \\
G_0(K[\pi_1/\pi_1^{(n_1+1)}\times\pi_2/\pi_2^{(n_2+1)}]) & \longrightarrow & \mathrm{Coker}\,\varphi_2 \\
\uparrow\downarrow & & \uparrow\downarrow \\
G_0(K[\pi_1^{(n_1)}/\pi_1^{(n_1+1)}\times\pi_2^{(n_2)}/\pi_2^{(n_2+1)}]) & \longrightarrow & \mathrm{Coker}\,\varphi_3\,.
\end{array}$$

By Lemma 4, Coker $\varphi_3\neq 0$. Since

$$(K(\zeta_p): K)(K(\zeta_p): K)/(K(\zeta_p): K)\nmid \prod_{i=1,2}|\pi_i:\pi_i^{(n_i)}|\,,$$

from Lemma 3 (a) it follows that Coker $\varphi_2\neq 0$ and so by Lemma 3 (d) we have Coker $\varphi_1\neq 0$. Q.E.D.

In case $2\nmid|\pi_1|\cdot|\pi_2|$, we can prove the converse to (5) by putting $K=Q$ in Theorem 7.

Corollary. 8 *Assume* $2\nmid|\pi_1|\cdot|\pi_2|$. *Then*

φ: $G_0(Q\pi_1)\otimes G_0(Q\pi_2)\xrightarrow{\sim} G_0(Q[\pi_1\times\pi_2])$ *if and only if* $(|\pi_1|, |\pi_2|)=1$.

Corollary 9. *Put* $|\pi|=\prod_{i=1}^{m} p_i^{e_i}$ *and suppose that* $p_i\nmid p_j-1$ *for any indices* $1\leqslant i, j\leqslant m$. *Then any splitting field of π contains the primitive $p_1\cdots p_m$-th root of unity.*

Proof. We can show this corollary by the same method as in Theorem 7. Q.E.D.

REMARK. If π is a nilpotent group, this result has been seen. For a given integer $n=p_1^{n_1}\cdots p_m^{n_m}$ all of groups of order n are nilpotent if and only if $p_j \nmid p_i^{n_i-t}-1$ for all t such that $n_i>t\geqslant 0$ and all i, j.

(IV) Here we consider 2-groups. In this case the groups with a cyclic subgroup of index 2 are important. For any character of 2-groups is induced by the character of such groups. (See [4] p. 73 (14.3).) Such groups can be classified as follows. Put $|\pi|=2^{n+1}$,

I $\quad \pi=\langle s \mid s^{2^{n+1}}=1\rangle$.

II $\quad \pi=\langle s, t \mid s^{2^n}=1,\ t^2=1,\ tst^{-1}=s\rangle$

III $\quad \pi=\langle s, t \mid s^{2^n}=1,\ t^2=s^{2^{n-1}},\ tst^{-1}=s^{-1}\rangle, \qquad n\geqslant 2$.

IV $\quad \pi=\langle s, t \mid s^{2^n}=1,\ t^2=1,\ tst^{-1}=s^{-1}\rangle, \qquad n\geqslant 2$.

V $\quad \pi=\langle s, t \mid s^{2^n}=1,\ t^2=1,\ tst^{-1}=s^{1+2^{n-1}}\rangle, \qquad n\geqslant 3$.

VI $\quad \pi=\langle s, t \mid s^{2^n}=1,\ t^2=1,\ tst^{-1}=s^{-1+2^{n-1}}\rangle, \qquad n\geqslant 3$.

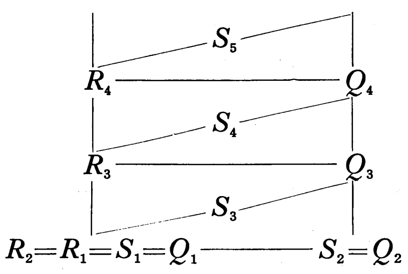
Theorem 10. *Let π_1 and π_2 be arbitrary two groups of the above types. Then $\varphi\colon G_0(Q\pi_1)\otimes G_0(Q\pi_2)\xrightarrow{\sim} G_0(Q[\pi_1\times\pi_2])$ if and only if*

(a) *π_1 is a group of type* (I, $n=0$), (II, $n=1$) *or* (IV, $n=2$) *and π_2 is any,*

(b) *π_1 is of type* (I, $n=1$), (II, $n=2$), (III, $n=2$), (V, $n=3$) *or* (VI, $n=3$) *and π_2 is of type IV,*

(c) *π_1 is of type* (I, $n=1$), (II, $n=2$) *or* (V, $n=3$) *and π_2 is of type* VI.

Let $Q_k=Q(\cos \pi/2^{k-1}+i\sin \pi/2^{k-1})$, $R_k=Q(\cos \pi/2^{k-1})$ and $S_k=Q(i\sin \pi/2^{k-1})$.



First, we shall write out the division algebras which are contained within $Q\pi$. (See, Feit [4] p. 63–p. 66.)

If π is of type I, $\{Q_i\}_{1\leqslant i\leqslant n+1}$ are all of the division algebras of $Q\pi$. When π is of type II, $\{Q_i\}_{1\leqslant i\leqslant n}$ are all of the division algebras. If π is of type III, then $\{D, R_i\}_{1\leqslant i\leqslant n-1}$ are all of the division algebras where D is the division algebra of a faithful irreducible representation of π. Hence the center of D is R_n. If π

is of type IV, $\{R_i\}_{1\leq i\leq n}$ are all of the division algebras. When π is of type V and $n=3$, then Q_1 and Q_2 are only division algebras of π. If $n>3$, Q_3 is one of the division algebras of π. And if π is of type VI, $\{S_n, R_i\}_{1\leq i\leq n-1}$ are all of the division algebras.

Lemma 11. *Let χ be a faithful irreducible character of the group of type* III. *Then $m_{S_k}(\chi)=1$ for $k\geqq 2$.*

In case $k=2$, we can see the proof of Lemma 11, for example, in Feit [4]. In case $k>2$, we can prove it similarly.

Proof of Theorem 10. a) When π_1 is of type (I, $n=0$), (II, $n=1$) or (IV, $n=2$), Q is a splitting field of π_1. Therefore φ is an isomorphism.
(b) If π_i is of type I, II or V, Q_2 is one of the division algebras of π_i, $i=1, 2$. Then Coker $\varphi\neq 0$, because $Q_2\otimes_Q Q_2\cong Q_2\oplus Q_2$.
(c) If π_1 is of type I, II or V and π_2 of type III, then Coker $\varphi\neq 0$ because $Q_2\otimes_Q D\cong(Q_2)_2$.
(d) If π_1 is of type (I, $n=1$), (II, $n=2$) or (V, $n=3$) and π_2 is of type IV, the division algebra of π_1 is Q_1 or Q_2 and the division algebra of π_2 is one of $\{R_i\}_{1\leq i\leq n}$. Since $Q_2\otimes_Q R_i\cong Q_i$ for $3\leqq i\leqq n$ and $Q_2\otimes_Q R_i\cong Q_2$ for $i<3$, we obtain Coker $\varphi=0$. If π_1 is of type (I, $n>1$), (II, $n>2$) or (V, $n>3$) and π_2 of type IV, Coker $\varphi\neq 0$, because Q_3 is one of the division algebras of π_1 and $Q_3\otimes_Q R_3\cong Q_3\oplus Q_3$.
(e) If π_1 is of type (I, $n=1$), (II, $n=2$) or (V, $n=3$) and π_2 is of type VI, Coker $\varphi=0$. For the division algebra of π_2 is one of $\{S_n, R_i\}_{1\leq i\leq n-1}$, $n\geqq 3$ and $Q_2\otimes_Q S_n\cong Q_n$ for $n\geqq 3$, $Q_2\otimes_Q R_i\cong Q_i$ for $3\leqq i\leqq n-1$ and $Q_2\otimes_Q R_i\cong Q_2$ for $i<3$. If π_1 is of type (I, $n>1$), (II, $n>2$) or (V, $n>3$) and π_2 is of type VI, then Coker $\varphi\neq 0$ because $Q_3\otimes_Q S_3\cong Q_3\oplus Q_3$ and $Q_3\otimes_Q R_3\cong Q_3\oplus Q_3$.
(f) If π_1 is of type (III, $n=2$), the division algebra of π_1 is Q_1 or D with center Q. Since $D\otimes_Q R_i$ is a division algebra also for all i, we have Coker $\varphi=0$ if π_2 is of type IV. If $n>2$, there exists a division algebra of π_1 with center R_3. From the fact that $R_3\otimes_Q R_3\cong R_3\oplus R_3$, it follows that Coker $\varphi\neq 0$ for the group π_2 of type IV.
(g) Assume that π_1 is of type III and π_2 of type VI. Since $D\otimes_Q S_n\cong(S_n)_2$ by Lemma 11, we obtain Coker $\varphi\neq 0$.
(h) Suppose that π_1 is of type IV. If π_2 is of type (VI, $n=3$), the division algebra of π_2 is Q_1 or S_3. Since $R_i\otimes_Q S_3\cong Q_i$ for $3\leqq i\leqq n$ or S_3 for $i<3$, Coker $\varphi=0$. If π_2 is of type (VI, $n=4$), then S_4 is a division algebra of π_2 and if π_2 is of type (VI, $n>4$), R_4 is a division algebra of π_2. Since $R_3\otimes_Q S_4\cong S_4\oplus S_4$ and $R_3\otimes_Q R_4\cong R_4\oplus R_4$, in both cases Coker $\varphi\neq 0$. Q.E.D.

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