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Osaka J. Math.
8 (1971), 291-297

ON THE WHITEHEAD GROUP OF THE DIHEDRAL GROUP OF ORDER 2p

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(Received December 17, 1970)

1. In [5], Lam has shown that the Whitehead group $\mathrm{Wh}(S_3)$ of the symmetric group S_3 on three letters is zero. His method is the interesting one which combines the induction theorem with some concrete computations.

The object of this paper is to show the following results.

Theorem. *If G is the dihedral group of order $2p$, p an odd prime, then the Whitehead group* $\mathrm{Wh}(G)$ *of G is torsion free.*

Hence, from the generalized unit theorem by Bass ([1]), we can easily see that $\mathrm{Wh}(G)$ is a free abelian group of rank $(p-3)/2$. In case $p=3$, G is isomorphic to S_3, so our result includes Lam's as a special case.

Our method is essentially based on his idea. However, using some techniques in algebraic K-theory we have been able to simplify the computations, for example, of the reduced norm.

Let $\mathrm{K}_1(ZG)$ be the Whitehead group of the integral group ring ZG of a finite group G. We shall denote by $\mathrm{Wh}(G)$ the cokernel of the natural homomorphism

$$\pm G \xrightarrow{\subset} GL_1(ZG) \xrightarrow{\subset} GL(ZG) \longrightarrow \mathrm{K}_1(ZG)\,,$$

and call it the "Whitehead group of the group G". For any Z-order A in a finite semi-simple Q-algebra, we shall denote by $\mathrm{SK}_1(A)$ the kernel of the reduced norm of $\mathrm{K}_1(A)$, and for any two-sided ideal $\mathfrak{a}$ of A, $\mathrm{SK}_1(A, \mathfrak{a})$ will denote the inverse image of $\mathrm{SK}_1(A)$ with the natural homomorphism $\mathrm{K}_1(A, \mathfrak{a}) \to \mathrm{K}_1(A)$.

2. The following notations will be fixed throughout this paper.

p=any odd prime
G=the dihedral group generated by the elements s and t under the defining relations $s^p=t^2=1$ and $ts=s^{-1}t$
ζ=a primitive p th root of unity
$L=Q(\zeta)$ the cyclotomic field over the rational number field Q
$L_0=Q(\vartheta)$, $\vartheta=\zeta+\zeta^{-1}$, the maximal real subfield of L
$R=Z[\zeta]$ the integral closure in L over the ring Z of rational integers
$R_0=Z[\vartheta]$ the integral closure in L_0 over Z

Let τ be the automorphism of order 2 of L such that $\zeta^\tau=\zeta^{-1}$ the complex conjugate of ζ, and consider the twisted group ring

$$\sum = L+L\tau, \quad \tau l = l^\tau\tau \qquad \text{for any } l\in L\,.$$

Then $\sum$ is a simple Q-algebra with the center L_0, and is identified with the full matrix algebra $M_2(L_0)$ over L_0, where the identification can be set up by the following "choice of coordinates"

$$\phi(\zeta)=\begin{pmatrix} 0 & 1 \\ -1 & \vartheta \end{pmatrix}, \quad \phi(\tau)=\begin{pmatrix} 1 & 0 \\ \vartheta & -1 \end{pmatrix}. \tag{1}$$

Now, we have the decomposition of the group ring QG into the simple components

$$QG = Q\oplus Q\oplus\sum$$

with the projections

$$\begin{matrix} j_1(s)=1 & j_2(s)=1 & & j_3(s)=\zeta \\ j_1(t)=1\,, & j_2(t)=-1 & \text{and} & j_3(t)=\tau\,. \end{matrix}$$

Let

$$\mathfrak{O} = Z\oplus Z\oplus\mathfrak{o}$$

be a maximal order in QG, where $\mathfrak{o}$ is the maximal order in $\sum$ which is identified with $M_2(R_0)$ under the identification (1), i.e.,

$$\phi(\mathfrak{o}) = M_2(R_0)\,. \tag{2}$$

If we set

$$\Lambda = R+R\tau$$

a Z-order in $\sum$ contained in $\mathfrak{o}$, then we see clearly that the integral group ring ZG is projected onto

$$j_1(ZG)=Z, \quad j_2(ZG)=Z, \quad \text{and} \quad j_3(ZG)=\Lambda\,,$$

respectively.

Recall, from [1] that the reduced norm Nrd is the map defined on $K_1(ZG)$, with values in the unit group of the maximal order in the center of QG. In the present case, this is concretely described as follows:

$$\mathrm{Nrd} = \varinjlim_n \mathrm{N}_n\,, \quad \text{where}$$

$$\mathrm{N}_n;\ GL_n(ZG)/[GL_n(ZG),\, GL_n(ZG)] \to U(Z)\times U(Z)\times U(R_0)\,, \tag{3}$$
$$\mathrm{N}_n = (\det\circ j_1,\ \det\circ j_2,\ \det\circ\phi\circ j_3)\,.$$

For any abelian group K, we shall denote by Tor K the "torsion subgroup" of K. Since Tor $\mathrm{Wh}(G) = \mathrm{Tor}\ \mathrm{K}_1(ZG)/\pm G$ and $\mathrm{SK}_1(ZG)$ is contained in Tor $\mathrm{K}_1(ZG)$ (in general, $\mathrm{SK}_1(\ )$ is finite (c.f. [1])), then our theorem is obtained if we show that $\mathrm{Nrd}(\mathrm{Tor}\ \mathrm{K}_1(ZG)) = \mathrm{Nrd}(\pm G)$ and $\mathrm{SK}_1(ZG) = 0$.

Before start the computations, we shall provide the following lemmas.

Lemma 1. *Let* $\mathfrak{c} = (\zeta-1)^2\mathfrak{o}$ *be the two-sided ideal of* $\mathfrak{o}$ *generated by the element* $(\zeta-1)^2$ *in* R. *Then* $\mathfrak{c}$ *is contained in* Λ, *and is identified with the ideal* $M_2((2-\vartheta)R_0)$ *of* $M_2(R_0)$, *under the identification* (2) *of* $\mathfrak{o}$ *with* $M_2(R_0)$.

Proof. Let actually calculate the image $\phi((\zeta-1)^2)$ by the rule (1). Then

$$\phi((\zeta-1)^2) = (2-\vartheta)\begin{pmatrix} 0 & -1 \\ 1 & -\vartheta \end{pmatrix}.$$

Since $\begin{pmatrix} 0 & -1 \\ 1 & -\vartheta \end{pmatrix}$ is a unit in $M_2(R_0)$, we get the equality $\phi(\mathfrak{c}) = (2-\vartheta)M_2(R_0) = M_2((2-\vartheta)R_0)$. This shows the second assertion of the lemma. To see that $\mathfrak{c} \subseteq \Lambda$, it therefore sufficies to show that $M_2((2-\vartheta)R_0) \subseteq \phi(\Lambda)$. Let

$$x = (\alpha+\beta\zeta)+(\gamma+\delta\zeta)\tau, \qquad \alpha, \beta, \gamma, \delta \in L_0$$

be any element of $\sum$. Then by an easy computation, it is seen that the four entries $\alpha', \beta', \gamma', \delta'$ of the matrix $\phi(x)$ are given by the equation

$$(\alpha, \beta, \gamma, \delta)X = (\alpha', \beta', \gamma', \delta')$$
$$X = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & \vartheta \\ 1 & 0 & \vartheta & -1 \\ \vartheta & -1 & \vartheta^2-1 & -\vartheta \end{pmatrix} \tag{4}$$

Here, $\det X = -(\vartheta+2)(2-\vartheta)$ and the element $(\vartheta+2)$ is a unit in R_0. Then, for any $\begin{pmatrix} \alpha' & \beta' \\ \gamma' & \delta' \end{pmatrix} \in M_2((2-\vartheta)R_0)$, we can find (uniquely) in R_0^4 a vector $(\alpha, \beta, \gamma, \delta)$ which satisfies the equation (4). This shows that any element of $M_2((2-\vartheta)R_0)$ is contained in the image of Λ by ϕ. This is the first assertion. Hence the lemma has been proved.

Let S be the normal subgroup of G generated by the element s and denote by $I(S)$ the augmentation ideal of the group ring ZS. Then the natural epimorphism $G \to G/S$ induces the exact sequence

$$0 \longrightarrow I(S)ZG \xrightarrow{\ i\ } ZG \xrightarrow{\ f\ } Z(G/S) \longrightarrow 0\,. \tag{5}$$

Lemma 2. *Set* $I = I(S)ZG$, *then* $\mathrm{K}_1(ZG/I^2, I/I^2)$ *has exponent* p.

Proof. Since in the ring ZG/I^2 the ideal I/I^2 has square zero, it is contained in the radical of ZG/I^2. Then $K_1(ZG/I^2, I/I^2)$ is a quotient of $GL_1(ZG/I^2, I/I^2)=1+I/I^2$ ([2], Chapter IX, (1.3)), and the later is isomorphic to the additive group I/I^2.

On the other hand, the additive group $I(S)/I^2(S)$ is isomorphic to the multiplicative group S. But S is of order p, so $pI(S)\subseteq I^2(S)$. Hence I/I^2 has exponent p. Therefore, $K_1(ZG/I^2, I/I^2)$ has also exponent p, which proves the lemma.

Lemma 3 (Lam [6]). *G has the Artin exponent* 2.

Proof. G is of order $2p$, and the p-Sylow subgroup S is cyclic. Then the lemma is immediate from [6].

3. Now, we shall compute the reduced norm of Tor $K_1(ZG)$.

Proposition 4. Nrd (Tor $K_1(ZG)$)$=$Nrd ($\pm G$).

Proof. Set $I=I(S)ZG$, and consider the exact sequence

$$K_1(ZG, I) \xrightarrow{i_*} K_1(ZG) \xrightarrow{f_*} K_1(Z(G/S))$$

induced from the exact sequence (5). Since the quotient group G/S is cyclic of order 2, it is known that $K_1(Z(G/S))=\pm(G/S)$ (Higman [4], c.f. [2]). Then for any $x\in$ Tor $K_1(ZG)$, there exist an element y of $K_1(ZG, I)$ and an element $\pm g$ of $\pm G$ such that

$$x = i_*(y)+(\pm g)\,. \tag{6}$$

Since x is a torsion element, then so $i_*(y)$ is. Hence by Lemma 3 and the induction theorem ([5]), we can see that $\mathrm{Nrd}(i_*(y))$ has exponent 4. Set $\mathrm{Nrd}(i_*(y))=(y_1, y_2, y_3)\in U(Z)\times U(Z)\times U(R_0)$. Then we get the equality $y_3^4=1$ in R_0. But R_0 is included in the real numbers, so y_3 must be equal to ± 1. Thus each component y_i is of the form ± 1, which implies the equality (p is odd !)

$$\mathrm{Nrd}(i_*(y)) = \mathrm{Nrd}(pi_*(y))\,. \tag{7}$$

On the other hand, we have the exact sequence (c.f. [2], Chapter IX, (1.2))

$$K_1(ZG, I^2) \xrightarrow{h_*} K_1(ZG, I) \longrightarrow K_1(ZG/I^2, I/I^2)\,.$$

Since by Lemma 2 $K_1(ZG/I^2, I/I^2)$ has exponent p, then there exists an element z of $K_1(ZG, I^2)$ such that $h_*(z)=py$. Then by the eqeality (7), to determine Nrd $(i_*(y))$, it suffices to do $\mathrm{Nrd}(i_*\circ h_*(z))$.

Any element z of $K_1(ZG, I^2)$ is represented by a matrix Z_n of $GL_n(ZG, I^2)$

for some n. Since S is generated by the element s, $I^2=I(S)^2ZG$ coincides with the ideal $(s-1)^2ZG$, which is projected by (j_1, j_2, j_3) into the ideal $0\oplus 0\oplus(\zeta-1)^2\mathfrak{o}$ of the maximal order $\mathfrak{O}=Z\oplus Z\oplus\mathfrak{o}$. Here, $\phi((\zeta-1)^2\mathfrak{o})=M_2((2-\vartheta)R_0)$ by Lemma 1. Therefore, by the formula (3) of the reduced norm, the first and the second components of $\mathrm{N}_n(Z_n)$ are both equal to 1, and the third component is congruent to 1 modulo $(2-\vartheta)R_0$. Thus, $\mathrm{Nrd}(i_*(y))=(1, 1, y_3)$, and $y_3\equiv 1 \mod (2-\vartheta)R_0$. But the ideal $(2-\vartheta)R_0$ does not divide 2, and y_3 is of the form ± 1, so that y_3 must be equal to 1. Hence, $\mathrm{Nrd}\,(i_*(y))=(1, 1, 1)$. Consequently, by the equality (6) we get the equalities $\mathrm{Nrd}\,(x)=\mathrm{Nrd}\,(i_*(y))\,\mathrm{Nrd}\,(\pm g)=\mathrm{Nrd}\,(\pm g)$. This shows the proposition.

4. Finally, we shall study the kernel of the reduced norm.

Let $\mathfrak{c}=(\zeta-1)^2\mathfrak{o}$ be the ideal of $\mathfrak{o}$ as in Lemma 1. Then, by the lemma, $\mathfrak{c}$ may be regarded as a two-sided ideal of Λ, so that we can consider the group $\mathrm{SK}_1(\Lambda, \mathfrak{c}^2)$.

Lemma 5 (c.f. [5]). $\mathrm{SK}_1(\Lambda, \mathfrak{c}^2)$ *is a p-group.*

Proof. For a sufficiently large integer n, set $T=SL_n(\Lambda, \mathfrak{c}^2)$. Since $\mathrm{SK}_1(\Lambda, \mathfrak{c}^2)$ is a quotient of $T/[T, T]$, it suffices to see that the latter is a p-group. But $\mathfrak{c}^2$ is an $\mathfrak{o}$-ideal, then $SL_n(\Lambda, \mathfrak{c}^2)=SL_n(\mathfrak{o}, \mathfrak{c}^2)$, and this is identified with $SL_{2n}(R_0, (2-\vartheta)^2R_0)$ by Lemma 1. Therefore, we may assume that $T=SL_{2n}(R_0, (2-\vartheta)^2R_0)$. Now, we see easily that $[T, T]\subseteq SL_{2n}(R_0, (2-\vartheta)^4R_0)$ and by the formula of ([2], Chapter V, (1.5)) we see that $[E_{2n}(R_0, (2-\vartheta)^2R_0), E_{2n}(R_0, (2-\vartheta)^2R_0)] \supseteq E_{2n}(R_0, (2-\vartheta)^4R_0)$, so

$$SL_{2n}(R_0, (2-\vartheta)^4R_0)\supseteq[T, T]\supseteq E_{2n}(R_0, (2-\vartheta)^4R_0)\,.$$

By the "congruence subgroup theorem" for R_0, the integers in the real field L_0 (Bass, Milnor and Serre [3], c.f. [2]), the extreme ends are equal. Thus we get the equality

$$T/[T, T]=SL_{2n}(R_0, (2-\vartheta)^2R_0)/SL_{2n}(R_0, (2-\vartheta)^4R_0)\,.$$

Moreover, this is isomorphic to a subgroup of the group $U=GL_{2n}(R_0/(2-\vartheta)^4R_0, (2-\vartheta)^2R_0/(2-\vartheta)^4R_0)$. Since the ideal $\bar{\mathfrak{c}}=(2-\vartheta)^2R_0/(2-\vartheta)^4R_0$ has square zero, U is isomorphic to the additive group $M_{2n}(\bar{\mathfrak{c}})$ consisting of matrices with entries from $\bar{\mathfrak{c}}$. But p^2 is divisible by $(2-\vartheta)^2$, so $M_{2n}(\bar{\mathfrak{c}})$ has exponent p^2. Therefore, $T/[T, T]$ has also exponent p^2. This shows the lemma.

Corollary. $\mathrm{SK}_1(\Lambda, (\zeta-1)^2\Lambda)$ *is a p-group.*

Proof. Since $\mathfrak{c}=(\zeta-1)^2\mathfrak{o}\subseteq\Lambda$, we see that $\mathfrak{c}^2\subseteq(\zeta-1)^2\Lambda\subseteq\mathfrak{c}$. Set $\mathfrak{c}_0=(\zeta-1)^2\Lambda$. Then we have the exact sequence

$$\mathrm{SK}_1(\Lambda, \mathfrak{c}^2)\longrightarrow\mathrm{SK}_1(\Lambda, \mathfrak{c}_0)\longrightarrow\mathrm{K}_1(\Lambda/\mathfrak{c}^2, \mathfrak{c}_0/\mathfrak{c}^2)\,,$$

where $\mathfrak{c}_0/\mathfrak{c}^2$ has square zero, and it has exponent p (p is divisible by $(\zeta-1)^2$). Then, we can repeat the same arguments as in the proof of Lemma 2 to conclude that $\mathrm{K}_1(\Lambda/\mathfrak{c}^2, \mathfrak{c}_0/\mathfrak{c}^2)$ has exponent p. Therefore, by our lemma $\mathrm{SK}_1(\Lambda, \mathfrak{c}_0)$ is a p-group.

Lemma 6. $\mathrm{SK}_1(ZG, I)$ *is a p-group, where* $I=I(S)ZG$.

Proof. Look at the exact sequence

$$\mathrm{SK}_1(ZG, I^2) \longrightarrow \mathrm{SK}_1(ZG, I) \longrightarrow \mathrm{K}_1(ZG/I^2, I/I^2)\,,$$

where $\mathrm{K}_1(ZG/I^2, I/I^2)$ has exponent p by Lemma 2. Then it suffices to see that $\mathrm{SK}_1(ZG, I^2)$ is a p-group.

Set $J=(1+s+\cdots+s^{p-1})ZG$. Then J is the kernel of the projection j_3; $ZG\to\Lambda$, and $I^2\cap J=0$. Therefore, by the "excision isomorphism theorem" on K_1 ([2], Chapter IX, (1.5)), we can see that the natural homomorphism

$$\mathrm{K}_1(ZG, I^2) \longrightarrow \mathrm{K}_1(ZG/J, I^2+J/J) = \mathrm{K}_1(\Lambda, (\zeta-1)^2\Lambda) \tag{8}$$

is an isomorphism, where the inverse map is obtained by taking the inverse images modulo J. However, we have seen in the proof of Proposition 4 that the first and the second components of the image of $\mathrm{K}_1(ZG, I^2)$ by the reduced norm are both equal to 1. Then we have the commutative diagram

$$\begin{tikzcd}
\mathrm{K}_1(ZG, I^2) \arrow[r, leftrightarrow, "\sim"] \arrow[d, "\mathrm{Nrd}"] & \mathrm{K}_1(\Lambda, (\zeta-1)^2\Lambda) \arrow[d, "\mathrm{Nrd}"] \\
\{1\}\times\{1\}\times U(R_0) \arrow[r, leftrightarrow, "\sim"] & U(R_0)\,.
\end{tikzcd}$$

Hence, the homomorphism $\mathrm{SK}_1(ZG, I^2)\to\mathrm{SK}_1(\Lambda, (\zeta-1)^2\Lambda)$ induced from the isomorphism (8) is also an isomorphism. Since $\mathrm{SK}_1(\Lambda, (\zeta-1)^2\Lambda)$ is a p-group by the above corollary, then $\mathrm{SK}_1(ZG, I^2)$ is a p-group. This proves the lemma.

Proposition 7. $\mathrm{SK}_1(ZG)=0$.

Proof. The exact sequence (5) induces the following exact sequence

$$\mathrm{SK}_1(ZG, I) \xrightarrow{i_*} \mathrm{SK}_1(ZG) \xrightarrow{f_*} \mathrm{SK}_1(Z(G/S))\,.$$

Since $\mathrm{SK}_1(Z(G/S))=0$ (recall that $\mathrm{K}_1(Z(G/S))=\pm(G/S)$), i_* is surjective. By the preceding lemma, $\mathrm{SK}_1(ZG, I)$ is a p-group, so that $\mathrm{SK}_1(ZG)$ is also a p-group. On the other hand, we apply the induction theorem to $\mathrm{SK}_1(ZG)$. Then by Lemma 3, $\mathrm{SK}_1(ZG)$ has exponent 4. Hence, $\mathrm{SK}_1(ZG)=0$, which proves the proposition.

This proposition shows that the reduced norm Nrd; $\mathrm{K}_1(ZG)\to U(Z)\times U(Z)\times U(R_0)$ is an injection, and in Proposition 4 it has been seen that

$\mathrm{Nrd}(\mathrm{Tor}\ K_1(ZG)) = \mathrm{Nrd}(\pm G)$. Hence, we obtain that $\mathrm{Tor}\ \mathrm{Wh}(G) = \mathrm{Tor}\ K_1(ZG)/\pm G = 0$, and complete the proof of the theorem.

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