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THE REPRESENTATION RINGS OF ORTHOGONAL GROUPS

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1. In this work the author presents the representation rings of orthogonal groups explicitely. As $O(2n+1)$ is a direct product of $SO(2n+1)$ and Z_2, $R(O(2n+1))$ is isomorphic to $R(SO(2n+1))\otimes R(Z_2)$, where $R(G)$ is the complex representation ring of a compact Lie group G. But the situation about $O(2n)$ is somewhat different. We prove the following

Theorem 1. *$R(O(2n))\cong Z[\lambda^1\bar{\rho}_{2n}, \cdots, \lambda^n\bar{\rho}_{2n}, \bar{\eta}_{2n}]$ with two generating relations $\lambda^n\bar{\rho}_{2n}\bar{\eta}_{2n}=\lambda^n\bar{\rho}_{2n}$ and $\bar{\eta}_{2n}^2=1$, where $\lambda^k\bar{\rho}_{2n}$ denotes the k-th exterior power of the standard representation $\bar{\rho}_{2n}$: $O(2n)\to U(2n)$ for $k=1, 2, \cdots, n$ and $\bar{\eta}_{2n}$ the determinant representation which is of 1-dimension.*

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2. Let us summarize here Segal's results in §1, 2 of [3] and Clifford's theorem [1].

Let G be a compact Lie group in this section. Segal introduced subgroups of G such that they are cyclic and of finite index in their normalizers, which are called Cartan subgroups, and proved that the number of the conjugacy classes of these subgroups is finite and that

(2. 1) *The restriction $R(G)\to\sum_S R(S)$ is injective, where S runs through the representatives of conjugacy classes of Cartan subgroups of G* ([3], p. 121).

Let $i: H\to G$ be the inclusion of a closed subgroup of G. When we regard $R(H)$ as an $R(G)$-module by the restriction i^*: $R(G)\to R(H)$, we can define a homomorphism i_*: $R(H)\to R(G)$ of $R(G)$-modules, which is called the induced representation homomorphism. In particular, we have

(2. 2) $$i_*(1)=\sum_{k\geq 0}(-1)^k[H^k(G/H, \boldsymbol{C})]$$

where $H^k(G/H, \mathbf{C})$ is the k-th cohomology group with complex coefficients of the manifold G/H and becomes a G-module by the induced actions of left actions of G on G/H for each $k \geq 0$, and we denote their isomorphism classes by the brackets ([3], p. 119).

Let N be a normal closed subgroup of G and i: $N \to G$ the inclusion of N. By $\mathfrak{D}(G)$ and $\mathfrak{D}(N)$ we denote the families of the equivalence classes of the irreducible representations of G and N repsectively. If G/N is isomorphic to Z_2, then we can state Clifford's theorem as follows.

(2. 3) *For any element ρ of $\mathfrak{D}(G)$, we have*

(i) *If $\rho \neq \rho\bar{\eta}$, then $i^*\rho \in \mathfrak{D}(N)$, $i^*\rho = C_\varepsilon(i^*\rho)$ and $i_*i^*(\rho) = \rho + \rho\bar{\eta}$,*

(ii) *If $\rho = \rho\bar{\eta}$, there exists an element σ of $\mathfrak{D}(N)$ satisfying that $i^*\rho = \sigma + C_\varepsilon(\sigma)$, $\sigma \neq C_\varepsilon(\sigma)$ and $i_*\sigma = \rho$, and conversely for any element σ of $\mathfrak{D}(N)$*

(iii) *If $\sigma \neq C_\varepsilon(\sigma)$, then $i_*\sigma \in \mathfrak{D}(G)$, $i_*\sigma = \bar{\eta} i_*\sigma$ and $i^*i_*(\sigma) = \sigma + C_\varepsilon(\sigma)$,*

(iv) *If $\sigma = C_\varepsilon(\sigma)$, there exists an element ρ of $\mathfrak{D}(G)$ satisfying that $i_*\sigma = \rho + \rho\bar{\eta}$, $\rho \neq \rho\bar{\eta}$ and $i^*\rho = \sigma$, where $\bar{\eta}$ is the 1-dimensional representation of G defined by the composition of the canonical 1-dimensional non-trivial representation of G/N and the natural projection from G to G/N, ε is an element of G such that εN generates G/N and $C_\varepsilon(\sigma)$ is the representation of N defined by acting $\varepsilon^{-1}h\varepsilon$ on the representation space of σ in stead of each element h of N.*

3. We put

$$\varepsilon_{2n} = \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \in O(2n) .$$

By [3], Prop. (1.5) and Proof of Prop. (1.2), we see that $O(2n)$ has only two Cartan subgroups

$$TSO(2n) \text{ and } TSO(2n-2) \times Z_2$$

up to conjugacy, where $TSO(m)$ denotes the standard maximal tours of $SO(m)$ and Z_2 the group generated by ε_{2n}.

Next, let i: $SO(2n) \to O(2n)$ be the inclusion map. Then we obtain

$$(3.1) \qquad i_*(1) = 1 + \bar{\eta}_{2n}$$

from (2.2) since we can regard $i_*(1)$ as a group algebra of Z_2 over $\mathbf{C}$.

By $R(SO(2n))^{Z_2}$ we denote a subalgebra of $R(SO(2n))$ consisting of invariant elements under the conjugacy $C_{\varepsilon_{2n}}$ by ε_{2n}. Using the notation of [2] we get the following lemma from Proof of Theorem 10.3 of [2], §13.

Lemma 1. $R(SO(2n))^{Z_2} \cong Z[\lambda^1\rho_{2n}, \cdots, \lambda^n\rho_{2n}]$.

Put

$$R_n = Z[\lambda^1\bar{\rho}_{2n}, \cdots, \lambda^n\bar{\rho}_{2n}, \bar{\eta}_{2n}]/(\lambda^n\bar{\rho}_{2n}(\bar{\eta}_{2n}-1), \bar{\eta}_{2n}^2-1).$$

We prove first the case $n=1$ of Theorem 1.

Lemma 2. $R(O(2)) \cong R_1$.

Proof. The relation $\bar{\eta}_2^2=1$ is easily seen from the definition of $\bar{\eta}_2$, and by means of the injection (2.1) of $R(O(2))$ into $R(SO(2)) \oplus R(Z_2)$ we see the relation $\bar{\rho}_2\bar{\eta}_2=\bar{\rho}_2$ and that R_1 is a subalgebra of $R(O(2))$.

When we restrict any irreducible representation of $O(2)$ to $SO(2)$, (2.3) shows that we have only two types. Let ρ be an irreducible representation of $O(2)$. When ρ is in the case of (2.3), (i), $i^*\rho$ is an irreducible representation of $SO(2)$, then it is of 1-dimension since $SO(2)$ is abelian and also invariant under the conjugcay C_{ε_2}. Therefore we get $i^*\rho=1$ and so $1+\bar{\eta}_2=\rho+\rho\bar{\eta}_2$ from (3.1). This shows that ρ is either $\bar{\eta}_2$ or 1. When ρ is in the case of (2.3), (ii), we have an irreducible representation σ of $SO(2)$ satisfying $i_*\sigma=\rho$. So if we show that every induced representation of $SO(2)$ is contained in R_1, then we will finish the proof.

Let α be the canonical 1-dimensional non-trivial representation of $SO(2)$. Then we have

$$i^*i_*(\alpha^{\pm m}) = \alpha^m+\alpha^{-m}$$

by (2.3), (iii) and

$$i^*(\bar{\rho}_2) = \alpha+\alpha^{-1}$$

and from these formulae and (2.1), we can deduce the identities

$$\bar{\rho}_2^{2m} = \textstyle\sum_{k=0}^{m-1} \binom{2m}{k} i_*(\alpha^{\pm(2m-2k)})+\binom{2m-1}{m}(1+\bar{\eta}_2),$$
$$\bar{\eta}_2^{2m+1} = \textstyle\sum_{k=0}^{m} \binom{2m+1}{k} i_*(\alpha^{\pm(2m+1-2k)}).$$

Consequently we see inductively that $i_*(\alpha^{\pm m})$ is contained in R_1 for all $m\geq 0$. q. e. d.

Lemma 3. *R_n is a subalgebra of $R(O(2n))$ for $n\geq 2$.*

Proof. Consider the inclusion map j: $SO(2n-2)\times O(2)\rightarrow O(2n)$. j^*: $R(O(2n))\rightarrow R(SO(2n-2)\times O(2))$, for $n\geq 2$, is injective by (2.1) since $O(2n)$ and $SO(2n-2)\times O(2)$ have the same Cartan subgroups $TSO(2n)$ and

$TSO(2n-2)\times Z_2$. Now it follows from Lemma 1 and Lemma 2 that the image of j^* is contained in the subalgebra R described by

$$R = Z[\lambda^1\rho_{2n-2}, \cdots, \lambda^{n-1}\rho_{2n-2}, \bar{\rho}_2, \bar{\eta}_2]/(\bar{\rho}_2(\bar{\eta}_2-1), \bar{\eta}_2^2-1).$$

Put $x_h=\lambda^h\bar{\rho}_{2n}$, $x'_h=i^*(\lambda^h\bar{\rho}_{2n})$, $1\leq h\leq n$, $y_s=\lambda^s\rho_{2n-2}$, $1\leq s\leq n-1$ and $z=\bar{\rho}_2$. Then we have

$$\begin{aligned} &j^*(\bar{\eta}_{2n}) = \bar{\eta}_2 \\ &j^*(x_1) = y_1+z \\ (3.2)\qquad &j^*(x_k) = y_k+y_{k-1}z+y_{k-2}\bar{\eta}_2, \ k=2, 3, \cdots, n-1 \\ &j^*(x_n) = y_n+y_{n-1}z+y_{n-2}\bar{\eta}_2 = y_{n-2}(\bar{\eta}_2+1)+y_{n-1}z \end{aligned}$$

since $\lambda^2z=\bar{\eta}_2$ and $y_m=y_{2n-2-m}$ for $m=0, 1, \cdots, 2n-2$. We know the relation $\lambda^n\bar{\rho}_{2n}\bar{\eta}_{2n}=\lambda^n\bar{\rho}_{2n}$ from the last formula of (3.2). Since the relation $\bar{\eta}_{2n}^2=1$ is easily seen, what we have to show is that no other generating relation exists.

Any element ξ of R_n can be written as follows:

$$\xi = f+g\bar{\eta}_{2n}+x_nh$$

where $f, g\in Z[x_1, \cdots, x_{n-1}]$ and $h\in Z[x_1, \cdots, x_{n-1}, x_n]$. Let us denote the restrictions of f, g and h to $SO(2n)$ by f', g' and h' respectively. If $\xi=0$, then $f'+g'+x'_nh'=0$ and so $f'+g'=0$, $x'_nh'=0$ by Lemma 1. We have then $f+g=0$ and $x_nh=0$. Thus we conclude that

$$f(\bar{\eta}_{2n}-1) = 0.$$

Next we prove by induction on k that if $f(\bar{\eta}_{2n}-1)=0$ for $f\in Z[x_1, \cdots, x_k]$, $1\leqq k\leqq n-1$, then $f=0$. In case of $k=1$, when we put $f(x_1)=\sum_{i\geqq 0}a_ix_1^i$,

$$\textstyle\sum_{i\geqq 0}a_iy_1^i(\bar{\eta}_2-1) = 0$$

follows from (3.2) and $z\bar{\eta}_2=z$. Since there exists no relation between y_1 and $\bar{\eta}_2$ in R, $a_i=0$ for all $i\geqq 0$ and therefore $f=0$. Suppose that the assertion is as stated for $k<l$. In case of $k=l$, put $f=\sum_{i=0}^N f_ix_l^i$ where $f_i\in Z[x_1, \cdots, x_{l-1}]$, $0\leqq i\leqq N$, then we get

$$f_N(\bar{\eta}_{2n}-1) = 0$$

by considering the image of $f(\bar{\eta}_{2n}-1)=0$ by j^* and comparing the coefficient of y_l^N, because $j^*(x_m)=y_m+y_{m-1}z+y_{m-2}\bar{\eta}_2$, $2\leqq m\leqq n-1$, by (3.2). Hence $f_N=0$ follows from the inductive hypothesis. Similarly $f_i=0$ is proved successively for $i=0, 1, \cdots, N-1$ and so $f=0$. This completes the induction. q.e.d.

By $\rho(m, m-1)$ we denote the inclusion map $O(m-1)\to O(m)$.

Lemma 4. $\rho(2n, 2n-1)^*(R_n)=R(O(2n-1))$.

Proof. Since $O(2n-1)$ is isomorphic to $SO(2n-1)\times Z_2$ where Z_2 is generated by

$$\begin{bmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{bmatrix} \in O(2n-1)\,.$$

we have

$$R(O(2n-1))\cong Z[\lambda^1\bar{\rho}_{2n-1}, \cdots, \lambda^{n-1}\bar{\rho}_{2n-1}, \bar{\eta}_{2n-1}]/(\bar{\eta}^2_{2n-1}-1)$$

by [2], §13, Theorem 10.3 and moreover

$$\begin{aligned} &\rho(2\mathrm{n}, 2n-1)^*(\bar{\eta}_{2n}) = \bar{\eta}_{2n-1} \\ (3.3)\qquad &\rho(2n, 2n-1)^*(\lambda^k\bar{\rho}_{2n}) = \lambda^k\bar{\rho}_{2n-1}+\lambda^{k-1}\bar{\rho}_{2n-1},\ 1\leq k\leq n-1 \\ &\rho(2n, 2n-1)^*(\lambda^n\bar{\rho}_{2n}) = \lambda^n\bar{\rho}_{2n-1}+\lambda^{n-1}\bar{\rho}_{2n-1} \\ &\qquad = \lambda^{n-1}\bar{\rho}_{2n-1}(\bar{\eta}_{2n-1}+1) \end{aligned}$$

where the last formula follows from $\lambda^k\bar{\rho}_{2,-1}=\lambda^k\rho_{2n-1}\bar{\eta}^k{}_{2n-1}$ and $\lambda^k\rho_{2n-1}=\lambda^{2n-1-k}\rho_{2n-1}$, $0\leq k\leq 2n-1$. Clearly (3.3) shows Lemma 4.

4. Proof of Theorem 1

From (2.2) and $O(2n)/O(2n-1)\approx S^{2n-1}$ as $O(2n)$-manifold we have

$$\rho(2n, 2n-1)_*(1) = [H^0(S^{2n-1}, \boldsymbol{C})]-[H^{2n-1}(S^{2n-1}, \boldsymbol{C})]\,.$$

Then $[H^0(S^{2n-1}, \boldsymbol{C})]=1$ obviously and $[H^{2n-1}(S^{2n-1}, \boldsymbol{C})]=\bar{\eta}_{2n}$ since the actions by the elements of $SO(2n)$ on $H^{2n-1}(S^{2n-1}, \boldsymbol{C})$ are trivial and ε_{2n} reverses the orieintation of the manifold S^{2n-1}. Therefore we obtain

$$(4.1)\qquad \rho(2n, 2n-1)_*(1) = 1-\bar{\eta}_{2n}\,.$$

It follows from (3,1) and (4.1) that

$$(4.2)\qquad \begin{aligned} &i_*i^*(\xi) = (1+\bar{\eta}_{2n})\xi \\ &\rho(2n, 2n-1)_*\rho(2n, 2n-1)^*(\xi) = (1-\bar{\eta}_{2n})\xi \end{aligned}$$

for any $\xi\in R(O(2n))$. When we restrict i^* and $\rho(2n, 2n-1)^*$ to R_n, we see that $i^*: R_n\to R(SO(2n))^{Z_2}$ and $\rho(2n, 2n-1)^*: R_n\to R(O(2n-1))$ are surjective by Lemma 1 and Lemma 4. Therefore there exist elements ξ_k, $k=1, 2$, of R_n satisfying

$$i^*(\xi_1) = i^*(\xi) \text{ and } \rho(2n, 2n-1)^*(\xi_2) = \rho(2n, 2n-1)^*\,(\xi)$$

and then we have

$$i_*i^*(\xi) = (1+\bar{\eta}_{2n})\xi_1$$

and $$\rho(2n, 2n-1)_*\rho(2n, 2n-1)^*(\xi)=(1-\bar{\eta}_{2n})\xi_2 .$$

Since $(1+\bar{\eta}_{2n})\xi_1$ and $(1-\bar{\eta}_{2n})\xi_2$ are elements of R_n, it follows from (4.2) that 2ξ is contained in R_n. Here, using the following lemma, it is proved that ξ is contained inR_n. This completes the proof of Theorem 1.

Lemma 5. *The j^*-image of R_n into R is a direct summand.*

Proof. From (3.2) it follows that

$$y_k = (-1)^k z^k + R(k),\ 1 \le k \le n-1$$

and $$j^*(x_n) = (-1)^{n-1} z^n + R(n)$$

where $R(k)$ is the linear combination of $j^*(\bar{\eta}_{2n})$ and z^h, $0 \le h \le k-1$, with the polynomials of $j^*(x_s)$, $1 \le s \le k$, as coefficients for $k=1, 2, \cdots, n$. This implies

(4. 3) $$R = R_n \cdot 1 + R_, \cdot z + \cdots + R_n \cdot z^{n-1}$$

when we regard R as an R_n-module by j^*. That is, any $\xi' \in R$ can be represented as

$$\xi' = \sum_{l=0}^{n-1} f_l z^l ,\quad f_l \in R_n \text{ for all } l .$$

Then we prove that if $\xi'=0$, then $f_0=0$.

Put

$$R(TSO(2n)) = Z[\alpha_1, \cdots, \alpha_n, \alpha_1^{-1}, \cdots, \alpha_n^{-1}]/(\alpha_1\alpha_1^{-1}-1, \cdots, \alpha_n\alpha_n^{-1}-1)$$

and $\beta_k=\alpha_k+\alpha_k^{-1}$ where α_k, $1 \le k \le n$, are the canonical 1-dimensional non-trivial representations of $TSO(2n)$. Suppose $\xi'=0$, then we have

(4. 4) $$\sum_{l=1}^{n-1} f'_l \beta_n^l = -f'_0$$

since the restriction of z to $TSO(2n)$ is β_n, where f'_l, $0 \le l \le n-1$, denote the restriction of f_l to $TSO(2n)$ respectively. Since f'_0 is contained in the image of the injection $R(SO(2n)) \to R(TSO(2n))$, it is invariant under the actions by the elements of the Weyl group of $SO(2n)$. Therefore we get

(4. 5) $$\sum_{l=1}^{n-1} f'_l \beta_1^l = \cdots = \sum_{l=1}^{n-1} f'_l \beta_l^n = -f'_0$$

Here we put

$$R_{l,\,l+1}^k = \sum_{i=0}^k \beta^i \beta_{l+1}^{k-i} \text{ for } 1 \le l \le n-1,\ k \ge 0$$

and $$R_{l,\, l+1, \dots,\, l+s\,:\, l+s+1,\, l+s+2}^k$$

$$= \sum_{i_1=0}^k \sum_{i_2=0}^{k-i_1} \cdots \sum_{i_{s+2}=0}^{k-(i_1+\cdots+i_{s+1})} \beta_l^{i_1} \cdots \beta_{l+s+1}^{i_{s+2}} \beta_{l+s+2}^{k-(i_1+\cdots+i_{s+2})}$$

for $1\leq l\leq n-s-2$, $k\geq 0$. Then we have the following equalities

$$
(4.6)\quad
\begin{aligned}
&\beta_l^k-\beta_{l+1}^k=(\beta_l-\beta_{l+1})R_{l,l+1}^{k-1}\\
&R_{l,l+1}^k-R_{l,l+2}^k=(\beta_{l+1}-\beta_{l+2})R_{l\,:\,l+1,\,l+2}^{k-1}\\
&R_{l,\,l+1,\,\dots,\,l+s\,:\,l+s+1,\,l+s+2}^k-R_{l,\,l+1,\,\dots,\,l+s\,:\,l+s+1,\,l+s+3}^k\\
&=(\beta_{l+s+2}-\beta_{l+s+3})R_{l,\,l+1,\,\dots,\,l+s+1\,:\,l+s+2,\,l+s+3}^{k-1}
\end{aligned}
$$

From (4.5) and (4.6), we get

$$
(4.7)\quad
\begin{aligned}
&\text{(i) } f_k'+\textstyle\sum_{i=k+1}^{n-1}f_i'R_{1,\,2,\,\dots,\,k-1\,:\,k,\,k+1}^{i-k}=0\\
&\text{(ii) } f_k'+\textstyle\sum_{i=k+1}^{n-1}f_i'R_{1,\,2,\,\dots,\,k-1\,:\,k,\,k+2}^{i-k}=0
\end{aligned}
$$

for $k=1, 2, \cdots, n-2$. We prove (4.7) by induction on k. In case of $k=1$, we get

$$(\beta_1-\beta_2)(f_1'+\textstyle\sum_{i=2}^{n-1}f_i'R_{1,2}^{i-1})=0 \text{ and } (\beta_2-\beta_3)(f_1'+\textstyle\sum_{i=2}^{n-1}f_i'R_{1,3}^{i-1})=0$$

by (4.5) and also since f_i' is contained in a subalgebra $Z[\beta_1, \cdots, \beta_n]$ of $R(TSO(2n))$ for each i and $Z[\beta_1, \cdots, \beta_n]$ has no zero divisor, we can devide by $\beta_1-\beta_2$ and $\beta_2-\beta_3$ respectively. So

$$f_1'+\textstyle\sum_{i=2}^{n-1}f_i'R_{1,2}^{i-1}=0 \text{ and } f_1'+\textstyle\sum_{i=2}^{n-1}f_i'R_{1,3}^{i-1}=0\,.$$

Suppose that the assertion is as (4.7) for $k<j$. In case of $k=j$, by subtracting (ii) from (i) and dividing by
$\beta_{j+1}-\beta_{j+2}$ we get

$$f_{j+1}'+\textstyle\sum_{i=j+2}^{n-1}f_i'R_{1,\,2,\,\dots,\,j\,:\,j+1,\,j+2}^{i-j-1}=0\,.$$

Analogously, we can deduce

$$f_{j+1}'+\textstyle\sum_{i=j+2}^{n-1}f_i'R_{1,\,2,\,\dots,\,j\,:\,j+1,\,j+3}^{i-j-1}=0\,.$$

In particular, when we put $k=n-2$ in (4.7) and subtract (ii) from (i), we have

$$f_{n-1}'(R_{1,\,2,\,\dots,\,n-3\,:\,n-2,\,n-1}^1-R_{1,\,2,\,\dots,\,n-3\,:\,n-2,\,n}^1)=(\beta_{n-1}-\beta_n)f_{n-1}'=0\,.$$

Thus we obtain

$$f_{n-1}'=0$$

and also successively

$$f_{n-2}'=\cdots=f_0'=0$$

using (4.7). This implies that the restriction of f_0 to $TSO(2n)$ vanishes. Furthermore the character of $\sum_{l=1}^{n-1}f_lz^l$ at any generator g of $TSO(2n-2)\times Z_2$ is zero because the character of z at g is zero and hence the restriction of f_0 to

$TSO(2n-2)\times Z_2$ vanishes. Consequently we see

$$f_0=-\sum_{i=1}^{n-1}f_i z^i=0$$

by (2.1). This shows in (4.3) that R is a direct sum of $R_n\cdot 1$ and $R_r\cdot z+R_n\cdot z^2+\cdots+R_n\cdot^{n-1}$ and therefore completes the proof of Lemma 5.

5. Finally we prove the following corollaries.

Corollary 1. *$R(O(m))\cong RO(O(m))$ for any $m\geq 1$.*

Proof. This follows from the facts that the generators of $R(O(m))$ are all complexifications of some real representations and the complexification c: $RO(O(m))\to R(O(m))$ is injective. q. e. d.

Corollary 2. *We have*

(i) *$\rho(m, m-1)^*$ is surjective for any $m\geq 0$*

(ii) *$Ker\ \rho(2n, 2n-1)^*=(\lambda^n\bar{\rho}_{2n}-(1+\bar{\eta}_{2n})\sum_{i=0}^{n-1}(-1)^i\lambda^{n-1-i}\bar{\rho}_{2n})$*

$Ker\ \rho(2n+1, 2n)^*=((\bar{\eta}_{2n+1}-1)\sum_{i=0}^{n}(-1)^i\lambda^{n-i}\bar{\rho}_{2n+1})$.

Proof. We have

$$\rho(2n+1, 2n)^*(\bar{\eta}_{2r+1})=\bar{\eta}_{2n}\ ,$$
$$\rho(2n+1, 2n)^*(\lambda^k\bar{\rho}_{2n+1})=\lambda^k\bar{\rho}_{2n}+\lambda^{k-1}\bar{\rho}_{2n},\ 1\leq k\leq n\ .$$

From these and (3.3), we see (i) easily and also

$$\lambda^k\bar{\rho}_{2n-1}=\sum_{i=0}^{k}(-1)^i\rho(2n.\ 2n-1)^*\ \lambda^{k-i}\bar{\rho}_{2n},\ 1\leq k\leq n-1\ ,$$
$$\lambda^k\bar{\rho}_2\ =\sum_{i=0}^{k}(-1)^i\rho(2n+1,\ 2n)^*\ \lambda^{k-i}\bar{\rho}_{2n+1},\ 1\leq k\leq<n\ .$$

Using these formulae, we obtain (ii). q. e. d.

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