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COBORDISM GROUPS OF IMMERSIONS

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1. Introduction

In the previous paper [3] we have introduced the cobordism groups $\boldsymbol{I}(n, k)$, $\boldsymbol{E}(n, k)$ and $\boldsymbol{G}(n, k)$ of immersions, embeddings and generic immersions of n-manifolds into $(n+k)$-manifolds respectively. Mainly we have considered the cobordism group $\boldsymbol{G}(n, k)$, and we have obtained the following exact sequence:

$$0 \to \boldsymbol{E}(n, k) \to \boldsymbol{G}(n, k) \to \boldsymbol{B}(n-k, k) \to 0$$

where the group $\boldsymbol{B}(n, k)$ is the cobordism group of bundles over manifolds with involution defined in [3]

In general $\boldsymbol{E}(n, k)$ is isomorphic to the bordism group $\mathfrak{N}_{n+k}(MO(k))$, so we have studied the cobordism group $\boldsymbol{B}(n, k)$ in [3], [4], in order to study the cobordism group $\boldsymbol{I}(n, k)$, since $\boldsymbol{I}(n, k)$ is canonically isomorphic to $\boldsymbol{G}(n, k)$ in the meta-stable range (i.e. $2k > n+1$). Especially, as one of the consequence of the previous paper, the forgetting homomorphism

$$\alpha_* : \boldsymbol{E}(n, k) \to \boldsymbol{I}(n, k)$$

is injective in the meta-stable range.

One of the results of the present paper is that the homomorphism

$$\alpha_* : \boldsymbol{E}(n, k) \to \boldsymbol{I}(n, k)$$

is injective without restriction of the meta-stable range.

First we will introduce the notion "completely regular (p)-immersion" and study some properties of completely regular (p)-immersions in sections 2, 3. We will define the cobordism group $\boldsymbol{C}(n, k; p)$ of completely regular (p)-immersions in section 4, and we will show, in section 5, that the forgetting homomorphism

$$\alpha_p : \boldsymbol{C}(n, k; p) \to \boldsymbol{C}(n, k; p+1)$$

is injective. Therefore the homomorphism

$$\alpha_* : \boldsymbol{E}(n, k) \to \boldsymbol{I}(n, k)$$

is injective.

Next, in section 6, we will study the oriented cobordism groups of completely regular (p)-immersions. In the final section 7, we will return to the unoriented case and study the cokernel of the homomorphism α_p.

2. Definitions and notations

Let M, N be C^∞-differentiable manifolds.

2.1. First we will give some definitions.

DEFINITION 2.1. A C^∞-differentiable mapping f: $M \rightarrow N$ is *proper* if $f^{-1}(\partial N)=\partial M$ and there are coller neighborhoods

$$c : \partial M \times [0, 1) \rightarrow M$$
$$c' : \partial N \times [0,1) \rightarrow N$$

for which the following diagram is commutative:

$$\begin{array}{ccc} \partial M \times [0, 1) & \xrightarrow{c} & M \\ \downarrow (f \mid \partial M) \times 1 & & \downarrow f \\ \partial N \times [0, 1) & \xrightarrow{c'} & N \end{array}$$

DEFINITION 2.2. Subspaces $V_1, V_2, \cdots, V_p$ of a vector space V are *in general position* if

$$\dim (V_{i_1} \cap \cdots \cap V_{i_k}) = \sum_{r=1}^{k} \dim (V_{i_r}) - (k-1) \dim V$$

for $1 \leqq i_1 < i_2 < \cdots < i_k \leqq p$.

DEFINITION 2.3. A proper immersion f: $M \rightarrow N$ is *completely regular* if the subspaces $df(M_{x_1}), \cdots, df(M_{x_p})$ of N_y are in general position for $x_1, \cdots, x_p$ in M such that $y=f(x_1)=\cdots=f(x_p)$, where M_x is a tangent space of M at x, and df: $M_x \rightarrow N_{f(x)}$ is a differential of f.

DEFINITION 2.4. A completely regular immersion f: $M \rightarrow N$ is *completely regular* (p)*-immersion* if $f^{-1}(f(x))$ has at most p-elements for any point x in M.

Lemma 2.5. *Let $V_1, \cdots, V_p$ be subspaces of a vector space V, then the following conditions are equivalent*:

(a) *$V_1, \cdots, V_p$ are in general position,*

(b) *$V_{i_0} + (V_{i_1} \cap \cdots \cap V_{i_k}) = V$ for mutually distinct indices i_0, $i_1, \cdots, i_k$,*

(c) *orthogonal complements of $V_1, \cdots, V_p$ are linearly independent for some* *inner product of* V.

2.2. Next we will fix some notations.

$\boldsymbol{E}(M, N)$: the set of all proper embeddings from M to N,
$\boldsymbol{I}(M, N)$: the set of all proper immersions from M to N,
$\boldsymbol{CR}(M, N)$: the set of all proper completely regular immersions from M to N,
$\boldsymbol{CR}_{(p)}(M, N)$: the set of all proper completely regular (p)-immersions from M to N,
$C^{\infty}(M, N)$: the set of all proper C^{∞}-differentiable mappings from M to N.
Then

$$\boldsymbol{E}(M, N) = \boldsymbol{CR}_{(1)}(M, N) \subset \boldsymbol{CR}_{(2)}(M, N) \subset \cdots \subset \boldsymbol{CR}_{(p)}(M, N) \subset \boldsymbol{CR}_{(p+1)}(M, N) \subset \cdots,$$

$$\boldsymbol{CR}(M, N) = \bigcup_{p \geqq 1} \boldsymbol{CR}_{(p)}(M, N),\ \boldsymbol{CR}(M, N) \subset \boldsymbol{I}(M, N) \subset \boldsymbol{C}^{\infty}(M, N),$$

$$\boldsymbol{CR}_{(p)}(M, N) = \boldsymbol{CR}(M, N) \text{ for } p(\dim N - \dim M) > \dim M,$$

$$\boldsymbol{CR}(M, N) = \boldsymbol{I}(M, N) \text{ for } \dim M = \dim N.$$

3. Completely regular immersions

Let X be a set. Denote by $X^{(p)}$ the p-fold cartesian product of X, $\Delta^p X$ the diagonal set of $X^{(p)}$ and

$$\Delta_{(p)} X = \{(x_1, \cdots, x_p) \in X^{(p)} \mid x_i = x_j \text{ for some } i < j\}.$$

Let f: $X \rightarrow Y$ be a mapping. Denote by

$$f^{(p)} : X^{(p)} \rightarrow Y^{(p)}$$

the mapping defined by $f^{(p)}(x_1, \cdots, x_p) = (f(x_1), \cdots, f(x_p))$.

Then we have the following result from the definition of the transverse regularity condition.

Lemma 3.1. *Let M,N be C^{∞}-differentiable manifolds without boundary and f:$M \rightarrow N$ be a C^{∞}-differentiable mapping. Then the C^{∞}-differentiable mapping $f^{(p)}$: $M^{(p)} - \Delta_{(p)} M \rightarrow N^{(p)}$ is transverse regular over the diagonal $\Delta^p N$, if and only if*

$$dfM_{x_1} + (dfM_{x_2} \cap \cdots \cap dfM_{x_p}) = N_y$$

for $(x_1, \cdots, x_p) \in M^{(p)} - \Delta_{(p)} M$ such that $y = f(x_1) = \cdots = f(x_p)$.

As a corollary of this lemma we have the following result.

Theorem 3.2. *Let M,N be C^{∞}-differentiable manifolds without boundary and f:$M \rightarrow N$ be an immersion. Then*

(a) *f is completely regular if and only if* $f^{(p)}: M^{(p)}-\Delta_{(p)}M\to N^{(p)}$ *is transverse regular over the diagonal* $\Delta^p N$ *for all* $p\geqq 2$,

(b) *f is completely regular* (p)*-immersion if and only if* $f^{(k)}: M^{(k)}-\Delta_{(k)}M\to N^{(k)}$ *is transverse regular over the diagonal* $\Delta^k N$ *for* $2\leqq k\leqq p$ *and* $f^{(p+1)}(M^{(p+1)}-\Delta_{(p+1)}M)$ *does not meet the diagonal* $\Delta^{p+1}N$.

Corollary 3.3. $\boldsymbol{CR}(M, N)$ *and* $\boldsymbol{CR}_{(p)}(M, N)$ *are open subsets of* $\boldsymbol{I}(M, N)$ *with respect to the fine* C^1*-topology.*

Corollary 3.4. *Let* $f: M\to N$ *be a completely regular* (p)*-immersion and* $X=\{x\in M\,|\,f^{-1}(f(x))$ *has just p-elements*$\}$. *Then X is a closed submanifold of M with dimension* $\dim N-p(\dim N-\dim M)$.

Theorem 3.5. *Let M be a compact* C^∞*-differentiable manifold and N a* C^∞*-differentiable manifold. Then*

(a) *the set* $\boldsymbol{CR}(M,N)$ *is a dense open subset of* $\boldsymbol{I}(M, N)$ *with respect to the fine* C^1*-topology,*

(b) *let A be a closed subset of M and* $f: M\to N$ *be an immersion, if the restriction of f over A is completely regular and* $f(A)$ *does not intersect* $f(M-A)$, *then, as an arbitrarily closed* C^1*-approximation of f, there is a completely regular immersion* $g: M\to N$ *such that* $g=f$ *on A.*

Proof. This is an immediate corollary of Theorem 3.2 and the generalized transversality theorem (Theorem 1.10[1]).

Corollary 3.6. *Any immersion* $f: M\to N$ *is differentiably homotopic to a completely regular immersion.*

Proof. This follows from Theorem 3.5 and the fact that $\boldsymbol{I}(M, N)$ is locally contractible with respect to the fine C^1-topology (cf. [2]).

4. Cobordism of immersions

4.1. A completely regular (p)-immersion of dimension (n, k) is a triple $(f, M, N,)$ consisting of two closed C^∞-differentiable manifolds M, N of dimensions n, $n+k$ respectively and a completely regular (p)-immersion $f: M\to N$. We identify (f, M, N) with (f', M', N') if and only if there are diffeomorphisms $\varphi: M\to M'$ and $\psi: N\to N'$ for which $\psi f=f'\varphi$.

A completely regular (p)-immersion (f, M, N) of dimension (n, k) will be said to be *cobordant to zero* if there exists a triple (F, V, W) where:

(1) V and W are compact C^∞-differentiable manifolds of dimensions $n+1$, $n+k+1$ respectively, and

(2) $F: V\to W$ is a proper completely regular (p)-immersion such that $(F\,|\,\partial V, \partial V, \partial W)=(f, M, N)$.

Then we denote $\partial(F, V, W)=(f, M, N)$. Two completely regular (p)-immersions (f_0, M_0, N_0) and (f_1, M_1, N_1) of dimension (n, k) will be said to be *cobordant* if and only if the disjoint union $(f_0, N_0, M_0)+(f_1, M_1, N_1)$ is cobordant to zero.

This cobordism relation is an equivalence relation and denote by $\boldsymbol{C}(n, k; p)$ the set of equivalence classes under this relation of completely regular (p)-immersions of dimension (n, k). As usual, an abelian group structure is imposed on $\boldsymbol{C}(n, k; p)$ by disjoint union, which is called the cobordism group of completely regular (p)-immersions of dimension (n, k), and every element of $\boldsymbol{C}(n, k; p)$ is its own inverse.

4.2. In the above definition, if the term "completely regular (p)-immersion" is replaced by "embedding", "immersion" and "completely regular immerison", one may define the cobordism groups of embeddings $\boldsymbol{E}(n, k)$, immersions $\boldsymbol{I}(n, k)$ and completely regular immersions $\boldsymbol{C}(n, k)$, of dimension (n, k) respectively.

By definition $\boldsymbol{E}(n, k)=\boldsymbol{C}(n, k; 1)$ and there are natural forgetting homomorphisms

$$\begin{aligned} \alpha_p &: \boldsymbol{C}(n, k; p) \rightarrow \boldsymbol{C}(n, k; p+1) \\ \bar{\alpha}_p &: \boldsymbol{C}(n, k; p) \rightarrow \boldsymbol{I}(n, k) \\ \alpha_* &: \boldsymbol{E}(n, k) \rightarrow \boldsymbol{I}(n, k) \end{aligned}$$

such that $\alpha_*=\bar{\alpha}_p \circ \alpha_{p-1} \circ \cdots \circ \alpha_2 \circ \alpha_1$.

REMARK. (a) $\bar{\alpha}_p$ is an isomorphim for $kp>n+1$,
(b) α_1 is injective, since $\boldsymbol{C}(n, k; 2)=\boldsymbol{G}(n, k)$ the cobordism group of generic immersions defined in the previous paper ([3], Section 4).

In the next section, we will prove that the homomorphism α_p is injective for all $p \geqq 1$.

5. Splitting homomorphisms

5.1. Let $f: M \rightarrow N$ be a proper completely regular (p)-immersion where M, N are compact C^∞-differentiable manifolds of dimensions n, $n+k$ respectively. Let

$$X=\{x \in M \mid f^{-1}(f(x)) \text{ has just } p\text{-elements}\}$$

and $Y=f(X)$. Then X and Y are closed submanifolds of M, N respectively, and $\dim X=\dim Y=n-(p-1)k$. Moreover $f \mid X: X \rightarrow Y$ is a p-fold covering. Then

Lemma 5.1. *There are Riemannian metrics on M, N such that*

(a) *the differential* $df: M_x \to N_{f(x)}$ *is isometric for* $x \in X$,

(b) *the orthogonal complements of* $dfM_{x_1}, \cdots, dfM_{x_p}$ *in* N_y *are mutually orthogonal if* $y=f(x_1)=\cdots=f(x_p)$.

Proof. Firstly, define a Riemannian metric satisfying (b) on a neighborhood in N of each point of Y. Define a Riemannian metric on N satisfying the condition (b) by making use of a C^∞-differentiable partition of unity on N. Next, define a Riemannian metric on M as the induced metric by df.

5.2. Let $\nu(X)$ and $\nu(Y)$ be the normal bundles of the embeddings $X \subset M$, $Y \subset N$ respectively, with respect to the above Riemannian metrics. Denote by $E(\nu(X))$, $E(\nu(Y))$ the total spaces of these normal bundles, and $E_\varepsilon(\nu(X))$, $E_\varepsilon(\nu(Y))$ the set of all normal vectors with length $\leqq \varepsilon$.

Then the differential df maps $E(\nu(X))$ into $E(\nu(Y))$ and the following diagram is commutative:

$$\begin{array}{ccc} E(\nu(X)) & \xrightarrow{df} & E(\nu(Y)) \\ \downarrow & f|X & \downarrow \\ X & \longrightarrow & Y \end{array}$$

where the vertical mappings are bundle projections. Then we have the following result by an elementary method.

Lemma 5.2. *There is a differentiably homotopic approximation g of f such that*

(a) $g=f$ on X,

(b) $dg=df$ on $E(\nu(X))$,

(c) *the following diagram is commutative for some* $\varepsilon > 0$:

$$\begin{array}{ccc} E_{2\varepsilon}(\nu(X)) & \xrightarrow{dg} & E_{2\varepsilon}(\nu(Y)) \\ \downarrow exp & g & \downarrow exp \\ M & \longrightarrow & N \end{array}$$

where exp is the exponential mapping.

5.3. Under these notations, let

$$\begin{aligned} M_0 &= M - exp\ (int\ E_\varepsilon(\nu(X))), \\ N_0 &= N - exp\ (int\ E_\varepsilon(\nu(Y))), \\ \partial_1 M_0 &= exp\ (\partial E_\varepsilon(\nu(X))), \\ \partial_1 N_0 &= exp\ (\partial E_\varepsilon(\nu(Y))), \end{aligned}$$

Then $\partial_1 M_0$ and $\partial_1 N_0$ have fixed point free C^∞-differentiable involutions a, b

induced from the bundle involutions of $E(\nu(X))$, $E(\nu(Y))$ respectively, and the following diagram is commutative:

$$\begin{array}{ccc} \partial_1 M_0 & \xrightarrow{g/\partial_1 M_0} & \partial_1 N_0 \\ \downarrow a & & \downarrow b \\ \partial_1 M_0 & \xrightarrow{g/\partial_1 M_0} & \partial_1 N_0 \end{array}$$

Let M_1 be the quotient space of M_0 by the relation $x=a(x)$ for $x\in\partial_1 M_0$. Let N_1 be the quotient space of N_0 by the relation $y=b(y)$ for $y\in\partial_1 N_0$. Then M_1, N_1 have naturally C^∞-differentiable structures and a mapping g_1: $M_1\to N_1$ is induced from the mapping $g|M_0$: $M_0\to N_0$ which is proper completely regular $(p-1)$-immersion.

Theorem 5.3. *There is a homomorphism.*

$$\gamma_p : \boldsymbol{C}(n,\ k;\ p) \to \boldsymbol{C}(n,\ k;\ p-1)$$

such that $\gamma_{p+1}\circ\alpha_p=$*identity.*

Pfoof. By the above notations, the correspondence from (f, M, N) to (g_1, M_1, N_1) is cobordism invariant, and this defines a desired homomorphism.

Corollary 5.4. *The homomorphism* α_p: $\boldsymbol{C}(n, k; p)\to\boldsymbol{C}(n, k; p+1)$ *is injective for all* $p\geqq 1$ *and the image of* α_p *is a direct summand.*

Corollary 5.5. *The homomorphism* α_*: $\boldsymbol{E}(n, k)\to\boldsymbol{I}(n, k)$ *is injective.*

Proof. If $k>0$, then this follows from the above corollary. If $k=0$, then this follows directly from the definition of $\boldsymbol{E}(n, k)$ and $\boldsymbol{I}(n, k)$.

6. Oriented cobordism of immersions

6.1. By similar argument to the unoriented case one may define the oriented cobordism groups of completely regular (p)-immersions $\boldsymbol{C}^0(n, k; p)$, embeddings $\boldsymbol{E}^0(n, k)$, immersions $\boldsymbol{I}^0(n, k)$ and completely regular immersions $\boldsymbol{C}^0(n, k)$, of dimension (n, k) respectively, where we consider orientation preserving mappings if $k=0$.

By definition $\boldsymbol{E}^0(n, k)=\boldsymbol{C}^0(n, k;\ 1)$ and there are forgetting homomorphisms

$$\begin{aligned} \alpha_p^0 &: \boldsymbol{C}^0(n,\ k;\ p) \to \boldsymbol{C}^0(n,\ k;\ p+1) \\ \bar{\alpha}_p^0 &: \boldsymbol{C}^0(n,\ k;\ p) \to \boldsymbol{I}^0(n,\ k) \\ \alpha_*^0 &: \boldsymbol{E}^0(n,\ k) \to \boldsymbol{I}^0(n,\ k) \end{aligned}$$

such that $\alpha_*^0=\bar{\alpha}_p^0\circ\alpha_{p-1}^0\circ\cdots\circ\alpha_2^0\circ\alpha_1^0$.

REMARK. (a) $\bar{\alpha}^0_p$ is an isomorphism for $kp>n+1$,
(b) $\boldsymbol{C}^0(n, k; 2)=\boldsymbol{G}^0(n, k)$ the oriented cobordism group of generic immersions defined in the previous paper ([3], Section 10).

We could not find such a homomorphism as γ_p, so we do not know in general whether the homomorphisms α^0_p and α^0_* are injective or not. In the following we give some partial results.

6.2. First, we consider the case of low codimensions. Let s be a point of k-sphere S^k. Let M, N be oriented closed C^∞-differentiable manifolds of dimensions n, $n+k$ respectively, and define a mapping

$$f:\ M\rightarrow N+M\times S^k$$

by $f(x)=(x, s)$, then f is an embedding. The function

$$\iota:\ \Omega_n\oplus\Omega_{n+k}\rightarrow \boldsymbol{E}^0(n, k)$$

defined by $\iota([M], [N])=[f, M, N+M\times S^k)]$ is a well-defined homomorphism.

Lemma 6.1. *The homomorphism*

$$\alpha^0_*\circ\iota:\ \Omega_n\oplus\Omega_{n+k}\rightarrow \boldsymbol{I}^0(n, k)$$

is injective and the image of ι *is a direct summand of* $\boldsymbol{E}^0(n, k)$.

Proof. Let π: $\boldsymbol{I}^0(n, k)\rightarrow\Omega_n\oplus\Omega_{n+k}$ be a homomorphism defined by $\pi([f, M, N])=([M], [N])$, then $\pi\circ\alpha^0_*\circ\iota=$identity. Therefore we have the desired result.

Proposition 6.2. *The homomorphism*

$$\alpha^0_*:\ \boldsymbol{E}^0(n, k)\rightarrow \boldsymbol{I}^0(n, k)$$

is injective for $k=0$ *and* $k=1$.

Proof. In general $\boldsymbol{E}^0(n, k)$ is isomorphic to $\Omega_{n+k}(MSO(k))$. If $k=1$, then $MSO(1)$ is homotopy equivalent to the circle and hence $\Omega_*(MSO(1))$ is isomorphic to the tensor product $H_*(MSO(1); Z)\otimes\Omega_*$. Therefore $\boldsymbol{E}^0(n, 1)$ is isomorphic to $\Omega_n\oplus\Omega_{,+1}$ and α^0_* is injective by Lemma 6.1. If $k=0$, then the results follows directly from the definition of $\boldsymbol{E}^0(n, 0)$ and $\boldsymbol{I}^0(n, 0)$.

6.3. If $k>n+1$, then the homomorphism

$$\alpha^0_*:\ \boldsymbol{E}^0(n, k)\rightarrow \boldsymbol{I}^0(n, k)$$

is isomorphic by Remark in 6.1. We consider the case $k=n+1$ and $k=n$.

Proposition 6.3.

(a) α_*^0: $\boldsymbol{E}^0(n, n+1)\to\boldsymbol{I}^0(n, n+1)$ *is an isomorphism,*
(b) α_*^0: $\boldsymbol{E}^0(n, n)\to\boldsymbol{I}^0(n, n)$ *is injective.*

Proof. Since the homomorphism

$$\bar{\alpha}_2^0 : \boldsymbol{C}^0(n, k; 2) \to \boldsymbol{I}^0(n, k)$$

is an isomorphism in both case except $n=k=1$, and $\boldsymbol{E}^0(1, 1)=0$, it is sufficient to consider the homomorphism

$$\alpha_1^0 : \boldsymbol{E}^0(n, k) \to \boldsymbol{C}^0(n, k; 2)=\boldsymbol{G}^0(n, k).$$

There are exact sequences [3]:

$$\boldsymbol{G}^0(2s-1, 2s-1)\xrightarrow{\beta}\boldsymbol{B}^-(0, 2s-1)\xrightarrow{\partial}\boldsymbol{E}^0(2s-2, 2s-1)\xrightarrow{\alpha_1^0}\boldsymbol{G}^0(2s-2, 2s-1)\to 0,$$

$$\boldsymbol{G}^0(2s, 2s)\xrightarrow{\beta}\boldsymbol{B}^+(0, 2s)\xrightarrow{\partial}\boldsymbol{E}^0(2s-1, 2s)\xrightarrow{\alpha_1^0}\boldsymbol{G}^0(2s-1, 2s)\to 0.$$

Now define a mapping

$$f : S_1^n+S_2^n \to S^n\times S^n$$

by $f(x)=(x, s)$ for $x\in S_1^n$ and $f(y)=(s, y)$ for $y\in S_2^n$, where S_i^n is a copy of S^n and $s\in S^n$ is a base point. Then f is a completely regular (2)-immersion with unique double point and therefore the homomorphisms β are onto in the above sequences. Consequently the homomorphism α_1^0 is an isomorphism.

Next, since $\boldsymbol{B}^-(1, 2s-1)=0$ in the exact sequence

$$\boldsymbol{B}^-(1, 2s-1)\xrightarrow{\partial}\boldsymbol{E}^0(2s-1, 2s-1)\xrightarrow{\alpha_1^0}\boldsymbol{G}^0(2s-1, 2s-1)\xrightarrow{\beta}\boldsymbol{B}^-(0, 2s-1)\to 0,$$

the homomorphism α_1^0: $\boldsymbol{E}^0(2s-1, 2s-1)\to\boldsymbol{G}^0(2s-1, 2s-1)$ is injective and not onto since $\boldsymbol{B}^-(0, 2s-1)=\boldsymbol{Z}_2$.

Lastly, $\boldsymbol{B}^+(1, 2s)=Z_2$ in the following exact sequence

$$\boldsymbol{G}^0(2s+1, 2s)\xrightarrow{\beta}\boldsymbol{B}^+(1, 2s)\xrightarrow{\partial}\boldsymbol{E}^0(2s, 2s)\xrightarrow{\alpha_1^0}\boldsymbol{G}^0(2s, 2s).$$

We will prove that β is onto, and it is sufficient to show the existence of a completely regular (2)-immersion (f, M, N) of dimension $(2s+1, 2s)$ such that

$$X_f= \{x\in M \mid f^{-1}(f(x)) \text{ has 2-elements}\}$$

is diffeomorphic to the circle S^1. Let CP^s be the complex projective space and

$$f : S^1\times CP^s \to S^1\times CP^{2s}$$

a mapping defined by

$$f(e^{2i\theta}, \langle z_0, z_1, \cdots, z_s\rangle)$$
$$= (e^{4i\theta}, \langle z_0 e^{i\theta}, z_1\cos\theta, \cdots, z_s\cos\theta, z_1\sin\theta, \cdots, z_s\sin\theta\rangle)$$

where $\langle z_0, z_1, \cdots, z_s\rangle$ is a homogeneous coordinate of CP^s. Then f is a completely regular (2)-immersion and

$$X_f = S^1 \times \{\langle 1, 0, \cdots, 0\rangle\}$$

REMARK. By direct calculation, if $n+k \leqq 7$ but $(n, k) \neq (4, 2)$, then the homomorphism

$$\alpha_*^0 : \boldsymbol{E}^0(n, k) \to \boldsymbol{I}^0(n, k)$$

is injective, and the homomorphism

$$\alpha_1^0 : \boldsymbol{E}^0(4, 2) \to \boldsymbol{C}^0(4, 2; 2) = \boldsymbol{G}^0(4, 2)$$

is injective.

7. Bundles over covering spaces

7.1. Now we return to the unoriented case. The homomorphism

$$\alpha_p : \boldsymbol{C}(n, k; p) \to \boldsymbol{C}(n, k; p+1)$$

is injective and the image of α_p is a direct summand by Corollary 5.4, so we study now the cokernel of α_p. For this purpose, we introduce new cobordism groups as follows.

7.2. Let k, p be fixed non-negative integers. A pair of bundles over a covering space is a quadruple $(\xi, \eta, h, \bar{h})$, where

$$\xi : E(\xi) \to B(\xi), \quad \eta : E(\eta) \to B(\eta)$$

are C^∞-differentiable vector bundles over compact C^∞-differentiable manifolds with fibre dimensions pk, $(p+1)k$ respectively,

$$\bar{h} : B(\xi) \to B(\eta)$$

is a $(p+1)$-fold covering which is a proper C^∞-differentiable mapping, and

$$h : E(\xi) \to E(\eta)$$

is a C^∞-differentiable mapping covering $\bar{h}$. The following must be satisfied:

(1) h maps each fibre ξ_x over $x \in B(\xi)$ linearly one to one into a fibre $\eta_{\bar{h}(x)}$,

(2) for each y in $B(\eta)$ and $x_0, x_1, \cdots, x_p$ in $B(\xi)$ such that $y = \bar{h}(x_0) = \bar{h}(x_1) = \cdots = \bar{h}(x_p)$, subspaces $h(\xi_{x_0}), \cdots, h(\xi_{x_p})$ of a vector space η_y are in general position.

7.3. A quadruple $(\xi, \eta, h, \bar{h})$ is identified with a quadruple $(\xi', \eta', h', \bar{h}')$ if and only if there is a quadruple $(a, \bar{a}, b, \bar{b})$ of C^∞-diffeomorphisms

$$a : E(\xi) \to E(\xi'), \quad \bar{a} : B(\xi) \to B(\xi'),$$
$$b : E(\eta) \to E(\eta'), \quad \bar{b} : B(\eta) \to B(\eta'),$$

such that $b \circ h = h' \circ a$ and a, b are bundle mappings covering $\bar{a}$, $\bar{b}$ respectively.

For a quadruple $(\xi, \eta, h, \bar{h})$, denote by $\partial(\xi, \eta, h, \bar{h})$ a quadruplet consisting of the restrictions

$$\xi \mid \xi^{-1}(\partial B(\xi)) : \xi^{-1}(\partial B(\xi)) \to \partial B(\xi),$$
$$\eta \mid \eta^{-1}(\partial B(\eta)) : \eta^{-1}(\partial B(\eta)) \to \partial B(\eta),$$
$$h \mid \xi^{-1}(\partial B(\xi)) : \xi^{-1}(\partial B(\xi)) \to \eta^{-1}(\partial B(\eta)),$$
$$\bar{h} \mid \partial B(\xi) : \partial B(\xi) \to \partial B(\eta).$$

7.4. The cobordism group $\boldsymbol{B}(n, k; p)$ of pairs of bundles over a covering spaces of n-manifold may be now defined. If $B(\xi_0)$ and $B(\xi_1)$ are closed n-manifolds, then a quadruple $(\xi_0, \eta_0, h_0, \bar{h}_0)$ is cobordant to a quadruple $(\xi_1, \eta_1, h_1, \bar{h}_1)$ if and only if there is a quadruple $(\xi, \eta, h, \bar{h})$ as such that

$$\partial(\xi, \eta, h, \bar{h}) = (\xi_0, \eta_0, h_0, \bar{h}_0) + (\xi_1, \eta_1, h_1, \bar{h}_1)$$

where the symbol $+$ denotes disjoint union. Then this cobordism relation is an equivalence relation. Denote by $\boldsymbol{B}(n, k; p)$ the set of all cobordism classes. As usual an abelian group structure is imposed on $\boldsymbol{B}(n, k; p)$ by disjoint union, then every element is its own inverse.

REMARK. $\boldsymbol{B}(n, k; 1)$ is naturally isomorphic with the cobordism group $\boldsymbol{B}(n, k)$ of bundles over mainfolds with involution defined in the previous paper ([3], Section 3).

7.5. Now we define homomorphisms

$$\beta_p : \boldsymbol{C}(n, k; p+1) \to \boldsymbol{B}(n-pk, k; p),$$
$$\pi_p : \boldsymbol{B}(n, k; p) \to \boldsymbol{C}(n+pk, k; p+1).$$

(7.5.1) Let $a \in \boldsymbol{C}(n, k; p+1)$ be represented by a completely regular $(p+1)$-immersion $f: M \to N$. Let

$$X = \{x \in M \mid f^{-1}(f(x)) \text{ has just } (p+1)\text{-elements}\}$$

and $Y = f(X)$. Then there are Riemannian metrics on M, N satisfying the conditions of Lemma 5.1. Let $\nu(X)$, $\nu(Y)$ be the normal bundles of the embeddings $X \subset M$, $Y \subset N$ respectively, with respect to these Riemannian metrics, and then the differential df maps $E(\nu(X))$ into $E(\nu(Y))$. Define $\beta_p(a)$ the cobordism class of the quadruple $(\nu(X), \nu(Y), df\,|\,E(\nu(X)), f\,|\,X)$.

(7.5.2.) Let $b \in \boldsymbol{B}(n, k; p)$ be represented by a qudaruple $(\xi, \eta, h, \bar{h})$. Define $\pi_p(b)$ the cobordism class of a completely regular $(p+1)$-immersion defined by the mapping

$$P(h \oplus 1) : P(\xi \oplus \theta^1) \rightarrow P(\eta \oplus \theta^1)$$

where θ^1 is the trivial line bundle, $P(\xi \oplus \theta^1)$ and $P(\eta \oplus \theta^1)$ are the total spaces of the associated projective space bundles, and $P(h \oplus 1)$ is a mapping canonically induced from the mapping h.

Theorem 7.1. *There is an exact sequence*:

$$0 \longrightarrow \boldsymbol{C}(n, k; p) \xrightarrow{\alpha_p} \boldsymbol{C}(n, k; p+1) \xrightarrow{\beta_p} \boldsymbol{B}(n-pk, k; p) \longrightarrow 0.$$

Proof. The homomorphism α_p is injective by Corollary 5.4, and the homomorphism β_p is surjective since $\beta_p \circ \pi_p =$ identity by definition (cf. [3] Theorem A'). The exactness at $\boldsymbol{C}(n, k; p+1)$ is proved by the handle attaching construction (cf. [3], Section 5), so we omit the details.

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