SEVERAL REMARKS ON TRANSITIVE EXTENSIONS
OF FINITE PERMUTATION GROUPS

Eiichi BANNAI

1. Introduction

In [2] D. R. Hughes showed non-existence of transitive extensions of permu-
tation groups which are collineation groups of some block designs, by using the
following criterion (see [2], see also H. Lüneburg [4]):

Criterion of Hughes. Let G be a collineation group of a t-(v, k, λ) design $\mathscr{F}$,
t-ply transitive on the set of all the v points of $\mathscr{F}$ and transitive on the set of all the
b blocks of $\mathscr{F}$, and let G, regarded as a permutation group on the points, has a
transitine extension $\mathfrak{G}$. Let a subgroup V of G fix every point of a block and fix
no other points, and let V be contained in U, a stabilizer of t-i points of $\mathscr{F}$. If
V is an S-subgroup[1] with respect to the pair $(U, \mathfrak{G})$, then $\dfrac{v+1}{k+1} \cdot b$ is not an
integer.

In this note we will look the criterion from a view point of group theory pro-
per, and so we will reduce it into more handy and general one (i.e. Criterion A).
This Criterion A may be essentially familiar to experts, and it might have been
even actually used before. Nevertheless, it seems to the author that it has never
been exploited in the form we give and use here. In Section 3 we will show
that Criterion of Hughes is obtained from Criterion A. Finally we will give
several applications of our criterion.

2. Criterion A

For a permutation group G on a set Ω, $I(X)$ denotes the set of points which
are fixed by every element of a subset X of G, and c_X denotes the number of sub-
sets of G which are conjugate to X in G.

Criterion A. Let G be a permutation group on a set Ω, and let $\mathfrak{G}$ be a
transitive extension of (G, Ω). If X is an S-subset[2][3] with respect to the pair

---

1) That is, every subgroup of U which is conjugate to V in $\mathfrak{G}$ is already conjugate in U.

2) That is, every subset of G which is conjugate to X in $\mathfrak{G}$ is already conjugate in G.

3) For an element σ of G, the set $\{\sigma\}$ is an S-subset, if every element x of G, of the order
equal to σ and $I(x) = I(\sigma)$, is conjugate to σ in G.

$(G, \mathfrak{G})$, then $\dfrac{|\Omega|+1}{|I(X)|+1} \cdot c_X$ is not an integer,

This criterion is an immediate consequence of the following lemma:

Lemma 1. *Let* $(\mathfrak{G}, \Omega \cup \{\infty\})$ *be a transitive extension of* (G, Ω). *Then the number of subsets of* $\mathfrak{G}$ *which are conjugate to* X *in* $\mathfrak{G}$ *is given by* $\dfrac{|\Omega|+1}{|I(X)|+1} \cdot c_X'$, *where* c_X' *denotes the number of subsets of* G *which are conjugate to* X *in* $\mathfrak{G}$.

The proof of the lemma may be familiar, but we repeat it for completeness.

For each $\alpha \in \Omega \cup \{\infty\}$, the stabilizer $\mathfrak{G}_\alpha$ contains c_X' subsets which are conjugate to X, and each subset which is conjugate to X is contained in $(|I(X)|+1)$ $\mathfrak{G}_\alpha$'s. So we have the assertion.

3. Deduction of Hughes' Criterion from Criterion A

Now we will prove that Hughes' Criterion is obtained from Criterion A.

Since V, in the assumption of Hughes' Criterion, is an S-subgroup for $(U, \mathfrak{G})$, V is also an S-subgroup for $(G, \mathfrak{G})$. Let X be the subgroup generated by all the subgroups of G which are conjugate to V and fix every point of $I(V)$. Then X is also an S-subgroup for $(G, \mathfrak{G})$, because G is transitive on the blocks. Moreover every element of G which fixes a block $I(V)$ as a whole, normalizes the subset X, and so we have $c_X = b$. Hence $\dfrac{|\Omega|+1}{|I(X)|+1} \cdot c_X = \dfrac{v+1}{k+1} \cdot b$ is not an integer. Thus Hughes' Criterion is deduced.

4. Some applications

In this last section we show some applications of our Criterion A. The result of Proposition 3 may be new.

Proposition 1. (H. Zassenhaus [6], see also D. R. Hughes [2].)
Let Ω *be the set of the points of the* $(r-1)$-*dimensional projective space defined over a finite field* F_q, *and let* $PSL(r,q) \leqslant G \leqslant P\Gamma L(r,q), r \geqslant 3$. *Then* (G, Ω) *has no transitive extension, unless* $q=2$, *or* $r=3$ *and* $q=4$.

Proof. (This is a modification of the proof in [2].) Since $r \geqslant 3$ by the assumption, a collineation σ such that $I(\sigma) = q^{r-2} + q^{r-3} + \cdots + q + 1$ is necessarily an elation by the Theorem of C. W. Norman (see [2], or [4], (2.5)). There exist $(q^{r-1} + \cdots + q + 1)(q^{r-2} + \cdots + q + 1)(q-1)$ elations in all, and these are all contained in G and are all conjugate in G, because G contains $PSL(r,q)$. While $\dfrac{|\Omega|+1}{|I(\sigma)|+1}$ $\cdot c_\sigma = \dfrac{q^{r-1} + q^{r-2} + \cdots + q + 2}{q^{r-2} + q^{r-3} + \cdots + q + 2} \cdot (q^{r-1} + \cdots + q + 1)(q^{r-2} + \cdots + q + 1)(q-1) \in \mathbf{Z}$ implies $\dfrac{(q-2)(q-1)^2}{q^{r-2} + q^{r-3} + \cdots + q + 2} \in \mathbf{Z}$, because the G.C.D. of $q^{r-1} + \cdots + q + 2$ and $q^{r-2} + \cdots + q$

$+2$ divides $q-2$, and the $G.C.D.$ of $q^{r-1}+\cdots+q+1$ and $q^{r-2}+\cdots+q+2$ divides $q-1$. But this is not an integer unless $q=2$, or $r=3$ and $q=4$, or $r=3$ and $q=16$. Thus, unless $q=2$, or $r=3$ and $q=4$, or $r=3$ and $q=16$, by Criterion A G has no transitive extension. To complete the proof we have only to prove that the last case is impossible, so let us assume that $r=3$ and $q=16$, and let $\mathfrak{G}$ be a transitive extension of G. A collineation τ such that $I(\tau)=q+1+1=18$ and of order 5 is necessarily a homology, and there exist in all $(q^2+q+1)\cdot q^2\cdot 4=273.16^2\cdot 4$ homologies of order 5, and these are all contained in $PSL(3,16)$, hence in G. Since $PSL(3,16)$ is transitive on the set of non-incident point-line pairs, c_τ', the number of elements of G which are conjugate to τ in $\mathfrak{G}$, is $(q^2+q+1)\cdot q^2\cdot t$, for some t, $1\leqslant t\leqslant 4$. By Lemma 1, the number of elements of G which are conjugate to τ is equal to $\dfrac{(q^2+q+1)+1}{(q+1)+1+1}\cdot(q^2+q+1)\cdot q^2\cdot t=\dfrac{274}{19}273\cdot 256\cdot t$. But this is never an inetger, a contradiction. Thus the proof is completed.

(If one takes a pointwise stabilizer of a hyperplane instead of an elation as an S-subset X, one gets quite the same proof as in [2].)

Proposition 2. (H. Lüneburg [3], M. Suzuki [5].) *If G is the Ree group of order $(q^3+1)q^3(q-1)$ with $q=3^{2r+1}$, then G has, considered as a doubly transitive permutation group of degree q^3-1, no transitive extension.*

Proof. (This is a slight modification of the proof in [3].)
There exist $q^2(q^2-q+1)$ involutions in G and these are all conjugate to each other, and for an involution σ we have $I(\sigma)=q+1$. Now, $\dfrac{|\Omega|+1}{|I(\sigma)|+1}\cdot c_\sigma=\dfrac{q^3+2}{q+2}\cdot q^2(q^2-q+1)$ is not an integer for every $q=3^{2r+1}$. Thus we have the assertion by Criterion A.

Proposition 3. *Let $G=G_2(q)$ be the Dickson-Chevalley group of type G_2 over a field F_q. Let us assume that q is not a power of 2 or 3. Denote by P_i $(i=1,2)$ the parabolic subgroups of G defined by $P_1=\mathfrak{U}\mathfrak{H}\cup\mathfrak{U}\mathfrak{H}\omega(w_a)\mathfrak{U}w_a$ and $P_2=\mathfrak{U}\mathfrak{H}\cup\mathfrak{U}\mathfrak{H}\omega(w_b)\mathfrak{U}w_b$ (see, for notations and explanations, B, Chang [1]). Under these conditions, the permutation groups $(G,G/P_i)$, $i=1,2$, have no transitive extension.*

Proof. $|G|=q^6(q^6-1)(q^2-1)$ and $|P_1|=|P_2|=q^6(q^2-1)(q-1)$. There exist $q^4(q^4+q^2+1)$ involutions in G and they are all conjugate to each other (see, e.g. [1]). Analysing the proofs of the determination of the conjugacy classes of G in [1], one can easily compute that P_i $(i=1,2)$ contains $(q+2)q^4$ involutions, and that there exist q^2+3q+2 cosets of G/P_i $(i=1,2)$ which are fixed by an involution σ. Now, $\dfrac{|G/P_i|+1}{|I(\sigma)|+1}\cdot c_\sigma=\dfrac{(q^5+q^4+q^3+q^2+q+2)}{q^2+3q+3}\cdot q^4(q^4+q^2+1)\in \mathbf{Z}$ implies $\dfrac{(4q+17)(3q+5)}{q^2+3q+3}\in \mathbf{Z}$, because the $G.C.D.$ of $q^5+q^4+q^3+q^2+q+1$ and q^2+3q+

3 divides $4q+17$, the *G.C.D.* of q^4+q^2+1 and q^2+3q+3 divides $3q+5$, the *G.C.D.* of q^4 and q^2+3q+3 is 1, since q is not a power of 2 or 3 by the assumption. This implies moreover $\dfrac{35q+47}{q^2+3q+3} \in \mathbf{Z}$. But one can easily check that this is impossible, because q is not a power of 2 or 3. Hence we have the assertion by Criterion A.

For many other classes of finite permutation groups, one can show the non-existences of transitive extensions using Criterion A. Moreover, Criterion A is modified according to circumstances, and we can show the non-existences of (not necessarily transitive) extensions of some finite permutation groups. The author has determined, for instance, the primitive extensions of rank 3 of the group *PGL* (r,q) acting on the points of the $(r-1)$–dimensional projective space over F_q, where q is an arbitrary power of 2. This will be published elsewhere.

University of Tokyo

References

[1] B. Chang: *The conjugate classes of Chevalley groups of type* (G₂), J. Algebra **9** (1968), 190–211.

[2] D. R. Hughes: *Extensions of designs and groups: projective, symplectic and certain affine groups*, Math. Z. **89** (1965), 199–205.

[3] H. Lüneburg: *Some remarks concerning the Ree groups of type* (G₂), J. Algebra **3** (1966), 256–259.

[4] ————: Transitive Erweiterungen endlicher Permutationsgruppen, Lecture Notes in Mathematics 84, Springer-Verlag, 1969.

[5] M. Suzuki: *Transitive extensions of a class of doubly transitive groups*. Nagoya Math. J. **27** (1966), 159–169.

[6] H. Zassenhaus: *Uber transitive Erweiterungen gewisser Gruppen aus Automorphismen endlicher mehrdimensionaler Geometrien*, Math. Ann. **111** (1935), 748–759.