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Osaka J. Math.
8 (1971), 91-97

ON THE ANNIHILATOR IDEALS OF THE RADICAL OF A GROUP ALGEBRA

Dedicated to Professor K. Asano for his 60th birthday

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(Received July 31, 1970)
(Revised October 20, 1970)

1. Introduction

Let G be a finite group, and k a field of characteristic p. Let $\mathfrak{n}$ denote the Jacobson radical of the group algebra kG, and $r(\mathfrak{n})$ the right annihilator ideal of $\mathfrak{n}$. In this paper we shall show some connections between $r(\mathfrak{n})$ and p-elements of G. One of them will state that $r(\mathfrak{n})$ contains the sum of all p-elements of G (including the identity). This may be regarded in a sense as a refinement of Maschke's theorem. In fact, if p does not divide the order of G then the identity is the only p-element, which implies $r(\mathfrak{n})\ni 1$ and hence $\mathfrak{n}=0$. On the other hand, as is easily seen from a theorem of T. Nakayama on Frobenius algebras (see §2), $r(\mathfrak{n})$ is a principal ideal. We shall show that it is generated by an element which is left invariant by every automorphism of kG induced by that of G. As an application of this fact, we shall give a lower bound for the first Cartan invariant in terms of the chief composition factors of G. The present study owes heavily to some general results on Frobenius algebras and symmetric algebras, which will be summarized in the next section.

NOTATION. If A is a ring, rad(A) will denote the Jacobson radical of A. For a subset T of A, $r(T)$ and $l(T)$ will denote respectively the set of right annihilators and the set of left annihilators of T in A. If M is a subset of a finite group G, then $\Delta_M=\sum_{\sigma\in M}\sigma\in kG$.

2. Preliminary results

Let $A(\ni 1)$ be a finite dimensional algebra over a field k.

DEFINITION. A linear function λ ($\in A^*=\mathrm{Hom}_k(A, k)$) is called *non-singular* if its kernel contains no left or right ideals other than zero. While, λ is called *symmetric if* $\lambda(ab)=\lambda(ba)$ for all $a, b\in A$.

If λ is a linear function and $a\in A$, we denote by λ_a the linear function defined by $\lambda_a(x)=\lambda(xa)$, $x\in A$. One may remark here that f: $a\to\lambda_a$ is a left A-homomorphism from A into A^*. It is an (onto) isomorphism if and only if is non-singular. A is a Frobenius [symmetric] algebra if and only if it has (at least) one non-singular [symmetric, non-singular] linear function.

Theorem A (T. Nakayama [6], [8], see also [2]). *Let A be a Frobenius [symmetric] algebra, λ a non-singular [symmetric, non-singular] linear function on A, and $\mathfrak{z}$ a two-sided ideal of A. If $A/\mathfrak{z}$ is Frobenius [symmetric], and μ a non-singular [symmetric, non-singular] linear function on $A/\mathfrak{z}$ then there exists an element [central element] $c\in A$ such that $\mu\psi=\lambda_c$ and $r(\mathfrak{z})=cA$, where ψ is the natural map $A\to A/\mathfrak{z}$. Conversely, if there exists an element [central element] $c\in A$ such that $r(\mathfrak{z})=cA$ then $A/\mathfrak{z}$ is Frobenius [symmetric].*

Proof. As was noted above, there exists an element $c\in A$ such that $\mu\psi=\lambda_c$. We shall show $r(\mathfrak{z})=cA$. Since $\lambda(\mathfrak{z}c)=\lambda_c(\mathfrak{z})=0$ and λ is non-singular, it follows at once $cA\subset r(\mathfrak{z})$. On the other hand, if $xc=0$ then $\lambda_c(Ax)=\lambda(Axc)=0$. Since λ_c is non-singular as a linear function on $A/\mathfrak{z}$, it follows $x\in\mathfrak{z}$ and hence $l(cA)\subset\mathfrak{z}$. Recalling here that A is Frobenius, we have then $cA=r(l(cA))\supset r(\mathfrak{z})$. Now, suppose further both λ and μ are symmetric. Then $\lambda(xyc)=\lambda_c(xy)=\lambda_c(yx)=\lambda(yxc)=\lambda(xcy)$ for all $x, y\in A$. Therefore, $yc=cy$ and c is central. Next, we shall prove the converse. Suppose $r(\mathfrak{z})=cA$. Then λ_c gives rise to a non-singular linear function on $A/\mathfrak{z}$. If c is central, the linear function is evidently symmetric.

Theorem B (T. Nakayama [6]). *If $\mathfrak{z}$ is a two-sided ideal of a symmetric algebra A then $r(\mathfrak{z})=l(\mathfrak{z})$.*

Proof. Let λ be a symmetric, non-singular linear function on A. Then $r(\mathfrak{z})=\{x\in A\mid\lambda(\mathfrak{z}x)=0\}=\{x\in A\mid\lambda(x\mathfrak{z})=0\}=l(\mathfrak{z})$.

3. The generator of $r(\mathfrak{n})$

From now on, k will denote a field of characteristic p, and G a finite group of order $|G|=p^n g_0$, where $(p, g_0)=1$. Let $\nu_p(l)$ denote the exponent of p in the primary decomposition of an integer l. Let $\mathfrak{n}$ be the radical of the group algebra kG as before. To be easily seen, kG is a symmetric algebra through the following linear function λ which will be fixed throughout the subsequent study: $\lambda(\Sigma_{\sigma\in G}\, a_\sigma\sigma)=a_1$, where $a_\sigma\in k$ and 1 denotes the identity of G.

REMARK 1. Since there exists a splitting field for G which is finite separable over k, $kG/\mathfrak{n}$ is a separable algebra over k. In particular, if K is an arbitrary extension field of k and $\mathfrak{n}_K$ denotes the radical of KG, then $\mathfrak{n}_K=K\mathfrak{n}$ and $\mathfrak{n}_K\cap kG=\mathfrak{n}$. Similar relations hold for $r(\mathfrak{n})$ and the right annihilator ideal of $\mathfrak{n}_K$ in KG.

Let $\bar{\phi}_1, \bar{\phi}_2, \cdots, \bar{\phi}_r$ be the distinct irreducible characters over a suitable splitting field for G containing k, and let $\bar{\phi}=\Sigma_{i=1}^r \phi_i$. Then, it is clear that $\bar{\phi}(\sigma)$ is contained in the prime field for every $\sigma\in G$.

Proposition 1. *If* $v=\Sigma_{\sigma\in G}\,\bar{\phi}(\sigma^{-1})\sigma$ *then* $r(\mathfrak{n})=(kG)v$.

Proof. By the above remark, we may assume that k is a splitting field for G. Then, $kG/\mathfrak{n}$ is a direct sum of total matrix algebras over k: $kG/\mathfrak{n}=\Sigma_{i=1}^r(k)_{n_i}$. Let ψ and p_i denote respectively the natural map $kG\to kG/\mathfrak{n}$ and the projection $kG/\mathfrak{n}\to(k)_{n_i}$. Since the trace map tr_i: $(k)_{n_i}\to k$ is a symmetric, non-singular linear function, so is $\mu=\Sigma_{i=1}^r tr_i p_i$ on $kG/\mathfrak{n}$. Therefore, by Theorem A, there exists a central element $v=\Sigma_\sigma a_\sigma\sigma\in kG$ such that $\mu\psi=\lambda_v$ and $r(\mathfrak{n})=v(kG)$. Noting here that $\mu\psi=\bar{\phi}$ on G, we obtain $a_\sigma=\lambda(\sigma^{-1}v)=\lambda_v(\sigma^{-1})=\bar{\phi}(\sigma^{-1})$. This completes the proof.

REMARK 2. Let f be an arbitrary automorphism of G. Then, it permutes the irreducible characters by $\bar{\phi}_i\to\bar{\phi}_i^f$, where $\bar{\phi}_i^f(\sigma)=\bar{\phi}_i(\sigma^f)$, $\sigma\in G$. In particular, it follows $\bar{\phi}(\sigma^f)=\bar{\phi}^f(\sigma)=\bar{\phi}(\sigma)$ for all $\sigma\in G$. Hence, regarding f naturally as an automorphism of kG, we obtain $v^f=\Sigma_\sigma\,\bar{\phi}(\sigma^{-1})\sigma^f=\Sigma_\sigma\,\bar{\phi}((\sigma^f)^{-1})\sigma^f=\Sigma_\tau\,\phi(\tau^{-1})\tau=v$.

Now, let H be a normal subgroup of G, and $\mathfrak{m}=\mathrm{rad}\,kH$. Then $\mathfrak{l}=(kG)\mathfrak{m}=\mathfrak{m}(kG)$ is a nilpotent two-sided ideal of kG.

Corollary 1. *Under the above notation, there holds the following:*

(1) *$kG/\mathfrak{l}$ is a symmetric algebra over k.*

(2) *Let $(\mathfrak{l}:\mathfrak{n})=\{x\in kG\,|\,\mathfrak{n}x\subset\mathfrak{l}\}$. Then, for an arbitrary primitive idempotent e of kG there holds $(kG)e/\mathfrak{n}e\cong(\mathfrak{l}:\mathfrak{n})e/\mathfrak{l}e$.*

Proof. (1) Let $r_H(\mathfrak{m})$ be the right annihilator ideal of $\mathfrak{m}$ in kH. Let v be as in Proposition 1 applied to kH. Since G induces an automorphism group on H, Remark 2 proves that v lies in the center of kG. Hence, $r(\mathfrak{l})=kG\cdot r_H(\mathfrak{l})=(kG)v$ is a principal ideal generated by a central element. Theorem A proves therefore $kG/\mathfrak{l}$ is a symmetric algebra.

(2) Evidently, the residue class $\bar{e}$ of e modulo $\mathfrak{l}$ is a primitive idempotent of the symmetric algebra $kG/\mathfrak{l}$. Since $(\mathfrak{l}:\mathfrak{n})/\mathfrak{l}$ is the right annihilator ideal of $\mathfrak{n}/\mathfrak{l}=\mathrm{rad}(kG/\mathfrak{l})$, there holds then $(kG/\mathfrak{l})\bar{e}/(\mathfrak{n}/\mathfrak{l})\bar{e}\cong((\mathfrak{l}:\mathfrak{n})/\mathfrak{l})\bar{e}$, which proves $(kG)e/\mathfrak{n}e\cong(\mathfrak{l}:\mathfrak{n})e/\mathfrak{l}e$.

Under the above notation, if H is a non-trivial p-group, then it is well-known that $\mathfrak{m}$ coincides with the augmentation ideal $\mathrm{I}(H)=\{\Sigma_{\sigma\in H}\,a_\sigma\sigma\,|\,\Sigma_\sigma a_\sigma=0\}$, so that $r_H(\mathfrak{m})=kH\cdot\Delta_H=k\Delta_H$. We obtain therefore $r(\mathfrak{n})\subset r(\mathfrak{l})=kG\cdot\Delta_H\subset\mathfrak{l}$.

Lemma 1. *Let $P_1\supsetneqq H_1$ be normal subgroups of G such that P_1/H_1 is a*

p-group. Let $\mathfrak{l}_1=kG\cdot\mathrm{rad}\, kP_1$, and $\mathfrak{h}_1=\ker(kG\to k(G/H_1))$. Then, there holds the following:

(1) *If e is a primitive idempotent of kG not contained in $\mathfrak{h}_1$ then $(\mathfrak{l}_1+\mathfrak{h}_1:\mathfrak{n})e/(\mathfrak{l}_1+\mathfrak{h}_1)e\cong(kG)e/\mathfrak{n}e$ and $(\mathfrak{l}_1+\mathfrak{h}_1)e$ contains a submodule isomorphic to $(kG)e/\mathfrak{n}e$.*

(2) *If $P_2\supsetneqq H_2$ are normal subgroups of G containing P_1 such that P_2/H_2 is a p-group then $\mathfrak{l}_2+\mathfrak{h}_2\supset(\mathfrak{l}_1+\mathfrak{h}_1:\mathfrak{n})$, where $\mathfrak{l}_2=kG\cdot\mathrm{rad}\, kP_2$ and $\mathfrak{h}_2=\ker(kG\to k(G/H_2))$.*

Proof. (1) Since $\mathfrak{l}_1+\mathfrak{h}_1=k(G/H_1)\cdot\mathrm{rad}\, k(P_1/H_1)$ and P_1/H_1 is a non-trivial p-group, the above remark proves $r(\mathrm{rad}\, k(G/H_1))\subset\mathfrak{l}_1+\mathfrak{h}_1/\mathfrak{h}_1$. Let $\bar{e}$ be the residue class of e modulo $\mathfrak{h}_1$. Then, it is still primitive by assumption, and the former is evident by Corollary 1. Further, noting that $(\mathfrak{l}_1+\mathfrak{h}_1/\mathfrak{h}_1)\bar{e}\neq0$, it follows at once $0\neq(\mathfrak{l}_1+\mathfrak{h}_1)e\supset r(\mathfrak{n})e\cong(kG)e/\mathfrak{n}e$, proving the latter.

(2) As in (1), we obtain $r(\mathrm{rad}\, k(G/H_2))\subset\mathfrak{l}_2+\mathfrak{h}_2/\mathfrak{h}_2$. If $\mathfrak{p}_1=\ker(kG\to k(G/P_1))$ then $(\mathfrak{l}_1+\mathfrak{h}_1:\mathfrak{n})+\mathfrak{p}_1/\mathfrak{p}_1\subset(\mathfrak{l}_1+\mathfrak{p}_1/\mathfrak{p}_1:\mathfrak{n}+\mathfrak{p}_1/\mathfrak{p}_1)=r(\mathrm{rad}\, k(G/P_1))$. Since the natural map $k(G/P_1)\to k(G/H_2)$ is an epimorphism, it sends $r(\mathrm{rad}\, k(G/P_1))$ into $r(\mathrm{rad}\, k(G/H_2))$. Therefore, $(\mathfrak{l}_1+\mathfrak{h}_1:\mathfrak{n})+\mathfrak{h}_2/\mathfrak{h}_2\subset r(\mathrm{rad}\, k(G/H_2))\subset\mathfrak{l}_2+\mathfrak{h}_2/\mathfrak{h}_2$.

Theorem 1. *Let m be the number of the chief composition factors of G which are (non-trivial) p-groups. Then the first Cartan invariant c_{11} is at least $m+1$.*

Proof. We take a primitive idempotent e of kG such that $(kG)e/\mathfrak{n}e$ is isomorphic to the trivial kG-module $k\cong k\Delta_G$. Then, $e\Delta_G$ being non-zero, e is not contained in $\ker(kG\to k(G/N))$ ($=$ the ideal generated by $\{1-\eta\mid\eta\in N\}$) for any normal subgroup N of G. Hence, by Lemma 1, we can easily see that $(kG)e$ possesses at least $m+1$ composition factors isomorphic to k, namely, $c_{11}\geqslant m+1$.

4. The sum of all p-elements

First, we shall introduce some notations. Let $\bar{\varepsilon}$ be a primitive g_0-th root of unity over the prime field of characteristic p. In what follows, whenever we consider Brauer characters, it is assumed that there is defined (and fixed)a homomorphism $Z[\varepsilon]\to k[\bar{\varepsilon}]$ such that $\bar{\varepsilon}$ is the image of a primitive g_0-th root of unity ε in the complex number field. As is well-known, there exists a unique (up to isomorphisms) indecomposable projective module P_i such that $P_i/\mathfrak{n}P_i$ affords the irreducible character $\bar{\phi}_i$. Let $\bar{\eta}_i$ be the character of the representation afforded by P_i and u_i the dimension of P_i. As is well-known, u_i is divisible by p^n. Let $u_i=p^n h_i$. We may assume, after a suitable change of index if necessary, the first $u_1, u_2, \cdots, u_t$ are all such that $\nu_p(u_i)=n$. Let ϕ_i and η_i be the Brauer characters of ϕ_i and η_i, respectively.

Noting that $h_j=0$ in k for $t<j\leq r$ and $\phi_i(\sigma)=\phi_i(\sigma')$ for the p-regular part σ' or σ, the orthogonality relation ([3] p. 561)

$$\Sigma_{i=1}^{r} \eta_i(\sigma^{-1})\phi_i(\tau) = \begin{cases} |C_G(\sigma)| & \text{if } \sigma \text{ is conjugate to } \tau, \\ 0 & \text{otherwise} \end{cases}$$

implies

$$(*) \qquad \Sigma_{i=1}^{t} h_i\bar{\phi}_i(\tau) = \begin{cases} g_0 & \text{if } \tau \text{ is a } p\text{-element}, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2. *Assume that k is a splitting field for G. Let $\mathfrak{z}=\Sigma_e (kG)e+\mathfrak{n}$, where e runs over the primitive idempotents such that ν_p $(\dim_k (kG)e)>n$. Let c be the sum of all p-elements of G. Then there holds $r(\mathfrak{z})=(kG)c$.*

Proof. We use the same notation as in the proof of Proposition 1. Then, $kG/\mathfrak{n}=\Sigma_{i=1}^{t} (k)_{n_i}+\Sigma_{j=t+1}^{r} (k)_{n_j}$ (direct sum), where $n_i=\dim_k P_i/\mathfrak{n}P_i$. It is clear that $\mathfrak{z}$ is the inverse image of $\Sigma_{j=t+1}^{r}(k)_{n_j}$ by the natural map ψ. Hence, it is a two-sided ideal and there holds $kG/\mathfrak{z}\cong\Sigma_{i=1}^{t}(k)_{n_i}$. We set here $\mu=\Sigma_{i=1}^{t} h_i\, tr_i\, p_i$. Since $h_i \neq 0$ in k, μ is a symmetric, non-singular linear function on $kG/\mathfrak{z}$. Then, by Theorem A, there exists a central element $c'\in kG$ such that $\mu\psi'=\lambda_{c'}$ and $r(\mathfrak{z})=(kG)c'$, where ψ' is the natural map $kG\rightarrow kG/\mathfrak{z}$. Now, by making use of $(*)$, we can prove $c'=g_0 c$ in the same way as in the last part of the proof of Proposition 1. This completes the proof.

Since $\mathfrak{z}\supset\mathfrak{n}$, we obtain in particular $r(\mathfrak{n})\ni c$. However, in virtue of Remark 1, this holds without assuming that k is a splitting field. Thus, we have shown

Theorem 2. *$r(\mathfrak{n})$ contains the sum of all p-elements of G.*

Corollary 2. *Let $x=\Sigma_\sigma a_\sigma \sigma$ be an element of kG, and $x(p)$ the sum of the coefficient a_σ of p-elements σ. If x is in $\mathfrak{n}$ then $x(p)=0$.*

Proof. Note that $x(p)$ is equal to the coefficient of 1 in xc. If x is in $\mathfrak{n}$ then $xc=0$, and hence $x(p)=0$.

If e is a primitive idempotent of kG, then $r(\mathfrak{n})e$ is a minimal left ideal of kG and contains $(kG)ce$.

Corollary 3. *Let e be a primitive idempotent of kG. If ν_p $(\dim_k(kG)e)=n$ then $r(\mathfrak{n})e=(kG)ce$. The converse holds, provided k is a splitting field for G.*

Proof. First, we assume that k is a splitting field. By Lemma 2, if $ce=0$, or what is the same, if $e\in l(c)=\mathfrak{z}$, then $\nu_p(\dim_k (kG)\mathrm{e})>n$, and coversely. We have seen therefore that $ce\neq 0$ and ν_p $(\dim_k (kG)e)=n$ are equivalent.

Secondly, we assume that $\nu_p(\dim_k (kG)e)=n$. Let K be a splitting filed for G containing k, and $e=\Sigma_j e_j$ a decomposition of e into orthogonal primitive idempotents of KG. Then, by assumption there exists at least one e_i such that $\nu_p(\dim_k(KGe_i)=n$. Since $ce_i\neq 0$ by the first step, we have $ce\neq 0$, completing the proof.

Now, if G is p-solvable then $\nu_p(\dim_k (kG)e)=n$ for every primitive idempotent e (P. Fong [5]), whence it follows $\mathfrak{z}=\mathfrak{n}$ and therefore $r(\mathfrak{n})=r(\mathfrak{z})=(kG)c$.

Corollary 4. *If G is p-solvable then $r(\mathfrak{n})=(kG)c$.*

5. Some nilpotent ideal of kG

The present section is independent of the preceding ones. Let T be a subgroup of G, and $\mathfrak{m}$ a left nilpotnet ideal of kT. Let $\mathfrak{m}^\sigma=\{\sigma^{-1}x\sigma \mid x\in\mathfrak{m}\}$ for $\sigma\in G$, and $r_T(\mathfrak{m})$ the right annihilator ideal of $\mathfrak{m}$ in kT.

Proposition 2. *Let $\tilde{\mathfrak{m}}=\bigcap_{\sigma\in G} kG\cdot\mathfrak{m}^\sigma$. Then there hold the following:*
(1) *$\tilde{\mathfrak{m}}$ is a nilpotent two-sided ideal of kG.*
(2) *$r(\tilde{\mathfrak{m}})=\Sigma_{\sigma\in G}\, r_T(\mathfrak{m})^\sigma kG$.*
(3) *If $\mathfrak{m}$ is a two-sided ideal of KT, then $\tilde{\mathfrak{m}}=\bigcap_{\sigma\in G}\mathfrak{m}^\sigma\cdot kG$.*

Proof. (1) For every $\tau\in G$, there holds $\tilde{\mathfrak{m}}\tau\subset\bigcap_\sigma kG\cdot\mathfrak{m}^{\sigma\tau}=\tilde{\mathfrak{m}}$, and hence $\tilde{\mathfrak{m}}$ is a two-sided ideal. Accordingly, $\mathfrak{m}^2\subset\mathfrak{m}(kG\cdot\mathfrak{m})=\mathfrak{m}\cdot\mathfrak{m}\subset kG\cdot\mathfrak{m}^2$, so that $\tilde{\mathfrak{m}}^t\subset kG\cdot\mathfrak{m}^t$ for every positive integer t. We see therefore $\tilde{\mathfrak{m}}$ is nilpotent.

(2) Since kT is Frobenius, there holds $\mathfrak{m}=l_T(r_T(\mathfrak{m}))$. Then, one will easily see that $kG\cdot\mathfrak{m}=l(r_T(\mathfrak{m})\cdot kG)$ and $kG\cdot\mathfrak{m}^\sigma=l(r_{T^\sigma}(\mathfrak{m}^\sigma)\cdot kG)=l(r_T(\mathfrak{m})^\sigma\cdot kG)$. Hence, $\tilde{\mathfrak{m}}=\bigcap_\sigma l(r_T(\mathfrak{m})^\sigma\cdot kG)=l(\Sigma_\sigma\, r_T(\mathfrak{m})^\sigma\cdot kG)$. Since kG is Frobenius, our assertion is clear by the last.

(3) Using freely the fact that the left annihilator ideal of a two-sided ideal in a symmetric algebra coincides with the right one (Theorem B), we obtain $\tilde{\mathfrak{m}}=l(\Sigma_\sigma\, r_T(\mathfrak{m})^\sigma\cdot kG)=r(\Sigma_\sigma\, l_T(\mathfrak{m})^\sigma\cdot kG)=r(\Sigma_\sigma\, kG\cdot l_T(\mathfrak{m})^\sigma)=\bigcap_\sigma\mathfrak{m}^\sigma\cdot kG$.

Theorem 3. *Let Ω be the set of all Sylow p-subgroups of G. Then, $r(\mathfrak{n})\subset\Sigma_{S\in\Omega}\,\Delta_S\cdot kG$.*

Proof. In proposition 2, we set $T=S\in\Omega$ and $\mathfrak{m}=\mathrm{rad}\,(kS)=I(S)$. Since every Sylow p-subgroup is conjugate to each other, Proposition 2 proves that $\tilde{\mathfrak{m}}=\bigcap_{S\in\Omega} kG\cdot I(S)$ is contained in $\mathfrak{n}$. We obtain therefore $r(\mathfrak{n})\subset r(\tilde{\mathfrak{m}})=\Sigma_{S\in\Omega}\Delta_S\cdot kG$.

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