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DOUBLY TRANSITIVE GROUPS IN WHICH THE MAXIMAL NUMBER OF FIXED POINTS OF INVOLUTIONS IS FOUR

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1. Introduction

Doubly transitive groups in which any involution fixes at most three points have been classified by H. Bender [3], C. Hering [14] and J. King [16], [17]. In this paper we shall prove the following results.

Theorem. *Let G be a doubly transitive group on $\Omega=\{1, 2,\cdots, n\}$. Assume that the maximal number of fixed points of involutions in G is four. Then, if $n\not\equiv 0$* (mod 8), *one of the following holds:*
(a) $n=6$ *and G is* S_6,
(b) $n=10$ *and G is* S_6 *or* $P\Gamma L(2, 9)$,
(c) $n=12$ *and G is* M_{11} *or* M_{12}(*the Mathieu group of degree* 11 *or* 12),
(d) $n=28$ *and G is* $P\Gamma L(2, 8)$,
(e) $n=28$ *and G is* $PSU(3, 3^2)$ *or* $P\Sigma U(3, 3^2)$.

Corollary. *Let G be a doubly transitive group on $\Omega=\{1,2,\cdots, n\}$. If every involution in G fixes four points in Ω, then one of the following holds*:
(*a*) $n=12$ *and G is* M_{11},
(b) $n=28$ *and G is* $P\Gamma L(2, 8)$,
(c) $n=28$ *and G is* $PSU(3, 3^2)$ *or* $P\Sigma U(3, 3^2)$.

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2. Definitions and notations

A permutation group G on Ω is called semi-regular if every non-identity element of G has no fixed point, and G is called regular if G is transitive and semi-regular. If a set X of permutations on Ω fixes a subset Δ of Ω the restriction of X on Δ will be denoted by X^{Δ}. Further we use the following notations which are standard.

S_n = symmetric group of degree n,
A_n = alternating group of degree n,

$GF(q)$ = finite field with q elements,
$P\Gamma L(m, q)$ = m-dimensional projective semi-linear group over $GF(q)$,
$PSL(m, q)$ = m-dimensional projective special linear group over $GF(q)$,
$PSU(3, q^2)$ = 3-dimensional projective special unitary group over $GF(q^2)$,
$P\Sigma U(3, q^2)$ = automorphism group of $PSU(3, q^2)$,
Aut G = automorphism group of a group G,
$G_{ij\cdots r}$ = pointwise stabilizer in G of points $i, j \cdots, r$,
$G_{\{ij\cdots r\}}$ = global stsbilizer in G of the set of points $i, j, \cdots, r$,
$I(X)$ = totality of points fixed by a set X of permutations,
$\alpha_i(x)$ = number of i-cycles of a permutation x,
$o(x)$ = order of a permutation x,
$N_G(X)$ = normalizer of X in G,
$C_G(X)$ = centralizer of X in G,
$\langle X, Y\rangle$ = subgroup generated by X and Y,
$[X, Y]$ = commutator of X and Y,
$|X|$ = cardinality of a set X.

3. Proof of Corollary

It suffices by our theorem to prove that $n \not\equiv 0 (\mathrm{mod}\ 8)$ in the case G satisfies the assumption of Corollary. Assume by way of contradiction that $n \equiv 0$ (mod 8). Then the length of any orbit of an S_2-subgroup P of G on Ω is divisible by eight (see [22], Theorem 3. 4′). Then since a central involution of P fixes a point on Ω, it fixes more than four points on Ω, contrary to the assumption.

4. Proof of Theorem

We begin with some lemmas on permutation groups.

Lemma 1 (J. Alperin [1]). *Let the group G be transitive on $\Omega=\{1, 2, \cdots, n\}$ and let H be a subgroup of the stabilizer G_1. If the conjugates $H^g (g\in G)$ which are contained in G_1 make up k different conjugacy classes of subgroups of G, then the normalizer $N_G(H)$ of H has exactly k orbits on $I(H)$.*

We remark that lemma 1 also holds valid if a subgroup H of G_1 in the above is replaced by a subset K of G_1. In fact, Alperin's proof in [1] does not make use of that H is a subgroup.

Lemma 2 (H. Nagao). *Let X be a semi-regular permutation group on $\Omega = \{1, 2, \cdots, n\}$. If a permutation group A on Ω normalizes X and fixes at least one point, then the order of $C_X(A)$ is not greater than the number of fixed points of A. If X is regular, then the order of $C_X(A)$ equals the number of fixed points of A.*

Proof. Suppose A fixes the point 1. Let x be an element of X and a an element of A. If x takes 1 to i and a takes i to j, then $a^{-1}ax$ takes 1 to j. Since $a^{-1}xa \in X$ and X is semi-regular, $x = a^{-1}xa$ if and only if $j = i$, i.e. $i \in I(a)$. Thus we have $|C_X(A)| \leq |I(A)|$. If X is regular, then for any fixed point i of A there is a unique element of X which takes 1 to i. Hence we have $|C_X(A)| = |I(A)|$.

Lamma 3 (D. Livingstone and A. Wagner [18]). *Let G be k-fold transitive on $\Omega = \{1, 2, \cdots, n\}$, and let H be the stabilizer of k points in Ω. Assume that an S_p-subgroup P of H fixes precisely the given k points. Then for a point i in a minimum P-orbit on $\Omega - I(P)$, $N_G(P_i)^{I(P_i)}$ is k-fold transitive.*

The proof of the above lemma is seen in p. 400-401 of [18].

Lemma 4. *Let Y be a cyclic 2-group which acts regularly on $\Omega = \{1, 2, \cdots, n\}$, and assume that Y normalizes a four group U which is semi-regular on Ω. Then $|Y| = |\Omega| = 4$.*

Proof. Assume that $n = 2^m \gneq 4$, and let Δ_i $(1 \leq i \leq t)$ denote the orbits of of U on Ω. Then since Y permutes $\Delta_1, \Delta_2, \cdots, \Delta_t$ transitively we have $|Y : Y_{\{\Delta_1\}}| = t \geq 2$, from which it follows that $Y_{\{\Delta_1\}}$ centralizes U since $|Y : C_Y(U)| \leq 2$. Then since $Y_{\{\Delta_1\}}{}^{\Delta_1}$ is cyclic and U^{Δ_1} is self-centralizing, we have $|Y_{\{\Delta_1\}}| = |Y_{\{\Delta_1\}}{}^{\Delta_1}| \leqq 2$, which yields that $|Y| = t|Y_{\{\Delta_1\}}| \leq 2t = \frac{n}{2}$, a contradiction.

Now to prove our theorem the following two cases will be treated separately.
Case I. $n \equiv 2 \pmod 4$.
Case II. $n \equiv 0 \pmod 4$.

Case I. Since an involution in G fixing four points is an odd permutation in this case, $N = G \cap A_n$ is a normal transitive subgroup of index two. Furthermore since $|G_1 : N_1| = 2$ is prime to n-1, N_1 is transitive on $\Omega - \{1\}$ and hence N is doubly transitive on Ω. Then by a result of C. Hering [14], either of the following holds:

(i) $n = q+1$ and N contains $PSL(2. q)$,
(ii) $n = 6$ and N is A_6.

In case(ii), $n=6$ and G is S_6. In case(i) an involution in $G - N$ fixing four points acts as a field automorphism. Hence we have $4 = 1 + \sqrt{q}$ and hence $q = 9$. Then $n = 10$ and G is S_6 or $P\Gamma L(2, 9)$.

Case II. We have $n \equiv 4 \pmod 8$ in this case by our assumption that $n \not\equiv 0 \pmod 8$. In particular G contains no regular normal subgroup and hence by a theorem of Burnside ([7], P202) we have

(*) a (unique) minimal normal subgroup H of G is a primitive non abelian simple group.

In what follows we denote by H a (unique) minimal normal subgroup of G throughout.

Lemma 5. *We may assume G contains no normal subgroup of index two.*

Proof. Let G have a normal subgroup N of index two. Then, as seen in Case I, N is also doubly transitive on Ω. If N contains an involution fixing four points we take N in place of G. If N contains no involution fixing four points, the results of H. Bender [3] and C. Hering [14] yield, similarly to Case I, that $n=6$ or 10, which is not the case.

Lemma 6. *An S_2-subgroup of H is not dihedral.*

Proof. This follows from a result of Gorenstein and Walter [11] and a result of Lüneburg ([19], Satz 1).

Lemma 7. *If H has a quasi-dihedral S_2-subgroup and H contains an involution a such that $C_H(a)$ is solvable then $n = 12$ and G is M_{11}.*

Proof. By a result of Alperin, Brauer and Gorenstein [2] H is $PSL(3, 3)$ or M_{11}. If H is $PSL(3, 3)$, since $|\mathrm{Aut}\, PSL(3, 3) : PSL(3, 3)| = 2$, we have G is $PSL(3, 3)$ or Aut $PSL(3, 3)$. But it is easily seen that $PSL(3, 3)$ and Aut $PSL(3, 3)$ have no doubly transitive representation satisfying our assumption. If H is M_{11}, since Aut $M_{11} \simeq M_{11}$, we have G is M_{11} on 12 points.

Lemma 8. *If H has an elementary abelian S_2-subgroup and H contains an involution a such that $C_H(a)$ is solvable then $n = 28$ and G is $P\Gamma L(2, 8)$.*

Proof. By a result of Walter [21] H is $PSL(2, q)$ for a suitable q. Then the result of Lüneburg [19] applies.

Lemma 9. *If H has a wreathed S_2-subgroup of order 32 and H contains an involution a such that $C_H(a)$ is solvable then $n = 28$ and G is $PSU(3, 3^2)$ or $P\Sigma U(3, 3^2)$.*

Proof. By a result of P. Fong [8] H is $PSU(3, 3^2)$, and hence G is $PSU(3, 3^2)$ or $P\Sigma U(3, 3^2)$. It is easy to see that $PSU(3, 3^2)$ and $P\Sigma U(3, 3^2)$ have no other doubly transitive representation than the usual one of degree 28.

Now we consider the following two cases.

Subcase 1. an S_2-subgroup of G_{12} fixes two points.

Subcase 2. an S_2-subgroup of G_{12} fixes four points.

Subcase 1. Let P be an S_2-subgroup of G. Then since $n \equiv 4 \pmod 8$ P has an orbit of length four, say $\Delta = \{1,2,3,4\}$ (see [22], Theorem 3.4′) and then $R = P_1$ is an S_2-subgrou of G_1. We may assume $I(R) = \{1, 2\}$. Then R contains an element b of the form $b = (1)(2)(34)\cdots$. We assume first that any element in R of the form $(1)(2)(34)\cdots$ is an involution. Set $S = R_{34}$. Then $N_G(S)^{I(S)}$ is doubly transitive by Lemma 3 and hence $N_G(S)^{I(S)} = S_4$. Since any element in the coset $b\,S$ is an involution b inverts every element of S and hence S is abelian. On the other hand, since b fixes exactly two points on $\Omega-\Delta$, and since S is semi-regular on $\Omega-\Delta$ Lemma 2 yields $|C_S(b)| = 2$. Thus S must have a unique involution and hence S is cyclic. This implies that $C_G(S)^{I(S)} \geqq A_4$ and any element t of $N_G(S)$ with $t^\Delta \in A_4$ lies in $C_G(S)$. Now since $|P^\Delta| = 8$, P contains an element y of the form $(1324)\cdots$. First assume $o(y) \geqq 8$. Then by Lemma 5 y has an odd number of cycles of the same length on $\Omega-\Delta$. Then since the length of any P-orbit on $\Omega-\Delta$ is of 2-power P fixes the set of the points of a cycle of y, say $(5, 6, \cdots, l)$ on $\Omega-\Delta$ and in particular so does S. Then since $y^4 \in S$ we have $o(y) = |S|, 2|S|$ or $4|S|$. If $o(y) = |S|$ either y or the generator z of S is an odd permutation, contrary to Lemma 5. Also if $o(y) = 2|S|$, since $[y^2, S] = 1$, setting $\Gamma = \{5, 6, \cdots l,\}$ we have $(y^2)^\Gamma = (z^m)^\Gamma$ for some odd integer m. Then $y^2 z^{-m} = (12)(34)\cdots$ fixes every point of Γ, contrary to the assumption of our theorem. Thus we have $o(y) = 4|S|$ and hence $\langle y \rangle$ is a cyclic subgroup of P of index two. Then the structure of P is known (see [10], Theorem 5.4.4) and since P contains a non normal dihedral subgroup R, P is dihedral or quasi-dihedral. We may assume by Lemma 6 that P is quasi-dihedral. Let a be an involution of S. Then since $C_G(a)^{I(a)} \leqq S_4$ and since $C_G(a)_{I(a)}$ hsa a cyclic S_2-subgroup it follows that $C_G(a)$ is solvable. Then Lemma 7 yields that G is M_{11} on 12 points.

Now let $o(y) = 4$. Then since $n \equiv 4 (\mathrm{mod}\ 8)$ y has four fixed points or two 2-cycles on $\Omega-\Delta$ by Lemma 5 and hence y^2 fixes four points on $\Omega-\Delta$. On the other hand $y^2 \in A_4$ implies $[y^2, S] = 1$ and hence $|S| \leq 4$ by Lemma 2. Thus we have $|P \cap H| \leq |P| \leq 32$, and by a result of P. Fong $P \cap H$ is dihedral, quasi-dihedral, elementary or wreathed of order 32. Then Lemma 6, Lemma 7, Lemma 8 or Lemma 9 applies, respectively.

Now we assume that R contains an element x of the form $(1)(2)(34)\cdots$ which is not an involution. Then since $n \equiv 4 (\mathrm{mod}\ 8)$, x has an odd number of cycles of the same length on $\Omega-\Delta$ by Lemma 5. In particular $o(x) \geq 8$, and P fixes the set of the points of a cycle of x, say$(56\cdots k)$ on $\Omega-\Delta$. Then since $x^2 \in S$, $\Gamma = \{5, 6, \cdots k\}$ consists of one or two orbits of S. If Γ consists of two orbits of S then we have $S = \langle x^2 \rangle$ is cyclic. In this case the similar argument to the above applies. Thus we may assume Γ consists of one orbit of S. Then $o(x) = |S| = |\Gamma| \geq 8$ and since Γ is a P-orbit an exponent of P is equal to $o(x)$. In particular P contains a normal four-group U(see [10], P215. Exercise 9). Then since $o(x) \geq 8$, Lemma 4 implies that an involution c in U fixes a point on Γ, and hence we have $|C_S(c)| \leq 4$ by Lemma 2. Then since $|S : C_S(c)| \leq |S : C_S(U)| \leq 2$ it follows that $|S| \leq 8$ and hence $|S| = 8$. If $S \cong Z_4 \times Z_2$, P contains a normal four group which is semi-regular on Γ, contrary to Lemma 4. Also if S is cyclic S contains an odd permutation. Thus S must be quarternion or dihedral of order eight. Now we shall determine the structure of $P \cap H$. If $|P \cap H| = 32$, G contains a normal subgroup of index two, contrary to Lemma 5. Let $|P \cap H| \leq 16$. Then $P \cap H$ is dihedral, quasi-dihedral or elementary. Since the transitivity of H on Ω implies the transitivity of $P\cap H$ on Δ(see [22], Theorem 3.4′) we have $|S\cap H|\leq|R\cap H|\leq 4$. If $S\cap H$ is a four-group, since it is normal in P, Lemma 4 yields a contradiction. Hence $S\cap H$ is cyclic. If $S\cap H=1$ we have $R\cap H=1$ and hence $P\cap H\leq 4$. Also if $S\cap H\neq 1$, H contains an involution a such that $C_H(a)$ is solvable. In either case Lemma 6, Lemma 7 or Lemma 8 applies. Thus we may assume $|P|=|P\cap H|=64$. Now let $\Gamma=\{5, 6, \cdots, 12\}$ be a P-orbit of length eight and put $T=P_5$. Then $|T|=8$, T is faithful on Δ and hence T^{Δ} is dihedral of order eight. T also acts faithfully on S and P is a semi-direct product of S by T, where, as is seen above, S is quarternion or dihedral. Now we shall determine the action of T on S. Let $T=\langle y, b\rangle$ with $o(y)=4$, $o(b)=2$ and $y^b=y^{-1}$. We may assume y and b are of the form

$y=(1324)(5)(6)\cdots,$

$b=(1)(2)(34)(5)(6)\cdots.$

First let S be quarternion with generators w and u. Since S contains three subgroups of order four, T normalizes one of them, say$\langle w\rangle$. Then since T is faithful on S and since Aut $S\cong S_4$ we have $w^y=w$ and $w^b=w^{-1}$. We may now assume $u^y=wu^{-1}$. Then since $\alpha_1(b^{\Omega-\Delta})=2$ implies by Lemma 2 that b centralizes no element of S of order four we have $u^b=wu$ or $w^{-1}u$ and we may assume $u^b=wu$ by taking b^y in place of b, if necessary. Next let S be dihedral with generators z and e where $o(z)=4$, $o(e)=2$ and $z^e=z^{-1}$. Then we have $z^y=z$, $z^b=z^{-1}$ and we may assume $e^y=ez$. If b induces an inner automorphism on S then bf with some f in S centralizes S which is impossible since $(bf)^{\Gamma}$ is an odd permutation. Hence we have $e^b=ez$ or ez^{-1} and we may assume $e^b=ez^{-1}$ by taking b^y in place of b, if necessary. We remark the two 2-groups obtained as above are both isomorphic to an S_2-subgroup of M_{12}. In fact, the two groups are isomorphic by the correspondence; $y\leftrightarrow y$, $b\leftrightarrow b$, $y^2u\leftrightarrow e$ and $w\leftrightarrow z$. We claim that G is M_{12} on 12 points in this case (and consequently the case S is quarternion occurs). In order to prove this it suffices to show that G is isormorphic to M_{12} since M_{12} has no doubly transitive representation other than the usual one of degree 12. We now assume by way of contradiction

(∗∗) G has an S_2-subgroup isomorphic to that of M_{12}, but G is not M_{12}.

Lemma 10. *P contains exactly four involutions of the form* $(1)(2)(34)\cdots$, *and exactly six or two involutions of the form* $(12)(34)\cdots$ *according as S is quarternion or dihedral.*

Proof. This follows immediately from the action of T on S.

Lemma 11. *G has a single conjugacy class of involutions. The elements of order eight of G with a 2-cycle are all conjugate.*

Proof. If G has more than one conjugacy class of involutions then so does

H since $|G:H|$ is odd. Then a result of Brauer and Fong ([6], Corollary 6B) yields H is M_{12} and hence G is M_{12}, contrary to (**). From the action of T on S we observe that bu (or be) is of order eight, bu (or be) is conjugate to all its odd power under $\langle b, y^2\rangle$ and that G_{12} has a quasi-dihedral (or dihedral) S_2-subgroup of order sixteen, from which the last half follows.

Lemma 12. *G_{12} has a single conjugacy class of involutions. A 2-element of G which has both a 2-cycle and a fixed point is of order two or eight. If x is such an element of order eight x has one 2-cycle and two fixed points and the centralizer $C_G(x)$ of x has $\langle x\rangle$ as its S_2-subgroup so that it has a normal 2-complement.*

Proof. By Lemma 11 the involutions of G_{12} are all conjugate under G. Now let a be a central involution of S. Then since $N_G(S)^\Delta = S_4$ we have $C_G(a)^\Delta = S_4$, which implies the first assertion by Lemma 1. If an element of order four in G has a fixed point then it has no 2-cycle by Lemma 5. The last part of Lemma 12 follows from the structure of an S_2-subgroup of M_{12}.

Lemma 13. *Let a be a central involution of S. Then an element d of $C_G(a)$ with $d^\Delta = (ijkl)$ or $(i)(j)(kl)$ fixes at most one orbit of $C_G(a)_\Delta$ on Ω–Δ and an element f of $C_G(a)$ with $f^\Delta = (ij)(kl)$ fixes at most three of them.*

Proof. Let $T = \langle y, b\rangle$ be as avobe. Then $\langle d, C_G(a)_\Delta\rangle$ is conjugate to $\langle y, C_G(a)_\Delta\rangle$ or $\langle b, C_G(a)_\Delta\rangle$ in $C_G(a)$ according as $d^\Delta = (ijkl)$ or $(i)(j)(kl)$ and $\langle f, C_G(a)_\Delta\rangle$ is conjugate to $\langle y^2, C_G(a)_\Delta\rangle$ in $C_G(a)$. Thus to prove Lemma 13 it suffices to show that b and y fix at most one orbit of S on Ω–Δ and y^2 fixes at most three of them since every orbit of $C_G(a)_\Delta$ on Ω–Δ consists of an odd number of S-orbits on Ω–Δ. By Lemma 10 P contains exactly four involutions of the form (1)(2)(34)$\cdots$ and exactly six (or two) involutions of the form (12)(34)$\cdots$. Let $\Gamma_i (1 \leq i \leq l)$ be the orbits of S on Ω–Δ. Then since b has a fixed point on Ω–Δ b fixes some Γ_i, say Γ_1. Then $\alpha_1(b^{\Gamma_1}) = 2$ and the transitivity of S on Γ_1 of length eight imply that any of four involutions in P of the form (1)(2)(34)... has its fixed points on Γ_1 and hence $\langle b, S\rangle$ is semiregular on Ω–Δ–Γ_1. Thus b fixes Γ_1 only and the similar argument yields that y^2 fixes at most three of Γ_i's. Now y also fixes Γ_1 and y has all of its fixed points on Γ_1. (Note that y has no 2-cycle). Assume y fixes another Γ_i, say Γ_2. Then since y^{Γ_2} and u^{Γ_2}(or e^{Γ_2}) are even permutations we have $(yu)^{\Gamma_2}$ (or $(ye)^{\Gamma_2}$) is of order at most four and hence $(yu)^4$ (or $(ye)^4$) $\neq 1$ fixes eight points, contrary to the assumption of our theorem.

Now since $C_G(a)_1{}^\Delta = S_3$, $C_G(a)_1$ contains a subgroup L of index two such that $L^\Delta = A_3$. Put $K = C_G(a)_\Delta$ and let t and s denote the number of orbits of L and K on Ω–Δ, respectively. Then

Lemma 14. $t = 1$ *and* $s = 3$.

Proof. We make use of a similar argument to H. Nagao ([20], p. 336-337).

Since G is doubly transitive by a theorem of Frobenius [9] we have

$$(1) \qquad \sum_{g\in G} \alpha_2(g) = \frac{1}{2}|G|.$$

On the other hand we have

$$(2) \qquad \sum_{g\in G} \alpha_2(g) \geq |G:C_G(a)| \sum_{f\in C_G(a)}' \alpha_2(af) + |G:C_G(x)| \sum_{h\in C_G(x)}' \alpha_2(xh),$$

where $a = w^2$(or x^2), $x = bu$(or be) and the summation in $\sum'$ is taken over all 2-regular elements of $C_G(a)$ or $C_G(x)$. We now calculate the right hand side of (2). First Lemma 12 yields

$$(3) \qquad |G:C_G(x)| \sum_{h\in C_G(x)}' \alpha_2(xh) = \frac{1}{8}|G|.$$

Secondly since a 2-regular element of $C_G(a) - K$ lies in precisely one of four $C_G(a)$ -conjugates of L, we have

$$(4) \qquad \begin{aligned} &|G:C_G(a)| \sum_{f\in C_G(a)}' \alpha_2(af) \\ &= |G:C_G(a)|\{4 \sum_{f\in L-K}' \alpha_2(af) + \sum_{f\in K} \alpha_2(af)\} \\ &= |G:C_G(a)|\{4 \sum_{f\in L}' \alpha_2(af) - 3 \sum_{f\in K}' \alpha_2(af)\} \end{aligned}$$

Now let $\alpha_1{}^*$ denote a permutation character of L or K acting on Ω–Δ. Then we have

$$(5) \qquad \begin{aligned} &\sum_{g\in L} \alpha_1{}^*(g) = t|L| \text{ and} \\ &\sum_{g\in K} \alpha_1{}^*(g) = s|K|. \end{aligned}$$

(see [12], Theorem 16.6.13). If v is a 2-singular element of L (or K), then v^{3m} with some integer m is an involution fixing 1, 2, 3 and 4. Therefore $\alpha_1{}^*(v^{3m}) = 0$ and hence $\alpha_1{}^*(v) = 0$. On the other hand if f is a 2-regular element then $\alpha_1{}^*(f) = \alpha_1{}^*((af)^2) = 2\alpha_2(af)$ since af has no fixed point on Ω–Δ. Thus by (4) and (5) we get

$$(6) \qquad \begin{aligned} |G:C_G(a)| \sum_{f\in C_G(a)}' \alpha_2(af) &= |G:C_G(a)|\{2t|L| - \frac{3}{2}s|K|\} \\ &= \left(\frac{1}{4}t - \frac{1}{16}s\right)|G|. \end{aligned}$$

Substituting (3) and (6) into (2) and comparing (2) with (1) we have

$$(7) \qquad 6 \geq 4t - s \geq t,$$

where the last inequality follows from $3t \geq s$. We now show that t is odd. By the theorem of Frobenius we have

$$\sum_{g\in G}\binom{\alpha_2(g)}{1}\binom{\alpha_1(g)}{1}=\frac{l}{2}|G| \tag{8}$$

for some integer l. On the other hand from Lemma 11 and Lemma 12 it follows

$$\sum_{g\in G}\binom{\alpha_2(g)}{1}\binom{\alpha_1(g)}{1}=|G:C_G(a)|\sum'_{f\in C_G(a)}\binom{\alpha_2(af)}{1}\binom{\alpha_1(af)}{1} \tag{9}$$
$$+|G:C_G(x)|\sum'_{h\in C_G(x)}\binom{\alpha_2(xh)}{1}\binom{\alpha_1(xh)}{1}.$$

Now by Lemma 12 we have

$$|G:C_G(x)|\sum'_{h\in C_G(x)}\binom{\alpha_2(xh)}{1}\binom{\alpha_1(xh)}{1}=\frac{1}{4}|G| \text{ and} \tag{10}$$

$$|G:C_G(a)\sum'_{f\in C_G(a)}\binom{\alpha_2(af)}{1}\binom{\alpha_1(af)}{1} \tag{11}$$
$$=|G:C_G(a)|\left\{4\sum'_{f\in K-L}\binom{\alpha_2(af)}{1}\binom{\alpha_2(af)}{1}+\sum'_{f\in K}\binom{\alpha_1(af)}{1}\binom{\alpha_1(af)}{1}\right\}$$
$$=|G:C_G(a)|\left\{4\sum'_{f\in L-K}\frac{1}{2}\alpha_1{}^*(f)+4\sum'_{f\in K}\frac{1}{2}\alpha_1{}^*(f)\right\}$$
$$=2|G:C_G(a)|\sum'_{f\in L}\alpha_1{}^*(f)$$
$$=\frac{t}{4}|G|$$

Then by (8)(9)(10) and (11) we have $2l=t+1$, which implies t is odd. Then by (7) one of the following holds:

(i) $t=1$ and $s=1$,
(ii) $t=1$ and $s=3$,
(iii) $t=3$ and $s=7$,
(iv) $t=3$ and $s=9$,
(v) $t=5$ and $s=15$.

If $s=1$, since a and b are conjugate in G_{12} by Lemma 12, G_{12} is transitive on $\Omega-\{1, 2\}$. Then the result of J. King yiedls that G is M_{12}, contrary to $(**)$. We remark that the semi-direct product of elementary abelian grou of order 3^3 by SL(3, 3) has no transitive extension. This follows from a result of Hering, Kantor and Seitz [15] or a direct calculation of the number of S_7-subgroups. Also it will be easily seen that (iii), (iv) or (v) in the above conflicts with Lemma 13. For instance assume that $t=3$ and $s=7$. We denote the orbits of $C_G(a)_\Delta$ on $\Omega-\Delta$ by $\sum_i$ $(1\leqq i\leqq 7)$. If $C_G(a)$ has three orbits on $\{\Sigma_i\text{'s}\}$, b would fix at least three of Σ_i's, contrary to Lemma 13. Thus the only case to be considered is that $C_G(a)$ fixes one Σ_i, say Σ_7 and permutes $\Sigma_1, \Sigma_2, \cdots \Sigma_6$, transitively. In this case we consider $C_G(a)_{\{\Sigma_1\}}{}^{\Delta}$. Since $|C_G(a)^{\Delta}:C_G(a)_{\{\Sigma_1\}}{}^{\Delta}| = |C_G(a) : C_G(a)_{\{\Sigma_1\}}| = 6$, it follows that $C_G(a)_{\{\Sigma_1\}}{}^{\Delta}$ is of order four. On the other hand since $C_G(a)_{\{\Sigma_1\}}$ fixes another $\Sigma_i(\neq\Sigma_1)$, $C_G(a)_{\{\Sigma_1\}}{}^{\Delta}$ is a regular four group by Lemma 13. Then we have $C_G(a)^{\Delta} \triangleright C_G(a)_{\{\Sigma_1\}}{}^{\Delta}$ and hence $C_G(a) \triangleright C_G(a)_{\{\Sigma_1\}}$. Then $C_G(a)_{\{\Sigma_1\}}$ fixes all Σ_i's, contrary to Lemma 13. The similar argument eliminates the possiblilities of case (iv) and case (v), completing the proof of Lemma 14.

Lemma 15. *G contans no element which has both a 2-cycle and a 3-cycle.*

Proof. Assume G contains an element z of the form (i j) (klm)$\cdots$. Then the 2-part of z is of order two and $d = z^{3r}$ with some odd integer r is an involution. Clearly z is in $C_G(d)$ and z fixes the 2-cycle (i j) of d. On the other hand since a and d are conjugate in G by Lemma 11, Lemma 14 implies that any element of $C_G(d)$ with a 3-cycle on $I(d)$ fixes no 2-cycle of d. This is a contradiction.

Now set $N = G_{\{12\}}$ and $M = G_{\{123\}}$. Then

Lemma 16. *G_{12} and N have two orbits on $\Omega-\{1, 2\}$ and M has one or two orbits on $\Omega-\{1, 2, 3\}$.*

Proof. By Lemma 14 K has three orbits Σ_1, Σ_2 and Σ_3 of the same length on $\Omega-\Delta$ and any orbit of G_{12} on $\Omega-\{1, 2\}$ is a union of some of $\{3, 4\}$, Σ_1, Σ_2 and Σ_3. If $\{5, 6\}$ is in Σ_1, since a and b are conjugate in G_{12} by Lemma 12 $\{3, 4\}$ and Σ_1 are contained in a G_{12}-orbit and b takes Σ_2 to Σ_3 by Lemma 13. Thus G_{12} has at most two, hence by (**) precisely two orbits of different length on $\Omega-\{1, 2\}$ and hence so does N. The last half is an immediate consequence of Lemma 14.

Lemma 17. *N has two conjugacy classes of involutions and two classes of elements of order eight. In particular involutions of N with 2-cycle (12) and elements of order eight with 2-cycle (12) are all conjugae in N respectively.*

Proof. We regard G as a (transitive) permutation group on the set of the unordered pairs of the points of Ω. Then N is the stabilizer of the pair $\{1, 2\}$ and the involutions in N are all conjugate under G by Lemma 11. On the other hand since $C_G(a)^{I(a)} = S_4$ and since $t = 1$ by Lemma 14 $C_G(a)$ has two orbits on the set of the unordered pairs which a fixes. Hence the first half follows from Lemma 1. Also the elements of order eight in N are all conjugate by Lemma 11, and $C_G(x)$ has two orbits on the set of the unordered pairs which x fixes. Thus the last half follows again from Lemma 1 and the remark after Lemma 1.

Lemma 18. *If c = (12) (3) (4)$\cdots$ is an involution in $C_G(a)$ then $C_M(c)$ is a subgroup of $C_N(c)$ of index four and any 2-regular element of $C_N(c)$ lies in $C_M(c)$.*

Proof. Clearly $C_M(c)$ is contained in N. (In fact, $C_M(c) = C_N(c) \cap G_3$). Since

$C_G(a)_{56}$ contains T, $C_G(a)_{56}$ is transitive on Δ. Then Lemma 14 implies the transitivity of $C_G(a)_{ij}$ on Δ for any 2-cycle (i j) of a. Then since a and c are conjugate, $C_G(c)_{12}$ is transitive on $I(c)$ and so is $C_N(c)$. This implies $|C_N(c) : C_M(c)| = 4$. Now let f be a 2-regular element of N. Then $f^{I(c)} = 1$ by Lemma 15, and hence f is in M.

Now we shall give a final contradiction. Let $\alpha_1{}^*$ and $\beta_1{}^*$ be permutation characters of N and M acting on $\Omega-\{1, 2\}$ and $\Omega-\{1, 2, 3\}$, respectively. By Lemma 16 we have

$$(12) \qquad \sum_{g\in N} \alpha_1{}^*(g) = 2|N| \quad \text{and}$$

$$(13) \qquad \sum_{g\in M} \beta_1{}^*(g) \leqq 2|M|.$$

From (12) and Lemma 16 we get

$$(14) \qquad \sum_{g\in N-G_{12}} \alpha_1{}^*(g) = \sum_{g\in N} \alpha_1{}^*(g) - \sum_{g\in G_{12}} \alpha_1{}^*(g) = 2|N| - |N| = |N|.$$

On the other hand, it follows from Lemma 12 and Lemma 17

$$(15) \qquad \sum_{x\in N-G_{12}} \alpha_1{}^*(x) = |N : C_N(c)| \sum_{f\in C_N(c)}' \alpha_1{}^*(cf) + |N : C_N(v)| \sum_{h\in C_N(v)}' \alpha_1{}^*(vh)$$

where c is an involution of N of the form (12) (3) (4)$\cdots$ and v is an element of N of order eight of the form (12) (3) (4)$\ldots$.

Now if we denote the set of 2-regular elemtns of a group X by X^* Lemma 15 and Lemma 12 imply

$$(16) \qquad |N : C_N(c)| \sum_{f\in C_N(c)}' \alpha_1{}^* (cf) = |N : C_N(c)| \times 4 |C_N(c)*|,$$

and

$$(17) \qquad |N : C_N(v)| \sum_{h\in C_N(v)}' \alpha_1{}^*(vh) = |N : C_N(v)| \times 2 |C_N(v)^*| = \frac{|N|}{4},$$

respectively. Then by (14), (15), (16) and (17) we have $|C_N(c)^*| = \frac{3}{16} | C_N(c) |$ and hence by Lemma 18

$$(18) \qquad |C_M(c)^*| = \frac{3}{4} |C_N(c)|.$$

On the other hand we have

$$(19) \qquad \sum_{g\in M} \beta_1{}^*(g) \geq \sum_{\substack{g\in M \\ g=(12)\cdots}} \beta_1{}^*(g) \geq |M : C_M(c)| \sum_{f\in C_M(c)}' \beta_1{}^*(cf) = |M : C_M(c)| \times 3 |C_M(c)^*|,$$

where the last equality follows from Lemma 15. Then substituting (18) into (19) we have

$$\sum_{g\in M}\beta_\iota^*(g) \geq \frac{9}{4}|M|, \tag{20}$$

which confiicts with (13).

Subcase 2. Let P and R be as in Subcase 1. Then R fixes four points say 1, 2, 3 and 4 on Ω and R is normal in P. By a theorem of Witt ([22], Theorem 9.4), $N_G(R)^{I(R)}$ is doubly transitive and hence we have $N_G(R)^{I(R)}=A_4$ since $N_G(R)^{I(R)}_{12}$ is of odd order. If G_{12} fixes more than two points it fixes four points on Ω and hence by a result of K. Harda [13] and (*), G is $P\Gamma L(2, 8)$ on 28 points. Thus we may assume G_{12} fixes exactly two points on Ω. Then the points 3 and 4 lie in the different G_{12}-orbits of odd length. We denote them by Γ_3 and Γ_4 respectively. Set $W=G_{12}{}^{\Gamma_3}$. Then W is transitive on Γ_3 and for any $i\in\Gamma_3$ $W_i{}^{\Gamma_3}$ is a strongly embedded subgroup of W in a sense of H. Bender [5] and hence by a result of Bender [5] either of the following cases occurs.

(i) An S_2-subgroup of W is cyclic or generalized quarternion,

(ii) W contains PSL(2, q), $S_z(q)$ or PSU(3, q^2) normally with odd index (as a permutation group of usual degree) where q is a suiitable power of 2.

Assume first (i) holds. Then $R\cong R^{\Gamma_3}$ is cyclic or generalized quarternion. If R is cyclic since $N_G(R)^{I(R)}=A_4$ we have $N_G(R)=C_G(R)$. Let $b=(12)\cdots$be an involution which is conjugate to an involution of R. We may assume b normalizes, therefore centralizes R. Then since R is cyclic and since $C_R(b)^{I(b)}\leq A_4$ we have $|C_R(b)|=|C_R(b)^{I(b)}|=2$ and hence $|R|=2$. Then $|P\cap H|\leq |P|\leq 8$, and so Lemma 6 or Lemma 8 applies. Now let R be generalized quarternion. Then involutions of G fixing four points are all conjugate. Note also that P contains a normal four group U in this case, for oteherwise P contains a cyclic subgroup $X=\langle x\rangle$ of index two with $x^{I(R)}\in A_4$ which implies $x^2\in R$ and hence $R=\langle x^2\rangle$, contrary to the assumption. Now since R is generalized quarternion we have $U^{I(R)}\neq 1$. If an involution c of U has a fixed point on $\Omega-I(R)$, since $c^{I(R)}\in A_4$ we have $|I(c)\cap(\Omega\text{-}I(R))|=4$, $C_R(c)^{I(c)}\leq A_4$ and hence $|C_R(c)|=|C_R(c)^{I(c)}|=2$. Then since $|R{:}C_R(c)|\leq|R:C_R(U)|\leq 2$ it follows that $|R|\leq 4$, a contradiction. Thus any involution of U has no fixed point on $\Omega-I(R)$. Now let c be an involution of U with $c^{I(R)}\neq 1$, say $c^{I(R)}=(12)\,(34)\cdots$ and let $b=(12)\cdots$ be an involution fixing four points. We may assume b normalizes R. Then since $\langle c, R\rangle$ and $\langle b, R\rangle$ are S_2-subgroups of $G_{\{1,2\}}$ they are conjugate and hence c also fixes an R-orbit Γ on $\Omega-I(R)$. Set $X=\langle c, R\rangle$ and let $m=|\Gamma|$. Then since $|X|=2m$ and since $X^\Gamma\leq A_m$ X contains an involution d fixing four points on Γ. Then since R is regular on Γ it follows from Lemma

2 that $|C_R(d)| = 4$ and hence $C_R(d)^{I(d)} \cong C_R(d) \cong Z_4$, a contradiction.

Now assume (ii) holds. Then W_3 is 2-closed and R is transitive on $\Gamma_3-\{3\}$. Let a be an involuiton of R, assume $a^{\Gamma_3} = (3)(56)\cdots$and let $b = (5)(6)\cdots$ be an involution commuting with a. Then we have $b^{I(R)} \in A_4$. If $b^{I(R)} = (12)(34)$, then b normalizes G_{12} and hence permutes Γ_3 to Γ_4, which is impossible. Thus we may assume $b^{I(R)} = (14)(23)$. Then since b normalizes G_{1234} b also normalizes an S_2-subgroup Q of G_{1234}. Then since Q is also an S_2-subgroup of G_{12}, Q^{Γ_3} is an S_2-subgroup of W_3, and hence we have $Q^{\Gamma_3} = R^{\Gamma_3}$ since W_3 is 2-closed. Thus $\Gamma_3-\{3\}$ is an orbit of Q and so b fixes $\Gamma_3-\{3\}$ as a whole. Then b has two or four fixed points on $\Gamma_3-\{3\}$. Suppose first b fixes two points on $\Gamma_3-\{3\}$. Then by a result of Zassenhaus ([23], Satz 5) Q has an cylic subgroup of index two. Then by the structure of S_2-subgroups of the groups in (ii) we have Q is a four group, and hence $|P|=16$. If $|P \cap H| = 16$, $P \cap H$ is dihedral, quasi-dihedral or elementary of order sixteen, while as is easily seen, P $(= P \cap H)$ contains a normal four group and an element of order four. This is a contradiction. Thus we have $|P \cap H| \leq 8$ and then Lemma 6 or Lemma 8 applies. Finally supose b fixes four points on $\Gamma_3-\{3\}$. Put $\Sigma=\{2\} \cup \Gamma_3$ and $L=\langle b, G_{12}\rangle$. Then $L \leq G_{\{\Sigma\}}$ and L^{Σ} is transitive, in particular, $L_2{}^{\Sigma}$ and $L_5{}^{\Sigma}$ are conjugate. Clearly Q^{Σ} is an S_2-subgroup of $L_2{}^{\Sigma}$ and Q^{Σ} is semi-regular on $\Sigma-\{2, 3\}$, while $L_5{}^{\Sigma}$ contains an involution b^{Σ} fixing four points, which is a contradiction.

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