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A NOTE ON THE CAPACITY OF RECURRENT MARKOV CHAINS

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1. Introduction

Let P be an irreducible recurrent transition function on a denumerable space S with strictly positive invariant measure α. For a kernel A, a function f and a measure μ on S, we define $Af(x)=\sum_y A(x, y)f(y)$, $\mu A(x)=\sum_y \mu(y)A(y, x)$, $\mu\cdot f=\sum_y \mu(y)f(y)$ and $\mu\cdot 1=\sum_y \mu(y)$. A kernel A on S is called a *weak potential kernel* if Af is bounded and satisfies $(P-I)Af=f$ for all null charge f. A left (right) equilibrium potential for a weak potential kernel A and a set E is the potential $\nu=\mu A$ ($g=Af$) satisfying $\mu=0$ ($f=0$) on $S-E$, $\mu\cdot 1=1$ ($\alpha\cdot f=1$) and $\nu=$constant $\times\alpha$($g=$constant) on E. The constant is denoted by $C(E)$ ($C^*(E)$) and is called the *left* (*right*) *capacity* of the set E with respect to (α, A). Its charge $\mu(f)$ is called the *left* (*right*) *equilibrium charge*. The existence of the equilibrium charge and various properties concerning its capacity were discussed in [3], [5] and [6].

In this paper we shall be concerned with the probablistic representation of the equilibrium charge and its capacity for some weak potential kernel. The argument depends on the notion of the approximate chain introduced by Hunt [2]. For a given transient transition function Q on S, a random chain (X, a, b) on a σ-finite measure space $(\Omega, \boldsymbol{B}, \boldsymbol{P})$ is called an *approximate Q-chain* if for every finite set E, (X, a, b) is reduced to a Q-chain by the hitting time σ_E of (X, a, b) for E and satisfies $\boldsymbol{P}[\sigma_E=-\infty]=0$. As was remarked by Hunt, this definition is equivalent to his original definition. In the following the approximate chains are denoted by (X, a, b) and distinguished only by the measure $\boldsymbol{P}$. Particularly if $a(\omega)=0$ a.e. and $\boldsymbol{P}[X_0=z]=I(x, z)$ then we shall use $\boldsymbol{P}_x$ in place of $\boldsymbol{P}$. Moreover the hitting time of (X, a, b) for a finite set E is denoted by σ_E. It is known that for any Q-excessive measure η, there corresponds an approximate Q-chain on $(\Omega, \boldsymbol{B}, \boldsymbol{P})$ satisfying $\eta(x)=\boldsymbol{E}[\sum_{a(\omega)\leqq n\leqq b(\omega)} I_{\{x\}}(X_n(\omega))]$ where I_E is the indicator function of the set E and $\boldsymbol{E}$ is the expectation with respect to $\boldsymbol{P}$. We shall call (X, a, b) ($\boldsymbol{P}$) the approximate Q-chain (measure) *canonically associated* with η. It was shown by T. Watanabe [4] that the transient capacity, in the sense of Kemeny and Snell [3], of a finite set E is equal to $\boldsymbol{P}[\sigma_E<\infty]$ where $\boldsymbol{P}$ is the measure canonically associated with α. We shall derive a similar representation in the recurrent case. From the representation, similar result to [3] concerning the capacity and the equilibrium potential for a wide class of weak potential kernels follows easily.

2. Probablistic representation of the capacity

In the following E denotes a finite subset of S and c a fixed state of S, and for simplicity we shall assume that $\alpha(c)=1$. Define ${}_cP(x, y)=P(x, y)-I(x, c)$ $P(c, y)$, ${}^cP(x, y)=P(x, y)-P(x, c)I(c, y)$, ${}_cG(x, y)=\sum_{n=0}^{\infty} {}_cP^n(x, y)$ and ${}^cG(x, y)=$ $\sum_{n=0}^{\infty} {}^cP^n(x, y)$. Since c is accessible from x by the Markov chain with transition function P, it follows that ${}_cG(x, c)=1$. Also by Derman-Harris relation ([5]) ${}^cG$ $(c, x)=\alpha(x)$. The weak potential kernel A is represented as

$$(2.1) \qquad \begin{aligned} A(x, y) &= -{}_cG(x, y)+h(x)\ \alpha(y)+\pi(y)+I(c, y) \\ &= -{}^cG(x, y)+(h(x)+I(x, c))\alpha(y)+\pi(y), \end{aligned}$$

where h and π are a function and a measure on S respectively. As is shown in the proposition 2.1, the equilibrium potential is uniquely determined. But for the existence it is necessary to restrict the kernel A as follows.

In this paper, following [3], we shall restrict to the case that the *kernel A is representable as* (2.1) *by h and π satisfying* (*i*) $h(c)=\pi(c)=0$, (*ii*) $\pi(\cdot)+I(c, \cdot)$ *is a ${}_cP$-excessive measure and* (*iii*) $h(\cdot)+I(\cdot, c)$ *is a ${}^cP$-excessive function.*

Proposition 2.1. *Left* (*right*) *equilibrium potential and hence left* (*right*) *capacity with respect to* (α, A) *is uniquely determined.*

Proof. Let $g_1=Af_1$ and $g_2=Af_2$ be two equilibrium potentials of the set E. Then, since f_1-f_2 is a null charge supported in E and $g_1-g_2=$constant on E, $f_1=f_2$ follows at once from the semi-reinforced maximum principle [4][1]. The uniqueness of the left equilibrium potential follows similarly.

Corollary. *If S is the set of all integers of dimension* 1 *or* 2 *and if* $P(x, y)=$ $P(0, y-x)$, $A(x, y)=A(0, y-x)$ *for all x, y in S then* $C(E+x)=C(E)$, *where* $E+x=\{y+x; y\in E\}$.

Proof. Let μ be the left equilibrium charge of the set E. Under the condition of the corollary, α is equal to 1 from [6] and hence for any $y\in E$

1) Semi-reinforced maximum principle: For every real number m and null charge f, if $Af\leqq m$ on $\{f>0\}$ then $Af\leqq m-f^-$ on S.

$$C(E) = \mu A(y) = \sum_{z \in E} \mu(z+x-x) A(z+x, y+x)$$
$$= \sum_{z \in E+x} \mu_x(z) A(z, y+x),$$

where $\mu_x(z)=\mu(z-x)$. Since the measure μ_x is supported in $E+x$ and $\mu_x \cdot 1=1$, the right hand side of the above equality is equal to $C(E+x)$.

Now we shall construct the equilibrium charge with respect to (α, A). Since $\pi(\cdot)+I(c,\cdot)$ and α are ${}_cP$-excessive, there are the approximate ${}_cP$-chains canonically associated with them. Let $\boldsymbol{P}^\pi$ and $\boldsymbol{P}^\alpha$ be the measure canonically associated with them, respectively. Define

$$\mu_E^\pi(x) = \begin{cases} \boldsymbol{P}^\pi[X_{\sigma_E} = x;\ \sigma_E < \infty] & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}$$

and μ_E^α similarly.

Proposition 2.2. *For any $x \in S$*

(2.2) $\pi(x)+I(c, x) \geqq \mu_E^\pi\, {}_cG(x)$ *and* $\alpha(x) \geqq \mu_E^\alpha\, {}_cG(x)$.
In particular, if $x \in E$ then

(2.3) $$\pi(x)+I(c, x) = \mu_E^\pi\, {}_cG(x) \text{ and } \alpha(x) = \mu_E^\alpha\, {}_cG(x).$$

Proof. From the definition of $\boldsymbol{P}^\pi$, for any $x \in S$

$$\pi(x)+I(c, x) = \boldsymbol{E}^\pi[\sum_{a(\omega) \leqq n \leqq b(\omega)} I_{\{x\}}(X_n(\omega))]$$
$$\geqq \boldsymbol{E}^\pi[\sum_{\sigma_E(\omega) \leqq n \leqq b(\omega)} I_{\{x\}}(X_n(\omega))] = \mu_E^\pi\, {}_cG(x).$$

where $\boldsymbol{E}^\pi$ is the expectation with respect to $\boldsymbol{P}^\pi$. In particular when $x \in E$, the equality holds.

Proposition 2.3. *For any finite set E, $0 \leqq \mu_E^\alpha \cdot 1 \leqq 1$ and $0 < \mu_E^\alpha \cdot 1 \leqq 1$. In particular, if $c \in E$ then $\mu_E^\pi \cdot 1 = \mu_E^\alpha \cdot 1 = 1$.*

Proof. If $c \in E$, by letting $x=c$ in (2.3) we obtain the equalities. In the general case, take a finite set F containing $E \cup \{c\}$. It then follows that $\mu_E^\pi \cdot 1 \leqq \mu_F^\pi \cdot 1 = 1$ and $\mu_E^\alpha \cdot \leqq \mu_F^\alpha \cdot 1 = 1$. The strict positivity of $\mu_E^\alpha \cdot 1$ follows from

$$0 < \alpha(x) \leqq (\mu_E^\alpha \cdot 1)\ {}_cG(x, x) \qquad \text{for } x \in E.$$

Set $K(E) = (1-\mu_E^\pi \cdot 1)/\mu_E^\alpha \cdot 1$, $\nu_E(x) = K(E)\ \mu_E^\alpha(x)$ and $C(E) = (\nu_E + \mu_E^\pi) \cdot h - K(E)$.

Theorem 2.1. *The measure $\mu_E^\pi + \nu_E$ is the left equilibrium charge and $C(E)$ is the left capacity of E with respect to (α, A).*

Proof. Obviously $\mu_E^\pi + \nu_E$ is a measure supported in E and satisfying

$(\mu_E^\pi+\nu_E)\cdot 1=1$. If $x\in E$ then

$$\begin{aligned}(\mu_E^\pi+\nu_E)A(x) &= -\nu_E\ {}_cG(x)+[(\mu_E^\pi+\nu_E)\cdot h]\alpha(x) \\ &= [-K(E)+(\mu_E^\pi+\nu_E)\cdot h]\alpha(x) = C(E)\ \alpha(x),\end{aligned}$$

by (2.3).

Next, we shall construct the right equilibrium charge with respect to (α, A). Since $h('x)=h(x)+I(x, c)$ is a ${}^cP$-excessive function, ${}^cP^{h'}$ defined by ${}^cP^{h'}(x, y)={}^cP(x, y)h'(y)/h'(x)$ if $h'(x)>0$, $=0$ otherwise, is also a transient transition function on S. Let $\boldsymbol{P}_x^h$ be the measure defining the Markov chain starting at x with transition function ${}^cP^{h'}$, $e_E^h(x)$ be the escape probability of the Markov chain from the set E, that is,

$$e_E^h(x)=\begin{cases}\boldsymbol{P}_x^h\ [X_n\notin E \text{ for all } n\geqq 1] & \text{if } x\in E\\ 0 & \text{otherwise}\end{cases}$$

and τ^E be the *last exit time* of the approximate chain (X, a, b) from E, that is,

$$\tau^E(\omega)=\begin{cases}\sup\{n;\ X_n\in E\} & \text{if } X_n\in E \text{ for some } n\\ -\infty & \text{otherwise.}\end{cases}$$

Define $\boldsymbol{P}_x^1$ and e_E^1 as above for $h'=1$.

Proposition 2.2′. *For any* $x\in S$

(2.4) $$h'(x)\geqq {}^cGf_E(x) \text{ and } 1\geqq {}^cGe_E^1(x)$$

In particular, if $x\in E$ *then*

(2.5) $$h'(x)={}^cGf_E(x) \text{ and } 1={}^cGe_E^1(x),$$

where $f_E(x)=h'(x)e_E^h(x)$.

Proof. Since $\boldsymbol{P}_x^h[\tau^E>-\infty]={}^cG^{h'}\ e_E^h(x)$ from [2] and E is a transient set with respect to $\boldsymbol{P}_x^h$, we have $1\geqq {}^cG^{h'}\ e_E^h(x)$ for $x\in S$. From this, (2.4) follows at once. (2.5) follows similarly.

Proposition 2.3′. *For any finite set* E, $0\leqq\alpha\cdot f_E\leqq 1$ *and* $0<\alpha\cdot e_E^1\leqq 1$. *In particular, if* $c\in E$ *then* $\alpha\cdot f_E=\alpha\cdot e_E^1=1$.

The proof is similar to the proposition 2.3 by noting $\boldsymbol{P}_c^h[\tau^E>-\infty]\leqq\boldsymbol{P}_c^h[\tau^F>-\infty]$ for $E\subset F$.

Set $K^*(E)=(1-\alpha\cdot f_E)/\alpha\cdot e_E^1$, $g_E(x)=K^*(E)e_E^1(x)$ and $C^*(E)=\pi\cdot(f_E+g_E)-K^*(E)$.

Theorem 2.1′. *The function* f_E+g_E *is the right equilibrium charge and* $C^*(E)$ *is the right capacity of* E *with respect to* (α, A).

The proof is similar to the theorem 2.1.

Proposition 2.4. $\mu_E^\alpha \cdot 1 = \alpha \cdot e_E^1$ *and* $\mu_E^\alpha \cdot h = \alpha \cdot f_E$.

Proof. Let ${}_cP^E(x, y)$ (${}^cP^E(x, y)$) be the probability of the Markov chain starting at x, with transition function ${}_cP$ (${}^cP$) and returning to E at y if $x \in E$ and $y \in E$, $=0$ otherwise. From (2.3) we obtain that $\mu_E^\pi(x) = (\pi + I(c, \cdot))\,(I - {}_cP^E)(x) = \pi(I - {}_cP^E)(x) + I(c, x)$ and $\mu_E^\alpha(x) = \alpha(I - {}_cP^E)(x)$. Similarly $f_E(x) = (I - {}^cP^E)h'(x) = (I - {}^cP^E)h(x) + I(x, c)$ and $e_E^1(x) = (I - {}^cP^E)1(x)$ follow from (2.5). Since ${}_cP^E(x, y) = {}^cP^E(x, y)$ for $x \neq c$ and $y \neq c$, the proposition follows.

Theorem 2.2. *For any finite set E, $C(E) = C^*(E)$.*

Proof. From (2.3) and (2.5)

$$\begin{aligned}\pi \cdot f_E &= (\mu_E^\pi\, {}_cG) \cdot f_E - f_E(c) \\ &= \mu_E^\pi \cdot ({}^cGf_E) + f_E(c)\,(\mu_E^\pi \cdot 1 - 1) - \mu_E^\pi(c)\alpha \cdot f_E \\ &= \mu_E^\pi \cdot h + f_E(c)\,(\mu_E^\pi \cdot 1 - 1) + \mu_E^\pi(c)(1 - \alpha \cdot f_E).\end{aligned}$$

Since $f_E(c)(\mu_E^\pi \cdot 1 - 1) = \mu_E^\pi(c)(1 - \alpha \cdot f_E) = 0$ from proposition 2.3 and proposition 2.3′, $\pi \cdot f_E = \mu_E^\pi \cdot h$ holds. Similarly $\mu_E^\pi \cdot 1 = \pi \cdot e_E^1 + \mu_E^\pi(c)$ and $\alpha \cdot f_E = \mu_E^\pi \cdot h + f_E(c)$ hold. Then by the definition of the capacity

$$\begin{aligned}C(E) &= \mu_E^\pi \cdot h + \nu_E \cdot h - K(E) = \mu_E^\pi \cdot h \\ &+ K(E)(\alpha \cdot f_E - 1 - f_E(c)) = \mu_E^\pi \cdot h - K(E)K^*(E)\alpha \cdot e_E^1 \\ &- K(E)f_E(c) = \mu_E^\pi \cdot h - K(E)K^*(E)\alpha \cdot e_E^1.\end{aligned}$$

Similarly $C^*(E) = \pi \cdot f_E - K^*(E)K(E)\mu_E^\alpha \cdot 1$. Hence from proposition 2.4, $C(E) = C^*(E)$ follows.

Theorem 2.3. $C(E) = \boldsymbol{E}^\pi[h(X_{\sigma_E})] - \dfrac{1}{\boldsymbol{P}^\alpha[\sigma_E < \infty]} \boldsymbol{P}^\pi[\sigma_E = \infty]\, \boldsymbol{P}_c^h[\tau^E = -\infty]$.

Proof. Obviously $\mu_E^\pi \cdot h = \boldsymbol{E}^\pi[h(X_{\sigma_E})]$. If $c \notin E$ then

$$\begin{aligned}\frac{1 - \mu_E^\pi \cdot 1}{\mu_E^\alpha \cdot 1}(\mu_E^\alpha \cdot h - 1) &= -\frac{1}{\mu_E^\alpha \cdot 1}(1 - \mu_E^\pi \cdot 1)\,(1 - \alpha \cdot f_E) \\ &= -\frac{1}{\boldsymbol{P}^\alpha[\sigma_E < \infty]} \boldsymbol{P}^\pi[\sigma_E = \infty] \boldsymbol{P}_c^h[\tau^E = -\infty].\end{aligned}$$

Generally $C(E)$ is not necessarily non-negative, but if $c \in E$ then $C(E)$ is non-negative since in this case the second term of the left hand side in theorem 2.3 is equal to 0. From this fact and the corollary to the proposition 2.1, it follows that $C(E)$ is always non-negative when P is a random walk, that is, if S, P and A satisfy the condition of the corollary to the proposition 2.1.

Theorem 2.4. *$C(E)$ is a non-negative, monotone increasing and alternating* *set function on the class of finite subsets of S containing c.*

Proof. For every finite set E containing c, $C(E)=\boldsymbol{E}^{\pi}[h(X_{\sigma_E})]=\mu_E^{\pi}\cdot h$. Since h' is a ${}^cP$-excessive function, the inequality

$$ {}^cH_{E\cup F}h'+{}^cH_{E\cap F}h'\leqq {}^cH_Eh'+{}^cH_Fh' $$

holds [1]. Hence

$$ {}_cH_{E\cup F}h+{}_cH_{E\cap F}h\leqq {}_cH_Eh+{}_cH_Fh, $$

where ${}^cH_E$ and ${}_cH_E$ are the réduite defined by ${}^cP$ and ${}_cP$ respectively. The above inequality combined with the equality $\mu_E^{\pi}=\mu_F^{\pi}\ {}_cH_E$ $(E\subset F)$, shows that

$$ \begin{aligned} C(E\cup F)+C(E\cap F) &= \mu_{E\cup F}^{\pi}({}_cH_{E\cup F}h+{}_cH_{E\cap F}h) \\ &\leqq \mu_{E\cup F}^{\pi}({}_cH_Eh+{}_cH_Fh) = C(E)+C(F). \end{aligned} $$

Also, since ${}^cH_Fh'\leqq h'$ and hence ${}_cH_Fh\leqq h$, it follows that

$$ C(E) = \mu_F^{\pi}\ {}_cH_E\cdot h\leqq \mu_F^{\pi}\cdot h = C(F) $$

for $E\subset F$.

REMARK. For the monotony of $C(E)$, it is not necessary to assume that each set E contains c.

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