Suzuki, S.
Osaka J. Math.
7 (1970), 375-396

ON LINEAR GRAPHS IN 3-SPHERE

SHIN'ICHI SUZUKI

(Received February 12, 1970)
(Revised March 7, 1970)

0. Introduction

Throughout this paper we work in the piecewise-linear category, consisting of simplicial complexes and piecewise-linear maps. By $(P\subset M)$ we denote a pair of complexes such that M has an arbitrary but fixed orientation if M is orientable and P is embedded as a subcomplex in M. $\boldsymbol{K}$ denotes a set of all connected finite 1-dimensional complexes. Then, for $K\in\boldsymbol{K}$ we will call $(K\subset S^3)$ a *linear graph,* or simply *graph,* in a 3-dimensional sphere S^3.

The purpose of the paper is to classify $\{(K\subset S^3)\,|\,K\in\boldsymbol{K}\}$ by an equivalence relation, which we will call a neighborhood-congruence. We will introduce a operation $\vee$ of composition in $\{(K\subset S^3)\,|\,K\in\boldsymbol{K}\}$ so that neighborhood-congruence classes of graphs form a commutative semi-group, and give the following as generalization of knots [14] and links [8].

Theorem 3.12. *In the semi-group of all neighborhood-congruence classes of linear graphs, factorization is unique.*

As an immediate application we can discribe socalled knotted solid tori of genus n in the 3-sphere S^3.

1. Definitions and notations

Throughout the paper, ∂M and ϑM denote the boundary and the interior of a manifold M, respectively. For a pair $(P\subset M)$, by $N(P;\, M)$ we denote a regular neighborhood of P in M, that is, we construct its second derived and take the closed star of P, see [9] and [12]. For any non-negative integer n, $\boldsymbol{K}(n)$ denotes a set of all connected finite 1-dim. complexes whose 1-dim. Betti number is n.

First let us explain an usual equivalence of pairs, see [2], [6].

1.1. DEFINITION. Two pairs $(P\subset M)$ and $(P'\subset M')$ are *congruent* iff there is a homeomorphism $h: M\to M'$ such that $h(P)=P'$ and h is orientation preserving if M is oriented.

Then it is trivial that the relation of congruence is an equivalence relation. We denote a congruence class of $(P\subset M)$ by $\langle P\subset M\rangle$, so $(P\subset M)$ is a representative of $\langle P\subset M\rangle$. In particular, congruent graphs are said to be of the *same type*, and each congruence class of graphs is a *graph type*. A graph type of $(K\subset S^3)$ is denoted by $\lambda=\langle K\subset S^3\rangle$.

Note: Two concepts of a graph and a graph type are essentially different. But little distinction will be drawn between them. In the following, sometimes one representative (i.e. graph) is convenient, sometimes another.

Next, we will give another equivalence, which is stated in §0.

1.2. DEFINITION. Two pairs $(P\subset M)$ and $(P'\subset M')$ are *neighborhood-congruent* (abbreviated by *N-congruent*), denoted by $(P\subset M)\overset{N}{\simeq}(P'\supset M')$, iff $(N(P;\ M)\subset M)$ and $(N(P';\ M')\subset M')$ are congruent.

Note that if $(P\subset M)$ and $(P'\subset M')$ are congruent, then $(P\subset M)\overset{N}{\simeq}(P'\subset M')$. So, the N-congruence can be defined for congruence classes of pairs, and sometimes we denote $\langle P\subset M\rangle\overset{N}{\simeq}\langle P'\subset M'\rangle$.

By the uniqueness of regular neighborhoods [9, Th. 2] and [13, Th. 1], the above definition does not depend upon the triangulations of M and M', and the regular neighborhoods $N(P;\ M)$ and $N(P';\ M')$. So, the relation of N-congruence is an equivalence relation, and we denote a N-congruence class of $(P\subset M)$ (or $\langle P\subset M\rangle$) by $[P\subset M]$. In particular, N-congruence classes of graphs are said to be the *same N-graph type*, and a N-graph type of $(K\subset S^3)$ (or $\lambda=\langle K\subset S^3\rangle$) will be denoted by $\Lambda=[K\subset S^3]$.

1.3. REMARK. By using an isotopy of pairs $(P\subset M)$, $(P'\subset M')$ and $(N(P;\ M)\subset M)$, $(N(P';\ M')\subset M')$, we can introduce the similar equivalence relations of 1.1 and 1.2, respectively, see [7], [9, p. 727]. But since an orientation preserving homeomorphism of S^3 onto itself is isotopic to the identity, for pairs $(P\subset S^3)$ the classification problems by the isotopy are the same as that by the orientation preserving homeomorphism.

For future reference we record the followings.

1.4. *Let* $(K\subset S^3)$ *be a graph. Then,* $N(K;\ S^3)$ *is a solid torus*1) T_n *of genus* n *provided that* $K\in\mathbf{K}(n)$.

1.5. *If* $K,\ K'\in\mathbf{K}$ *and* $(K\subset S^3)\overset{N}{\simeq}(K'\subset S^3)$, *then* $K,\ K'\in\mathbf{K}(n)$ *for some* n.

1.6. *Let* $(T_n\subset S^3)$ *be a solid torus of genus* n *in* S^3. *Suppose that* $K(\subset S^3)$ *and* K' $(\subset S^3)$ *are spines*2) *of* T_n, *then* $(K\subset S^3)\overset{N}{\simeq}(K'\subset S^3)$.

To characterize the N-graph types, it is convenient to introduce special

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1) Henkelkörper vom Geschlechte n, see [16], p. 219.

2) See [9], pp. 726–7.

linear graphs.

1.7. *n-leafed rose.* Let $\boldsymbol{C}(n)$ be a subset of $\boldsymbol{K}(n)$ whose elements are homeomorphic to the union of n topological circles $S_1^1, \cdots, S_n^1$ and a n-forest Ω joined as illustrated in Fig. 1. Especially, $\boldsymbol{C}(0)$ is considered to be one point ω. For brebity, we denote the vertices of Ω by $\omega, \omega_1, \cdots, \omega_n$ as shown in Fig. 1, and especially call the point ω (and its image) the node. Let $\boldsymbol{C}=\bigcup_{n\geq 0}\boldsymbol{C}(n)$. Of course, $\boldsymbol{C}(n)\subset\boldsymbol{K}(n)$ and $\boldsymbol{C}\subset\boldsymbol{K}$, and therefore, $\{(C\subset S^3)\,|\,C\in\boldsymbol{C}\}\subset\{(K\subset S^3)\,|\,K\in\boldsymbol{K}\}$. For $C\in\boldsymbol{C}(n)$, we will call a graph $(C\subset S^3)$ a *n-leafed rose,* or simply *rose*, and a graph type $\theta=\langle C\subset S^3\rangle$ a *rose type,* and a N-graph type $\Theta=[C\subset S^3]$ a *N-rose type.*

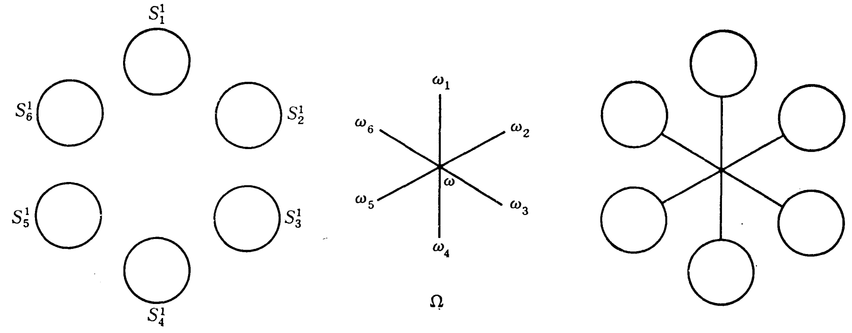

Fig. 1

1.8. *Knotted Solid Tori.* Let $\boldsymbol{T}(n)$ be a set of solid tori of genus n, and let $\boldsymbol{T}=\bigcup_{n\geq 0}\boldsymbol{T}(n)$. For $T\in\boldsymbol{T}$, a congruence class $\tau=\langle T\subset S^3\rangle$ of $(T\subset S^3)$ will be called a ***knot type*** of a solid torus. Note that two solid tori $(T\subset S^3)$ and $(T'\subset S^3)$ are congruent if and only if $(T\subset S^3)$ and $(T'\subset S^3)$ are N-congruent.

Since each $(T\subset S^3)$ of genus n has a n-leafed rose $C(\subset S^3)$ as its spine, we have the followings as consequences of 1.4, 1.5 and 1.6:

1.9. Proposition. *For any* $\Lambda=[K\subset S^3]$, *there is a representative* $\theta=\langle C\subset S^3\rangle$.

1.10. Proposition. *There are set identifications*

$$\begin{aligned}\{\Lambda=[K\subset S^3]\,|\,K\in\boldsymbol{K}\}&=\{\Theta=[C\subset S^3]\,|\,C\in\boldsymbol{C}\}\\&=\{\tau=\langle T\subset S^3\rangle\,|\,T\in\boldsymbol{T}\}.\end{aligned}$$

2. Knotting-genus of N-graph type

In this section we will introduce the knotting-genus of a N-graph type as generalization of genera of knots [15] and links.

2.1. *Spanning-surface for a link.* Let $L=(S_1^1\cup\cdots\cup S_n^1\subset S^3)$ be a (non-oriented) link with n components, that is, L is an union of mutually disjoint (non-oriented) 1-spheres $S_1^1, \cdots, S_n^1$ in S^3. Let $F_1, \cdots, F_u$ be mutually disjoint orientable surfaces in S^3. Then, a system of surfaces $F_L = F_1 \cup \cdots \cup F_u$ is said to be a *spanning-surface* for L, or L bounds a system of surfaces $F_L = F_1 \cup \cdots \cup F_u$ iff

(i) $\partial F_L = \{\partial F_1 \cup \cdots \cup \partial F_u\} = \{S_1^1 \cup \cdots \cup S_n^1\}$,
(ii) every component of F_L has non-void boundary, and
(iii) there are mutually disjoint u 3-cells $Q_1, \cdots, Q_u$ in S^3 such that $\partial Q_i \supset F_i$, $i=1, \cdots, u$.

Since Seifert's construction [15] of a surface spanning a given knot can be readily extended to a link, a spanning-surface for L always exists. Condition (iii) cannot be removed, as can be seen by the boundary links [6].

To a spanning-surface $F_L = F_1 \cup \cdots \cup F_u$ for L, we associate a pair $(u, v) \in N^* \times N^*$ of non-negative integers, where[3] $v = \sum_{i=1}^{u} g(F_i)$. On the other hand, we define a total order $<$ (or $>$) in $\{(u, v)\}$ ($\subset N^* \times N^*$) as follows:

(2.2) $(u, v) < (u', v')$ *if* $u > u'$ *or if* $u = u'$ *and* $v < v'$.

Then, for a link $L = (S_1^1 \cup \cdots \cup S_n^1 \subset S^3)$ we can define an invariant (u, v) as follows:

2.3. DEFINITION. L is of *knotting-genus* (u, v) iff there exists a spanning-surface F_L for L with (u, v), and for any spanning-surface F_L' for L with (u', v'), $(u, v) \leq (u', v')$.

Since $1 \leq u \leq n$ and $0 \leq v < \infty$, it is clear that the knotting-genus (u, v) is an invariant of a link type.

Using the spanning-surface and knotting-genus for a link, we will define a spanning-surface and knotting-genus for a rose type as follows:

2.4. *Spanning-surface for a rose type.* Let $\theta = \langle C \subset S^3 \rangle$ be a n-leafed rose type and $(C \subset S^3)$ be a representative of θ. Let $F_\theta = F_1 \cup \cdots \cup F_u$ be a system of orientable surfaces in S^3. F_θ is said to be a *spanning-surface* for θ, iff F_θ satisfies the conditions (i), (ii) in 2.1 for a non-oriented link $L = (\overline{C - \Omega} \subset S^3)$ and additional conditions below:

(iii)′ there are u 3-cells $Q_1, \cdots, Q_u$ in S^3 such that $\partial Q_i \supset F_i$ $\partial Q_i \cap C = \partial Q_i \cap \Omega = \omega$ and $Q_i \cap Q_j = \partial Q_i \cup \partial Q_j = \omega$ for $i \neq j$ and $i, j = 1, \cdots, u$.
(iv) $F_\theta \cap \Omega = \partial F_\theta \cap \Omega = \omega_1 \cup \cdots \cup \omega_n$.

Since the n-forest Ω is contractible in S^3, we may assume that there is a regular projection $\mathcal{P}$ of a rose C in a suitably chosen 2-sphere S_0^2 in S^3, in the

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[3] $g(F)$ denotes the genus of 2-manifold F.

sense of knot theory, such that $\mathscr{P}(\Omega)$ has no crossing. So, the existence of a spanning-surface for θ is easily derived from 2.1.

To a spanning-surface F_θ for θ, we associate a pair $(u, v) \in \boldsymbol{N}^* \times \boldsymbol{N}^*$ of non-negative integers in the same way as a spanning-surface for a link. Moreover we define a total order $<$(or $>$) in $\{(u, v)\}$ $(\subset \boldsymbol{N}^* \times \boldsymbol{N}^*)$ by (2.2) and define the knotting-genus of a n-leafed rose θ as follows:

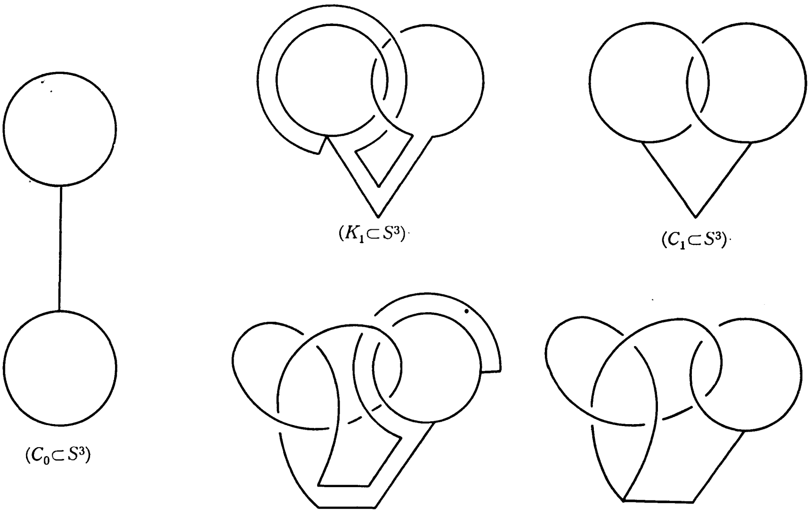
2.5. DEFINITION. A n-leafed rose type θ is of *knotting-genus* (u, v), iff there exists a spanning-surface F_θ for θ with (u, v), and for any spanning-surface F'_θ for θ with (u', v'), $(u, v) \leq (u', v')$. Especially, 0-leafed rose is considered to be of knotting-genus (0, 0).

Note that if the knotting-genus of a n-leafed rose type $\theta = \langle C \subset S^3 \rangle$ is (u, v), then $u \leq n$ and $v \geq g(\overline{C - \Omega} \subset S^3)$ where $g(\overline{C - \Omega} \subset S^3)$ is a genus of link $(\overline{C - \Omega} \subset S^3)$, see [15].

By virtue of 2.5, we have the followings:

2.6. DEFINITION. A N-rose type Θ of $\theta = \langle C \subset S^3 \rangle$ is of *knotting-genus* (u, v), iff there is a representative Θ of θ of knotting-genus (u, v), and for any representative θ' of Θ of knotting-genus (u', v'), $(u, v) \leq (u', v')$.

2.7. DEFINITION. The knotting-genera of a N-graph type Λ and a knot type of solid torus τ are defined by the set identifications of 1.10. That is, Λ is of *knotting-genus* (u, v) iff Θ is of knotting-genus (u, v) and $\Lambda = [C \subset S^3]$



$(C_0 \subset S^3)$

$(K_1 \subset S^3)$

$(C_1 \subset S^3)$

$(K_2 \subset S^3)$

$(C_2 \subset S^3)$

Fig. 2

for any representative $(C\subset S^3)$ of Θ, and τ is of *knotting-genus* (u, v) iff Θ is of knotting-genus (u, v) and $\tau=\langle N(C;\ S^3)\subset S^3\rangle$ for any representative $(C\subset S^3)$ of Θ.

2.8. REMARK. (1) For a graph $(K\subset S^3)$, we may define a spanning-surface, therefore the knotting-genus, directly by using a system of some kinds of surfaces in the similar way as 2.1 and 2.4. (2) Let F_θ be a spanning-surface for $\theta=\langle C\subset S^3\rangle$. Then, $F_\theta\cap\partial N(C;\ S^3)$ consists of mutually disjoint n simple loops, say $b_1, \cdots, b_n$, on $\partial N(C;\ S^3)$. In particular, $b_1,\cdots, b_n$ together generate the first integral homology group $H_1(N(C;\ S^3), Z)$.

2.9. EXAMPLES. We now list five examples of graphs. In Fig. 2, $\langle C_0\subset S^3\rangle$ is of knotting-genus (2, 0), $\langle C_1\subset S^3\rangle$ of (1, 0) and $\langle C_2\subset S^3\rangle$ of (1, 1). Particularly, any two of them are different graph type, but all of them are same N-graph type. So, $[C_i\subset S^3]=[K_j\subset S^3]$ is of knotting-genus (2, 0), $i=0, 1, 2$ and $j=1, 2$.

3. Unique decomposition theorem of N-graph type

In view of Definitions 2.5, 2.6 and 2.7, we have the following:

3.1. DEFINITION. A rose type θ is *prime* iff θ is of knotting-genus $(1, *)$. And a N-rose type Θ (resp. a N-graph type Λ resp. a knot type of solid torus τ) is *prime* iff Θ (resp. Λ resp. τ) is of knotting-genus $(1, *)$.

By the above definition, we have immediately the following:

3.2. Proposition. *Any* $\langle C\subset S^3\rangle$, $[C\subset S^3]$, $[K\subset S^3]$ *and* $\langle T\subset S^3\rangle$ *are prime provided that* $C\in \boldsymbol{C}(1)$, $K\in\boldsymbol{K}(1)$ *and* $T\in \boldsymbol{T}(1)$.

3.3. *Composition.* If graph types $\lambda_1=\langle K_1\subset S^3\rangle$ and $\lambda_2=\langle K_2\subset S^3\rangle$ are represented in a 3-sphere S^3 on opposite sides of a 2-sphere S_0^2 and have one point $\omega\in S_0^2$ in common, then we have a new graph type represented by a graph $(K_1\cup K_2\subset S^3)$. We will call the new graph type the *composition* of λ_1 and λ_2, and denote it by $\lambda_1\vee\lambda_2$, (see for knots [14], [5, §7], for links [8] and generally [7]). The composition of knot type of solid tori $\tau_1=\langle T_1\subset S^3\rangle$ and $\tau_2=\langle T_2\subset S^3\rangle$ can be defined in the similar way as graph types, that is, the *composition* $\tau_1\vee\tau_2$ of τ_1 and τ_2 is the knot type of solid torus $(T_1\cup T_2\subset S^3)$, where T_1 and T_2 are represented in a S^3 on opposite side of a 2-sphere S_0^2 and have a disk $D=\partial T_1\cap\partial T_2\subset S_0^2$ in common.

While, it is easily known that in general the composition of λ_1 and λ_2 is not uniquely determined. So, for rose types we give the following definition: the *composition* $\theta_1\vee\theta_2$ of $\theta_1=\langle C_1\subset S^3\rangle$ and $\theta_2=\langle C_2\subset S^3\rangle$ is the rose type of $(C_1\cup C_2\subset S^3)$, where C_1 and C_2 are represented in a S^3 on opposite side of a

2-sphere S_0^2 and have a common point $\omega = C_1 \cap C_2$ which is the nodes of $\Omega_1 \subset C_1$ and $\Omega_2 \subset C_2$. Then, we have:

3.4. Proposition. *In the set of all rose types* $\{\theta = \langle C \subset S^3 \rangle \mid C \in \boldsymbol{C}\}$, *the composition* $\vee$ *is well-defined, and moreover associative and commutative. Especially,* $\theta_0 = \langle \omega \subset S^3 \rangle$ *is an unit. Thus,* $\{\theta = \langle C \subset S^3 \rangle \mid C \in \boldsymbol{C}\}$ *forms a commutative semi-group under the operation* $\vee$.

3.5. Corollary. *We define the composition* $\Theta_1 \vee \Theta_2$ *of two N-rose type* Θ_1 *and* Θ_2 *by the N-rose type of the composition* $\theta_1 \vee \theta_2$ *of any representatives* θ_1 *of* Θ_1 *and* θ_2 *of* Θ_2. *Then the composition* $\vee$ *is well-defined in* $\{\Theta = [C \subset S^3] \mid C \in \boldsymbol{C}\}$.

Therefore, from 1.10 (or 1.9) we obtain at once the

3.6. Corollary. (1) *The composition* $\vee$ *is well-defined in* $\{\tau = \langle T \subset S^3 \rangle \mid T \in \boldsymbol{T}\}$.

(2) *We define the composition* $\Lambda_1 \vee \Lambda_2$ *of two N-graph types* Λ_1 *and* Λ_2 *by the N-graph type of any composition* $\lambda_1 \vee \lambda_2$ *of any representatives* λ_1 *of* Λ_1 *and* λ_2 *of* Λ_2. *Then, the composition* $\vee$ *is well-defined in* $\{\Lambda = [K \subset S^3] \mid K \in \boldsymbol{K}\}$.

We can now formulate our main theorem.

3.7. Theorem. *In the semi-group* $\{\theta = \langle C \subset S^3 \rangle \mid C \in \boldsymbol{C}\}$, *factorization is unique. That is, every* $\theta = \langle C \subset S^3 \rangle$ *is decomposable in an unique way into prime*

$$\theta_1 = \langle C_1 \subset S^3 \rangle, \cdots, \theta_u = \langle C_u \subset S^3 \rangle.$$

The existence of such a decomposition can be proved easily from 2.5(2.4), 3.1, 3.2 and the following:

3.8. Proposition. *Let* (u, v), (u_1, v_1) *and* (u_2, v_2) *be the knotting-genera of* θ, θ_1 *and* θ_2, *respectively. Suppose that*

$$\theta = \theta_1 \vee \theta_2 .$$

Then, $(u, v) = (u_1 + u_2, v_1 + v_2)$.

Proof. From Definition 2.5, it is obvious that $(u, v) \leq (u_1 + u_2, v_1 + v_2)$. So we must show that $(u, v) \geq (u_1 + u_2, v_1 + v_2)$.

Let $(C_1 \subset S^3)$ and $(C_2 \subset S^3)$ be representatives of θ_1 and θ_2, respectively, in a 3-sphere S^3 such that $C_1 \cap C_2 = \omega$, the nodes of $\Omega_1 \subset C_1$ and $\Omega_2 \subset C_2$, and $(C_1 \cup C_2 \subset S^3)$ is a representative of θ. And let S_0^2 be a 2-sphere in S^3 separating C_1 from C_2. If F_θ is a spanning-surface for $\theta = \theta_1 \vee \theta_2$, then the intersection of F_θ and S_0^2 consists of a finite number of simple loops in ∂F_θ. These loops can be capped to produce surfaces F_{C_1} and F_{C_2} spanning C_1 and C_2 respectively. Thus, if F_{C_1}, F_{C_2} and F_θ be with (u_1', v_1'), (u_2', v_2') and (u, v), respectively, clearly $u_1' + u_2' \geq u$ and $v_1' + v_2' \leq v$, thereby showing that $(u, v) \geq (u_1 + u_2, v_1 + v_2)$.

The uniqueness of the decomposition will clearly follow from the next lemma:

3.9. Lemma. *If $\theta_1 \vee \theta_2$ has θ as a prime component, then either θ_1 or θ_2 has θ as a prime component.*

Proof. To prove this, we start with a rose $(C_1 \cup C_2 \subset S^3)$ representing $\theta_1 \vee \theta_2$ and a 2-sphere S_0^2 that cut it in one point ω separating θ_1 from θ_2. Since θ is a component of $\theta_1 \vee \theta_2$, there exists a 3-cell Q in S^3 such that $\partial Q \cap (C_1 \cup C_2) = \omega$ and $Q \cap (C_1 \cup C_2)$ is a representative of θ. If $S_0^2 \cap \partial Q = \omega$, we can easily take a 3-cell Q_1 (or Q_2) in S^3 so that $Q_1 \cap Q = \partial Q_1 \cap \partial Q = \omega$ and $\vartheta Q_1 \cap C_1 \neq \emptyset$ (or $Q_2 \cap Q = \partial Q_2 \cap \partial Q = \omega$ and $\vartheta Q_2 \cap C_2 \neq \emptyset$), and so we are finished. If not, $S_0^2 \cap \partial Q$ consists of a finite number of disjoint simple loops $c_1, \cdots, c_\nu$, and a finite number of simple loops $d_1 \cdots, d_\mu$ such that $d_i \cap d_j = \omega$ for $i \neq j$ and $i, j = 1, \cdots, \mu$. Let $A(c_1), \cdots, A(c_\nu)$ be disks on ∂Q bounded by $c_1, \cdots, c_\nu$, respectively, such that $A(c_i) \not\ni \omega$, $i = 1 \cdots, \nu$.

Let c_1 be a minimal, i.e. there is no other c_i in $A(c_1)$. Let $B(c_1)$ be a disk on S_0^2 bounded by c_1 such that $B(c_1) \not\ni \omega$. Then, a 2-sphere $A(c_1) \cup B(c_1)$ bounds a 3-cell Q_1' in S^3. Since $A(c_1) \cap (C_1 \cup C_2) = \emptyset = B(c_1) \cap (C_1 \cup C_2)$, $Q_1' \cap (C_1 \cup C_2) = \emptyset$. Then we have a new 2-sphere $S_0^2 - B(c_1) \cup A(c_1)$ that cuts $(C_1 \cup C_2 \subset S^3)$ in one point ω separating θ_1 from θ_2, and again denote this 2-sphere by S_0^2. We may deform S_0^2 into general position in S^3 so that

$$S_0^2 \cap \partial Q \subset c_2 \cup \cdots \cup c_\nu \cup d_1 \cup \cdots \cup d_\mu .$$

By the repetition of the procedure we can get rid of all intersections c_1, $\cdots$ c_ν of $S_0^2 \cap \partial Q$.

Now, we will consider $d_1 \cup \cdots \cup d_\mu \subset S_0^2 \cap \partial Q$. First, we may assume that at least one of $d_1, \cdots, d_\mu$, say d_1, bounds a disk $B(d_1)$ on S_0^2 such that $B(d_1)$ does not contain any other d_i. Let $A(d_1)$ and $A'(d_1)$ be disks on S_0^2 bounded by d_1. Then we have two 2-spheres $S_1 = A(d_1) \cup B(d_1)$ and $S_1' = A'(d_1) \cup B(d_1)$ in S^3. We may deform S_1 and S_1' into general position in S^3 so that $S_1 \cap S_1' = \omega$. It will be noticed that S_1 and S_1' decompose one of $(C_1 \subset S^3)$ and $(C_2 \subset S^3)$ into two roses, one of them may be the trivial rose $(\omega \subset S^3)$, and

$$\partial Q \cap (S_1 \cup S_1') = d_2 \cup \cdots \cup d_\mu .$$

Repeating the procedure, we have 2μ 2-spheres $S_1, S_1', \cdots, S_\mu, S_\mu'$ in S^3 having the one point ω in common. It should be noted that these 2-spheres decompose $(C_1 \subset S^3)$ and $(C_2 \subset S^3)$ into severvl roses, and

$$\partial Q \cap (S_1 \cup S_1' \cup \cdots \cup S_\mu \cup S_\mu') = \omega .$$

Since θ is prime, we can take a new 3-cell, again denote it by Q, in S^3 such that

$$Q \cap (S_1 \cup S_1' \cup \cdots \cup S_\mu \cup S_\mu') = \omega \in \partial Q$$

and $Q \cap (C_1 \cup C_2)$ is a representative of θ. Thus, we can conclude that θ is one of prime components of θ_1 or θ_2.

From Definitions 2.6, 2.7 and 3.1, we have the followings, whose proofs are the same as that of 3.8 and 3.9 except for obvious modifications.

3.10. *Proposition* 3.8 *remains valid if* Θ *(or* Λ *or* τ*) is substituted for* θ.

3.11. *Lemma* 3.9 *remains valid if* Θ *(or* Λ *or* τ*) is substituted for* θ.

Thus, as an immediate consequence of 3.5, 3.6 and the above 3.10, 3.11, we have the main theorem in §0.

3.12. Theorem. *In the every semigroup* $\{\Theta=[C\subset S^3] \mid C\in \boldsymbol{C}\}$, $\{\Lambda=[K\subset S^3] \mid K\in \boldsymbol{K}\}$ *and* $\{\tau=\langle T\subset S^3\rangle \mid T\in \boldsymbol{T}\}$, *factorization is unique.*

4. Elementary ideals of a N-graph type

As generalization of the Alexander polynomial of knot [1], R.H. Fox [4] defined a sequence of elementary ideals, see [2, Chap. VII], and a sequence of polynomials, see [2, Chap. VIII], of a finitely presented group G. And S. Kinoshita [10], [11] discussed the Alexander polynomials of graphs. In this section, we will explain the Alexander matrix and the elementary ideals of linear groups. As in §3, the notions of roses and rose types are useful.

4.1. *Presentation of a group* $\pi_1(S^3-C)$. Now let $\mathcal{P}$ be a regular projection of a rose $C\subset S^3$ in a suitably chosen 2–sphere S_0^2 in S^3, in the sense of knot theory. Especially we may assume that $\mathcal{P}(\Omega)$ has no crossing and $\mathcal{P}(\Omega)\cap\mathcal{P}(S_1^1\cup\cdots\cup S_n^1)=\omega_1\cup\cdots\cup\omega_n$. We give a suitable orientation for each $S_1^1, \cdots, S_n^1$. Then, by using this projection and the orientation, we can obtain a *Wirtinger presentation* of the group $\pi_1(S^3-C)$. Let r be a number of the crossing points of $\mathcal{P}(S_1^1\cup\cdots\cup S_n^1)$. Then actually the presentation consists of $r+n$ generators $\boldsymbol{X}$ corresponding to the overpasses of $\mathcal{P}(S_1^1\cup\cdots\cup S_n^1)-(\omega_1\cup\cdots\cup\omega_n)$ and $r+1$ defining relations $\boldsymbol{R}$ corresponding to the r crossing points and Ω. The relation corresponding to a crossing point is the form

$$x_i\, x_j^{\varepsilon}\, x_{i+1}^{-1}\, x_j^{-\varepsilon} = 1$$

and the relation corresponding to Ω is the form

$$x_1 x_2^{-1} x_3 x_4^{-1} \cdots x_{2n-1} x_{2n}^{-1} = 1,$$

where x_i are the generators corresponding to the overpasses of Fig. 3. While, it is easily checked that one of the r relations corresponding to the crossing points is a consequence of the other $r-1$ and the relation corresponding to Ω. Since, for every non-split link L with n components, its link group $\pi_1(S^3-L)$ has deficiency 1, we can easily derived that:

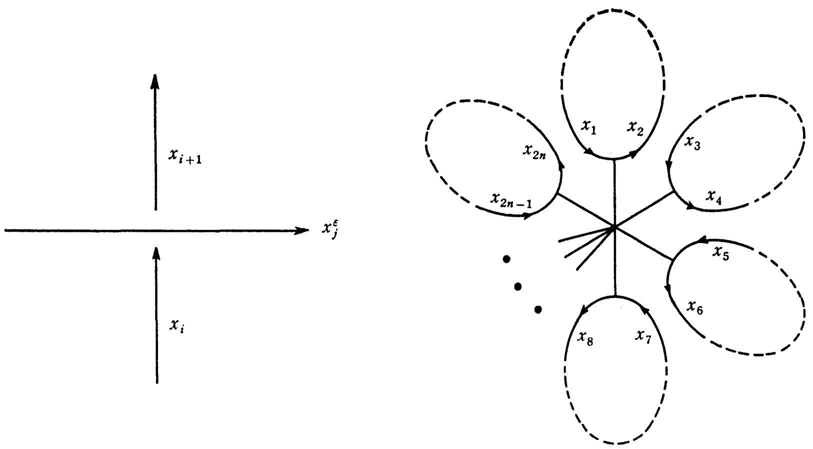


Fig. 3

4.2. *For a n-leafed rose* ($C\subset S^3$), *the fundamental group* $\pi_1(S^3-C)$ *has deficiency n.*

Of course, we can have a Wirtinger presentation of a group $\pi_1(S^3-K)$ of a graph ($K\subset S^3$) as the same way as a rose, see [5, §5], [10].

On the other hand, since $\pi_1(S^3-C)\cong\pi_1(S^3-N(C;\ S^3))$, for any N-congruent roses ($C\subset S^3$) and ($C'\subset S^3$), $\pi_1(S^3-C)\cong\pi_1(S^3-C')$. More generally, from 1.9 and 4.2 we have:

4.3. *The fundamental group* $\pi_1(S^3-K)$ *is a N-congruent invariant of a graph type* $\lambda=\langle K\subset S^3\rangle$, *and it has deficiency n if* $K\in\boldsymbol{K}(n)$.

In view of 4.3, for a N-graph type $\Lambda=[K\subset S^3]$ (resp. a N-rose type $\Theta=[C\subset S^3]$), we denote $\pi_1(S^3-K)$ (resp. $\pi_1(S^3-C)$) by $G(\Lambda)$ (resp. $G(\Theta)$), and call it a N-graph group (resp. a N-rose group). From the unique decomposition theorem 3.12, we have:

4.4. Proposition. *Suppose that* Λ *is of knotting-genus* (u, v). *Then* $G(\Lambda)\cong G(\Lambda_1)*\cdots*G(\Lambda_u)$, *that is* $G(\Lambda)$ *is a non-trivial free product of not finite groups* $G(\Lambda_1), \cdots, G(\Lambda_u)$, *where each* $G(\Lambda_i)$ *is a N-graph group of a prime N-graph type*, $i=1, \cdots, u$.

4.5. Corollary. *Suppose that* $K\in\boldsymbol{K}(n)$ *and* $\Lambda=[K\subset S^3]$ *is of knotting-genus* (n, v). *Then* $G(\Lambda)$ *is a non-trivial free product of n knot groups* $G_1, \cdots, G_n$.

4.6. *Elementary Ideals of a N-graph Type.* Let $Z[t]$ is the infinite cyclic, multiplicative group generated by t, and let $F[X]$ be the free group freely generated by $X=\{x_1, \cdots, x_{n+r}\}$. Then, the homomorphism $\psi(x_i)=t$, $i=1, \cdots, n+r$, has an unique linear extension to a homomorphism ψ: $JF[X]\rightarrow JZ[t]$ of the integral group ring [3]. Using a Wirtinger presentation $(\boldsymbol{X}|\boldsymbol{R})=(x_1, \cdots, x_{n+r}|r_1, \cdots, r_{r-1}, \Omega)$ of 4.1, we have a matrix

$$A = ||a_{ij}|| = \begin{pmatrix} \dfrac{\partial r_i}{\partial x_j} \\ \dfrac{\partial \Omega}{\partial x_j} \end{pmatrix}^{\psi}$$

over $JZ[t]$, where $\begin{pmatrix} \dfrac{\partial r_i}{\partial x_j} \\ \dfrac{\partial \Omega}{\partial x_j} \end{pmatrix}$ is the matrix of free derivatives [3]. We call A an *Alexander matrix* of the Wirtinger presentation $(\boldsymbol{X}|\boldsymbol{R})$of $G(\theta) = \pi_1(S^3-C)$ (or $G(\lambda)=\pi_1(S^3-K)$). It can be shown that

$$\sum_{j=1}^{r+n} a_{ij} = 0, \ i = 1, \cdots, n+r.$$

For an arbitary integer d, an ideal E_d of $JZ[t]$ generated by the determinants of all $(n+r-d)\times(n+r-d)$ minors of A will be called the d^{th} *elementary ideal* of the Wirtinger presentation $(\boldsymbol{X}|\boldsymbol{R})$.

The Alexander matrix and the d^{th} elementary ideal are not invariants of the abstract group $\pi_1(S^3-K)$. Nevertheless, from (4.6) of [2, p. 107] and the above 4.3, it can be shown that:

4.7. *The Alexander matrix and the sequence of elementary ideals are invariants of a graph type and of a N-graph type.*

Moreover, from 4.3 we claim:

4.8. *Let* $K\in\boldsymbol{K}(n)$ *and* $\Lambda=[K\subset S^3]$. *Then, if* $0\leq d<n$ *elementary ideals* $E_d(\Lambda)$ *are all equal to* 0, *see* [10, *Th.* 1]. *And, in general, the* n^{th} *elementary ideal* $E_n(\Lambda)$ *is not trivial.*

But $E_n(\Lambda)$ is not principal, in general. According to S. Kinoshita [10] and R.H. Crowell-R.H. Fox [2, Chap. VIII], we note the following:

4.9. *The* d^{th} *Alexander polynomial* $\Delta^{(d)}(t)$ *is the generator of the smallest principal ideal containing the* d^{th} *elementary ideal* E_d.

From 4.4, we have:

4.10. *The Alexander matrix* $A(\Lambda)$ *of a Wirtinger presentation of* $\pi_1(S^3-K)$ *is the form*

$$A(\Lambda)=\begin{Vmatrix} A(\Lambda_1) & & & 0 \\ & A(\Lambda_2) & & \\ & & \ddots & \\ 0 & & & A(\Lambda_u) \end{Vmatrix},$$

where $A(\Lambda_i)$ *is the Alexander matrix of a prime N-graph type* Λ_i, $i=1, \cdots, u$.

In particular, as a direct consequence of 4.5, we have:

4.11. Theorem. *Suppose that* $K\in\boldsymbol{K}(n)$ *and* $\Lambda=[K\subset S^3]$ *is of knotting-genus* (n, v). *Then, the* n^{th} *elementary ideal* $E_n(\Lambda)$ *is principal and its generator must be a product polynomial* $\Delta^{(n)}(t)=\Delta_{S_1^1}(t)\cdots\Delta_{S_n^1}(t)$ *of n knot Alexander polynomials* $\Delta_{S_1^1}(t), \cdots, \Delta_{S_n^1}(t)$.

5. Existence of non-trivial prime N-graph types

Since for any n, there is a non-split link L with n components, we have:

5.1. Theorem. *For any n, there exists a prime n-leafed rose type.*

In this section, we will prove the following:

5.2. Theorem. *For* $C\in\boldsymbol{C}(2)$, *there exists a prime N-rose type* $\Theta=[C\subset S^3]$. *So, for* $K\in\boldsymbol{K}(2)$ *and* $T\in\boldsymbol{T}(2)$, *there exist prime* $\Lambda=[K\subset S^3]$ *and* $\tau=\langle T\subset S^3\rangle$.

Proof. In order to prove, we give the following 2-leafed rose $(C\subset S^3)$ in Fig. 4. A Wirtinger presentation in which x_i and y_j correspond to the overpasses shown in Fig. 4 is the following:

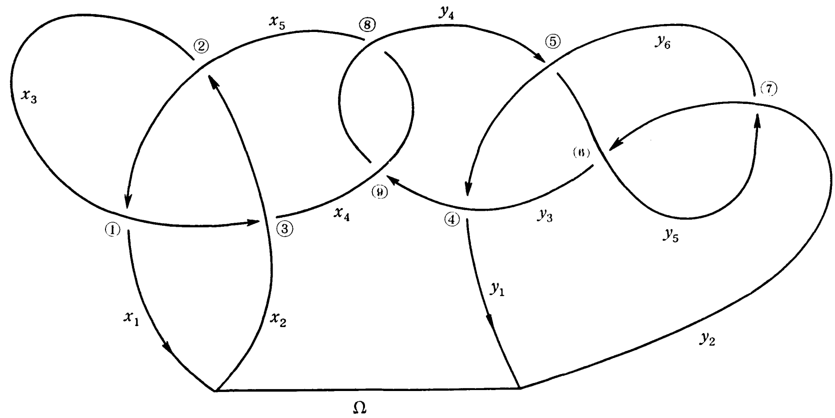


Fig. 4

$$\left(\begin{array}{l|llll} x_1,\ x_2,\ x_3, & \text{①} & x_1x_3 = x_3x_5, & \text{②} & x_3x_5 = x_5x_2, \\ x_4,\ x_5, & \text{③} & x_4x_2 = x_2x_3, & \text{④} & y_6y_3 = y_3y_1, \\ & \text{⑤} & y_4y_6 = y_6y_5, & \text{⑥} & y_3y_5 = y_5y_2, \\ y_1,\ y_2,\ y_3, & \text{⑦} & y_6y_2 = y_2y_5, & \text{⑧} & x_4\,y_4 = y_4x_5, \\ y_4,\ y_5,\ y_6 & \text{⑨} & y_3x_4 = x_4y_4 & \text{\textcircled{Ω}} & x_1x_2^{-1} = y_2y_1^{-1} \end{array}\right)$$

Any one of the relations ①, ②, ⋯, ⑨ is a consequence of the other nine, we may drop ⑤.

Substituting ② $x_3=x_5x_2x_5^{-1}$ in ① and ③, ④ $y_6=y_3y_1y_3^{-1}$ in ⑦, and ⑨ $y_4=x_4^{-1}y_3x_4$ in ⑧, we obtain

$$\left(\begin{array}{l|llll} x_1,\ x_2, & \text{①}' & x_1x_5x_2 = x_5x_2x_5, & \text{③}' & x_4x_2x_5 = x_2x_5x_2, \\ x_4,\ x_5, & \text{⑥} & y_3y_5 = y_5y_2, & \text{⑦}' & y_3y_1y_3^{-1} = y_2y_5y_2^{-1}, \\ y_1,\ y_2, & \text{⑧}' & x_4y_3x_4 = y_3x_4x_5, & \text{\textcircled{Ω}} & x_1x_2^{-1} = y_2y_1^{-1} \\ y_3,\ y_5 & & & & \end{array}\right).$$

Substitutions of ①′ $x_1=x_5x_2x_5x_2^{-1}x_3^{-1}$ in $\text{\textcircled{$\Omega$}}$, ③′ $x_4=x_2x_5x_2x_5^{-1}x_2^{-1}$ in ⑧′, and ⑦′ $y_5=y_2^{-1}y_3y_1y_3^{-1}y_2$ in ⑥ yield

$$\left(\begin{array}{l|ll} x_2,\ x_5, & \text{⑥}'' & y_2y_3y_2^{-1}y_3y_1 = y_3y_1y_3^{-1}y_2y_3 \\ y_1,\ y_2, & \text{⑧}'' & x_2x_5x_2x_5^{-1}x_2^{-1}y_3x_2x_5x_2 = y_3x_2x_5x_2x_5^{-1}x_2^{-1}x_5x_2x_5, \\ y_3 & \text{\textcircled{Ω}}' & x_5x_2x_5x_2^{-1}x_5^{-1}x_2^{-1} = y_2y_1^{-1} \end{array}\right)$$

From this Wirtinger presentation, we obtain the Alexander matrix

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\left\|\begin{array}{ccccc} x_2 & x_5 & y_1 & y_2 & y_3 \\ 0 & 0 & t^2-t & 1-2t & -t^2+3t-1 \\ t^4-2t^3+3t^2-2t+1 & -t^4+2t^3-3t^2+t & 0 & 0 & t-1 \\ -t^2+t-1 & t^2-t+1 & 1 & -1 & 0 \end{array}\right\| .$$

We can reduce A to an equivalent matrix[4] of simpler form

$$A \sim \left\|\begin{array}{ccccc} x_2 & x_5 & y_1 & y_2 & y_3 \\ 0 & 0 & t^2-t & t^2-3t+1 & 0 \\ 1-t & -t^4+2t^3-3t^2+t & 0 & 0 & 0 \\ 0 & t^2-t+1 & 1 & 0 & 0 \end{array}\right\|$$

---

[4] The equivalence of matrixes is Fox's equivalence [3], [4], see [2, Chap. VII].

$$\sim \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{ccccc} x_2 & x_5 & y_1 & y_2 & y_3 \\ \end{array}$$

$$\sim \left\| \begin{array}{ccccc} 0 & -(t^2-t+1)(t^2-t) & 0 & t^2-3t+1 & 0 \\ 1-t & -t^4+2t^3-3t^2+t & 0 & 0 & 0 \\ 0 & t^2-t+1 & 1 & 0 & 0 \end{array} \right\|$$

$$\begin{array}{cccc} x_2 & x_5 & y_2 & y_3 \end{array}$$

$$\sim \left\| \begin{array}{cccc} 0 & t(1-t)(t^2-t+1) & t^2-3t+1 & 0 \\ 1-t & -1 & 0 & 0 \end{array} \right\|$$

$$\sim \left\| \begin{array}{ccc} t^2-3t+1 & (t-1)^2(t^2-t+1) & 0 \end{array} \right\| .$$

Thus, the 2^{nd} elementary ideal E_2 is generated by two polynomials (t^2-3t+1) and $(t-1)^2(t^2-t+1)$, which are relatively prime in $JZ[t]$, integral group ring. So, E_2 is not principal, and by Theorem 4.11 $\theta = \langle C \subset S^3 \rangle$ and $\Theta = [C \subset S^3]$ are prime.

5.3. REMARK. I think, for any $C \in C(n)$ a prime N-rose type $\Theta = [C \subset S^3]$ may be constructed as the same way as the case $n=2$.

5.4. REMARK. Consider the following 2-leafed rose $(C \subset S^3)$ in Fig. 5. Then, a Wirtinger presentation of the group $\pi_1(S^3 - C)$ can be simplified to give

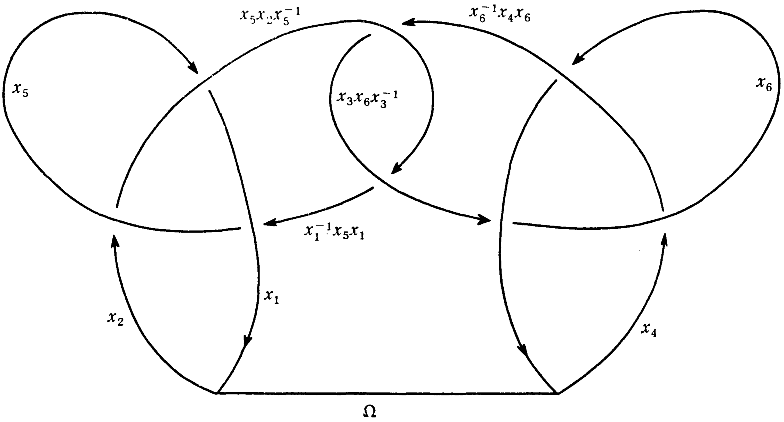


Fig. 5

$$\left(\begin{array}{c|l} x_1,\ x_2 & x_1x_5x_2 = x_5x_2x_5,\ x_6x_4x_6 = x_4x_6x_3, \\ x_3,\ x_4 & x_3x_6x_3^{-1}x_5x_2x_5^{-1} = x_5x_2x_5^{-1}x_6^{-1}x_4x_6, \\ x_5,\ x_6 & x_1x_2^{-1} = x_3^{-1}x_4 \end{array} \right)$$

and its Alexander matrix is follows:

$$\left\| \begin{array}{cccccc} 1 & t^2-t & 0 & 0 & -t^2+t-1 & 0 \\ 0 & 0 & -t^2 & t-1 & 0 & t^2-t+1 \\ 0 & t^2-t & 1-t & -1 & -t^2+2t-1 & 1 \\ 1 & -1 & t & -t & 0 & 0 \end{array} \right\|$$

$$\sim \left\| \begin{array}{ccc} t^2-t+1 & 0 & 0 \end{array} \right\| .$$

Thus, its 2^{nd} elementary ideal E_2 is principal generated by the 2^{nd} Alexander polynomial $\Delta^{(2)}=t^2-t+1$. But $\theta=\langle C\subset S^3\rangle$ and $\Theta=[C\subset S^2]$ may be prime.

In the remainder of this section, we shall give examples of linear graphs, which will seem to be of interest to some readers.

5.5. EXAMPLE. The first example is the following Fig. 6. Corresponding to the overpasses shown in Fig. 6, a presentation of $\pi_1(S^3-C)$ can be simplified to give

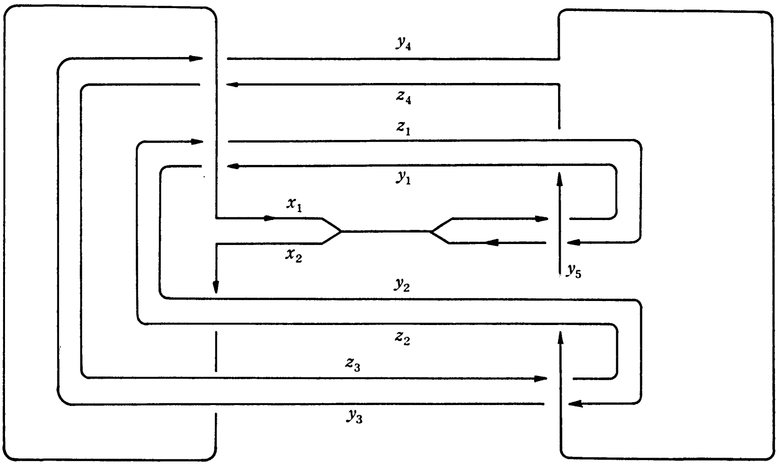


Fig. 6

[redacted]

$$\left(\begin{array}{l|l} x_1,\ x_2, & z_1y_5x_2^{-1}x_1y_5^{-1}y_1^{-1}=1, \\ y_1,\ y_2,\ y_3,\ y_4,\ y_5, & x_1y_1x_1^{-1}y_2^{-1}=1,\ z_2x_1z_1^{-1}x_1^{-1}=1, \\ z_1,\ z_2,\ z_3,\ z_4 & y_2y_4y_3^{-1}y_4^{-1}=1,\ y_4z_3y_4^{-1}z_2^{-1}=1, \\ & y_3x_1y_4^{-1}x_1^{-1}=1,\ x_1z_4x_1^{-1}z_3^{-1}=1, \\ & z_2y_4z_2^{-1}y_2y_5^{-1}y_2^{-1}=1,\ y_1y_5y_1^{-1}z_1z_4^{-1}z_1^{-1}=1, \\ & y_2x_2y_2^{-1}z_2z_3^{-1}y_3x_1^{-1}y_3^{-1}z_3z_2^{-1}=1 \end{array} \right)$$

Since the last relation is a consequence of the others, and may be discarded. As a result we may drop the 10^{th} row of the matrix and obtain

$$A \sim \begin{array}{c} \\ \\ \end{array}\begin{array}{ccccccccccc} x_1 & x_2 & y_1 & z_1 & y_2 & z_2 & y_3 & z_3 & y_4 & z_4 & y_5 \end{array}$$

$$A \sim \left\| \begin{array}{ccccccccccc} t & -t & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-t & 0 & t & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ t-1 & 0 & 0 & -t & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -t & 0 & t-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & t & 1-t & 0 & 0 \\ t-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -t & 0 & 0 \\ 1-t & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & t & 0 \\ 0 & 0 & 0 & 0 & t-1 & 1-t & 0 & 0 & t & 0 & -t \\ 0 & 0 & 1-t & t-1 & 0 & 0 & 0 & 0 & 0 & -t & t \end{array} \right\|$$

It is easily checked that the operations in the following reduction of A to an equivalent matrix of simpler form.

$$A \sim \begin{array}{cccccc} x_1 & z_1 & z_2 & z_3 & y_4 & z_4 \end{array}$$

$$A \sim \left\| \begin{array}{cccccc} 0 & -t & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & t & 0 & 0 \\ 0 & 0 & 0 & -1 & -t & 0 \\ 0 & t-1 & 1-t & 0 & t & 0 \end{array} \right\|$$

$$\sim \left\| \begin{array}{ccc} -t^2+2t-2 & 0 & 0 \end{array} \right\| .$$

Hence, the 2^{nd} elementary ideal E_2 is principal; E_2 generated by the 2^{nd} Alexander polynomial $\Delta^{(2)}=-t^2+2t-2$. Since $\Delta^{(2)}$ is not a reciprocal polynomial, by H. Seifert [15], see [2, Chap. IX], $\Delta^{(2)}$ is not a knot Alexander polynomial. So, by Theorem 4.11 the rose type $\theta=\langle C\subset S^3\rangle$ and the N-rose type $\Theta=[C\subset S^3]$ are prime.

5.6. EXAMPLE. (Figure 7). We obtain for the group $\pi_1(S^3-C)$ a presentation

$$\left(\begin{array}{l|l} x_1,\ x_2, & y_0y_1x_1^{-1}y_1^{-1}x_1y_1^{-1}=1,\ z_1x_1y_1^{-1}z_0^{-1}y_1x_1^{-1}=1, \\ y_0,\ y_1, & z_0y_1z_0^{-1}y_0z_1^{-1}y_0^{-1}=1,\ x_2y_0^{-1}z_0x_1^{-1}=1, \\ z_0,\ z_1 & x_1y_1^{-1}z_1x_2^{-1}z_1^{-1}y_1=1 \end{array} \right).$$

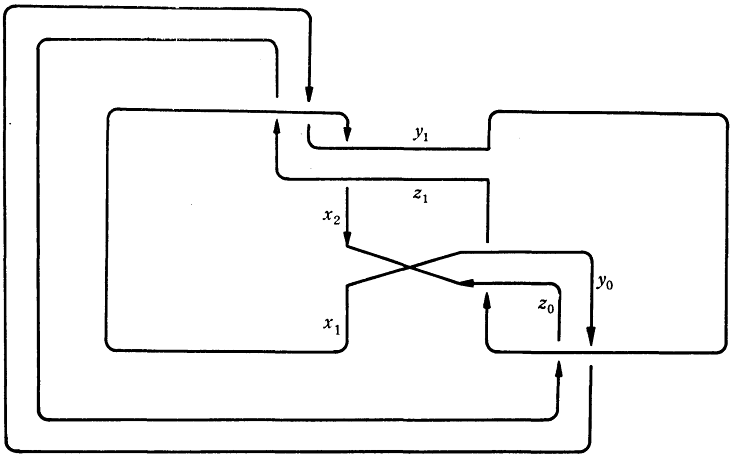


Fig. 7

Since the last relation is a consequence of the others, we obtain the Alexander matrix

$$A \sim \begin{array}{cccccc} x_1 & x_2 & y_0 & z_0 & y_1 & z_1 \\ \hline 1-t & 0 & 1 & 0 & t-2 & 0 \\ t-1 & 0 & 0 & -1 & 1-t & 1 \\ 0 & 0 & t-1 & 1-t & t & -t \\ -1 & 1 & -1 & 1 & 0 & 0 \end{array}$$

$$\sim \begin{array}{ccccc} x_1 & y_0 & z_0 & y_1 & z_1 \\ \hline 1-t & 1 & 0 & 0 & 0 \\ t-1 & 0 & -1 & 0 & 1 \\ 0 & t-1 & 1-t & 0 & -t \end{array}$$

$$\sim \left\| \begin{matrix} 2t-1 & 0 & 0 \end{matrix} \right\| .$$

Hence, the 2^{nd} elementary ideal E_2 is principal, and its generator is the 2^{nd} Alexander polynomial $\Delta^{(2)}=2t-1$. By the same reason, the rose type $\theta=\langle C\subset S^3\rangle$ and the N-rose type $\Theta=[C\subset S^3]$ are prime.

5.7. EXAMPLE. (Figure 8). This example generalizes Example 5.6. $K\in\boldsymbol{K}(n+1)$ and K has n nodes.

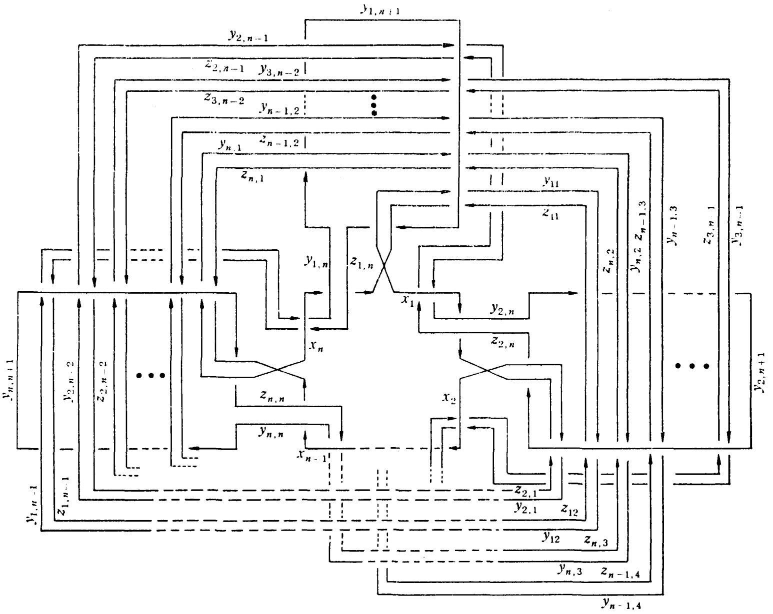


Fig. 8

A presentation in which x_i, y_{ij} and z_{ij} correspond to the overpasses shown in Fig. 8 is the following:

$$\left(\begin{array}{l|l} x_i\ :\ 1\leq i\leq n, & \boldsymbol{R}_i,\ \boldsymbol{Y}_i,\ \boldsymbol{Z}_i,\ \boldsymbol{O}_i,\ \boldsymbol{\Omega}_i, \\ y_{ij}\!:\ 1\leq i\leq n,\ 1\leq j\leq n+1, & \boldsymbol{Y}_{ij},\ \boldsymbol{Z}_{ij}\!: \\ z_{ij}\!:\ 1\leq i\leq n,\ 1\leq j\leq n & 1\leq i\leq n,\ 1\leq j\leq n-2 \end{array}\right),$$

where

$$\boldsymbol{R}_i\ :\ (y_{i,n+1}z_{i,1}y_{i,n+1})z_{i,1}^{-1}y_{i,1}y_{i,n+1}^{-1}z_{i,n}^{-1}(y_{i,n+1}y_{i,1}^{-1}y_{i,n+1}^{-1})=1,$$
$$i=1,\cdots,n,$$

$$\boldsymbol{Y}_i\ :\ (y_{i+n-1,n+1}^{-1}y_{i,n-1}y_{i+n-1,n+1})Y_i(x_{i+n-1}^{-1}y_{i,n}^{-1}x_{i+n-1})Y_i^{-1}=1,$$
$$Y_i=y_{i+1,n-1}^{-1}z_{i+1,n-1}y_{i+2,n-2}^{-1}z_{i+2,n-2}\cdots y_{i+n-1,1}^{-1}z_{i+n-1,1},$$
$i=1,\cdots,n$; indices are integers mod n,

$$\boldsymbol{Z}_i\ :\ (x_{i+n-1}^{-1}z_{i,n}x_{i+n-1})Z_i(y_{i+n-1,n+1}^{-1}z_{i,n-1}^{-1}y_{i+n-1,n+1})Z_i^{-1}=1,$$
$$Z_i=z_{i+n-1,1}^{-1}y_{i+n-1}z_{i+n-2,2}^{-1}y_{i+n-2,2}\cdots z_{i+1,n-1}^{-1}y_{i+1,n-1},$$
$i=1,\cdots,n$; indices are integers mod n,

$$\boldsymbol{O}_i\ :\ O_iy_{i,n+1}^{-1}O_i^{-1}y_{i,n}=1,$$
$$O_i=z_{i+n-1,1}^{-1}y_{i+n-1,1}z_{i+n-2,2}^{-1}y_{i+n-2,2}\cdots z_{i+1,n-1}^{-1}y_{i+1,n-1},$$
$i=1,\cdots,n$; indices are integers mod n,

$$\boldsymbol{\Omega}_i\ :\ y_{i,n}x_{i+n-1}y_{i,n}^{-1}z_{i,n}y_{i,n+1}y_{i,1}^{-1}z_{i,1}y_{i,n+1}^{-1}x_i^{-1}z_{i,n}^{-1}=1,$$
$$i=1,\cdots,n,$$

and

$$\boldsymbol{Y}_{ij}\colon\ y_{i,j}y_{i+j,n+1}y_{i,j+1}^{-1}y_{i+j,n+1}^{-1}=1,$$
$$i=1,\cdots,n \text{ and } j=1,\cdots,n-2;\ i+j \text{ are integers mod } n,$$
$$\boldsymbol{Z}_{ij}\colon\ y_{i+j,n+1}z_{i,j+1}y_{i+j,n+1}^{-1}z_{i,j}^{-1}=1,$$
$$i=1,\cdots,n \text{ and } j=1,\cdots,n-2;\ i+j \text{ are integers mod } n,$$

see Fig. 9.

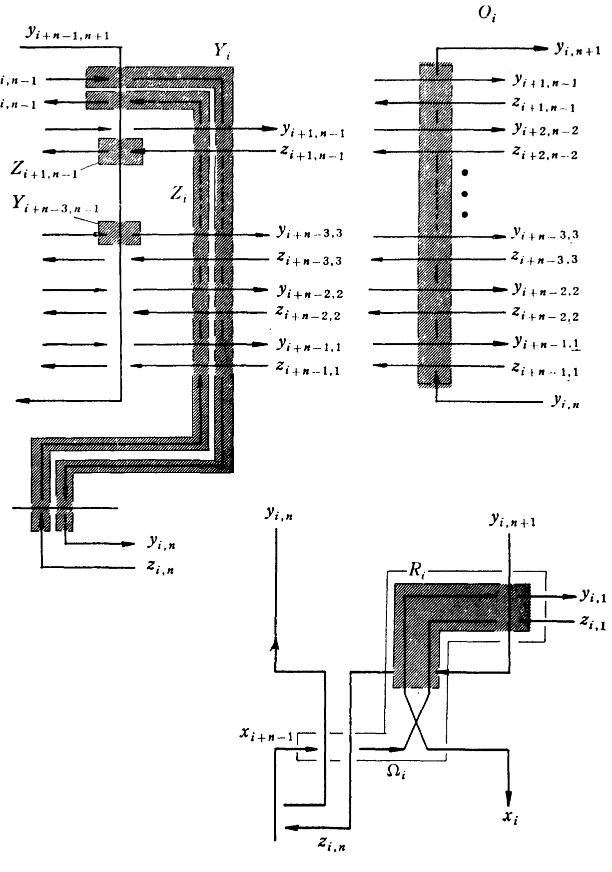


Fig. 9

The Alexander matrix A of the presentation can be written down. Especially, we have the following equivalent matrix of simpler form

$$
* \left\| \begin{array}{cccccccccc}
-t^n-t+1 & -t^n-t+1 & -t^n-t+1 & \cdots\cdots & -t^n-t+1 & -t^n-t+1 & 0 & \cdots\cdots & 0 \\
t^n-t^{n-1}+t & t-1 & t(t-1) & \cdots\cdots & t^{n-3}(t-1) & t^{n-2}(t-1) & 0 & \cdots\cdots & 0 \\
t^{n-2}(t-1) & t^n-t^{n-1}+t & t-1 & \cdots\cdots & t^{n-4}(t-1) & t^{n-3}(t-1) & 0 & \cdots\cdots & 0 \\
t^{n-3}(t-1) & t^{n-2}(t-1) & t^n-t^{n-1}+t & \cdots\cdots & t^{n-5}(t-1) & t^{n-4}(t-1) & 0 & \cdots\cdots & 0 \\
\cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\
\cdot & \cdot & \cdot & \cdots\cdots & \cdot & \cdot & \cdot & \cdots\cdots & \cdot \\
t(t-1) & t^2(t-1) & t^3(t-1) & \cdots\cdots & t^n-t^{n-1}+t & t-1 & 0 & \cdots\cdots & 0 \\
\hline
t-1 & t(t-1) & t^2(t-1) & \cdots\cdots & t^{n-2}(t-1) & t^n-t^{n-1}+t & * & \cdots\cdots & *
\end{array} \right\|
$$

Fig. 10

That the only first $n\times n$ minor determinant is not equal to 0 may be seen by setting $t=1$. We conclude that the $(n+1)^{th}$ elementary ideal is principapl and its $(n+1)^{th}$ Alexander polynomial $\Delta^{(n+1)}$ contains $-t^n-t+1$ as a factor. Note

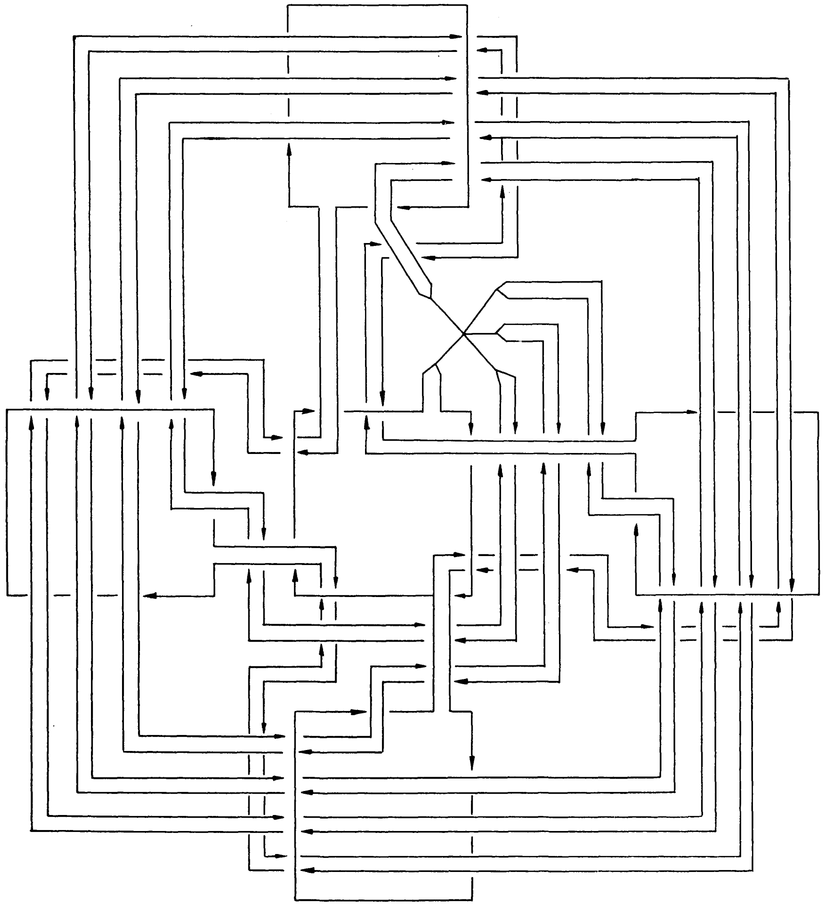


Fig. 11

that for any n, $-t^n-t+1$ can not be a knot Alexander polynomial.

Sliding the each node of K in Fig. 8 along the center circle (see Fig. 11 which shows the case $n=4$), we have a $(n+1)$-leafed rose $(C\subset S^3)$, whose group presentation and the Alexander matrix are the same of $(K\subset S^3)$. But we can not conclude that the rose type $\theta=\langle C\subset S^3\rangle$ and the N-rose type $\Theta=[C\subset S^3]$ are prime by the methods developed in the last two sections. The rose type $\theta=\langle C\subset S^3\rangle$ can, however, be shown to be prime by the following Theorem 5.8 and the Examples 5.6 and 5.7.

Let $C\in \boldsymbol{C}(n)$ and C' be a subcomplex of C. Then, $(C'\subset S^3)$ is said to be a *subrose* of a rose $(C\subset S^3)$ iff $C'\in \boldsymbol{C}$. Especially, a subrose $(C'\subset S^3)$ of $(C\subset S^3)$ is *proper* iff $C'\in \boldsymbol{C}(m)$, $C\in \boldsymbol{C}(n)$ and $m<n$.

5.8. Theorem. *Suppose that* $C\in \boldsymbol{C}(n)$ *and* $n\geq 3$. *For a rose type* $\theta=\langle C\subset S^3\rangle$ *to be prime, it is sufficient that* $\theta'=\langle C'\subset S^3\rangle$ *is prime for every proper subrose* $(C'\subset S^3)$ *of* $(C\subset S^3)$ *such that* $C'\in \boldsymbol{C}(2)$.

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