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COBORDISM GROUPS OF SEMI-FREE S^1- AND S^3-ACTIONS

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1. Introduction

Let G be a compact Lie group and M a differentiable manifold. Let f: $G\times M\rightarrow M$ be a differentiable action and F the set of all stationary points of f. Then we shall say that f is semi-free if the following condition holds: if $f(g, x)=x$ for some $x\in M-F$, then $g=e$ the identity element of G.

Now let G be a compact connected Lie group. We shall consider all semi-free G-actions (M^n, f) on closed oriented differentiable n-manifolds. Such a semi-free G-action (M^n, f) is cobordant to zero if and only if there is a semi-free G-action (W^{n+1}, g) on a compact oriented differentiable $(n+1)$-manifold W^{n+1} for which $(\partial W, g|\partial W)$ is equivariantly orientation preserving diffeomorphic to (M^n, f). From two semi-free G-actions (M_1^n, f_1) and (M_2^n, f_2) a disjoint union $(M_1^n\cup M_2^n, f)$ can be formed as usual. We say that (M_1^n, f_1) is cobordant to (M_1^n, f_2) if and only if the disjoint union of (M_1^n, f_1) and $(-M_2^n, f_2)$ is cobordant to zero in the above sense. By making use of the existence of an equivariant collared neighborhood, it is shown that corbodism is an equivalence relation, the cobordism class to which (M^n, f) belongs is denoted by $[M^n, f]$. The collection of such cobordism classes is denoted by $SF_n(G)$. An abelian group structure is imposed on $SF_n(G)$ by disjoint union. We shall call this group $SF_n(G)$ the cobordism group of semi-free G-actions of dimension n.

Let S^1 and S^3 be the unit spheres in the field of complex numbers $\boldsymbol{C}$ and of quaternions $\boldsymbol{H}$ respectively. In this paper we shall consider $SF_n(S^1)$ and $SF_n(S^3)$. We shall recall some well known results on semi-free actions in section 2. And in section 3 we shall prove the following result.

Theorem 3.2. *There are split exact sequences*:

$$0\rightarrow SF_n(S^1)\rightarrow\sum_k\Omega_{n-2k}(BU(k))\rightarrow\Omega_{n-2}(CP^\infty)\rightarrow 0,$$
$$0\rightarrow SF_n(S^3)\rightarrow\sum_k\Omega_{n-4k}(BSp(k))\rightarrow\Omega_{n-4}(HP^\infty)\rightarrow 0.$$

Next we shall show that another splitting homomorphism of $\sum_k\Omega_{n-2k}(BU$

$(k))$ onto $SF_n(S^1)$ is obtained by making use of associated projective bundles in section 4. And we have an isomorphism $CP_* : \sum_{k \neq 1} \Omega_{n-2k}(BU(k)) \to SF_n(S^1)$.

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2. Semi-free actions

In this section, we recall some well known results.

Let G be a compact Lie group and M a compact differentiable manifold. Let f: $G \times M \to M$ be a semi-free differentiable action. By the usual averaging process, there is a Riemannian metric on M with respect to which G is a group of isometries of M.

If $x \in M$ is a stationary point of f, then we have an induced orthogonal representation of G on M_x, the tangent space of M at $x \in M$.

Lemma 2.1. *Let $x \in M$ be a stationary point of a semi-free differentiable action f: $G \times M \to M$. Let ρ_x: $G \to GL(M_x)$ be an induced orthogonal representation and*

$$V = \{v \in M_x \mid \rho_x(g)v = v \text{ for any } g \in G\}.$$

Then the restriction on V of an exponential mapping exp: $M_x \to M$ is a local diffeomorphism of V into F, the set of all stationary points of f, and the orthogonal complement $V^\perp$ of V in M_x is a G-invariant subspace on which G acts freely except for the zero vector.

From this lemma we have a well known result that each connected component of F is a differentiable submanifold of M.

In particular, if we consider semi-free S^a-actions ($a=1$, 3), we obtain the following result (cf. [2], §38).

Lemma 2.2. *Let f:$S^a \times M \to M$ ($a=1, 3$) be a semi-free differentiable action. Let F^k denote the union of the k-dimensional components of the set of all stationary points of f. Then the normal bundle ν_k of an embedding $F^k \subset M$ has naturally a complex structure for $a=1$ and a quaternionic structure for $a=3$, such that the induced S^a-action on ν_k is a scalar multiplication.*

Proof. This follows from the fact that the real vector space $\boldsymbol{C}^1$ (resp. $\boldsymbol{H}^1$) is the only one irreducible real representation on which S^1 (resp. S^3) acts freely except for the zero vector. q.e.d.

REMARK. From this lemma, a codimension of each component of F in M is even for $a=1$ and divisible by 4 for $a=3$.

3. Cobordism of semi-free G-actions

Let G be a compact connected Lie group. We consider all semi-free differentiable G-actions (M^n, f) on closed oriented differentiable n-manifolds, where $f\colon G\times M^n\to M^n$ is a semi-free G-action.

Such a semi-free G-action (M^n, f) is cobordant to zero if and only if there is a semi-free G-action (W^{n+1}, F) on a compact oriented differentiable $(n+1)$-manifold W^{n+1} for which $(\partial W, F|\partial W)$ is equivariantly orientation preserving differomorphic to (M^n, f). From two semi-free G-actions (M_1^n, f_1) and (M_2^n, f_2) a disjoint union $(M_1^n \cup M_2^n, f)$ can be formed as usual. We say that (M_1^n, f_1) is cobordant to (M_2^n, f_2) if and only if the disjoint union of (M_1^n, f_1) and $(-M_2^n, f_2)$ is cobordant to zero in the above sense, where $-M_2^n$ is the negatively oriented manifold of M_2^n. By making use of the existence of an equivariant collared neighborhood (cf. [2], Th. 21.2), it is shown that cobordism is an equivalence relation, the cobordism class to which (M^n, f) belongs is denoted by $[M^n, f]$. The collection of such cobordism classes is denoted by $SF_n(G)$. An abelian group structure is imposed on $SF_n(G)$ by disjoint union. We call this group $SF_n(G)$ the cobordism group of semi-free G-actions of dimension n.

Let $f\colon G\times M\to M$ be a semi-free G-action and N a closed manifold. Then $f^N\colon G\times M\times N\to M\times N$ defined by $f^N(g, m, n)=(f(g, m), n)$ is also a semi-free G-action. This defines a right Ω_*-module structure on $SF_*(G)=\sum_{n\geq 0} SF_n(G)$.

In the above definition, if the term "semi-free" is replaced by "free", one may define a cobordism group $F_n(G)$ of fixed point free G-actions of dimension n and a right Ω_*-module structure on $F_*(G)=\sum_{n\geq 0} F_n(G)$.

Let $B(G)$ be a classifying space of G and k the dimension of G as a manifold. let (M^n, f) be a free G-action. Then $\pi\colon M^n\to M^n/G$ is a principal G-bundle by the differentiable slice theorem (cf. [1], ch. 8), and there exists a homotopy class of a classifying map $\bar{f}\colon M^n/G\to B(G)$. The correspondence $[M^n, f]\mapsto[M^n/G, \bar{f}]$ is well-defined homomorphism of $F_n(G)$ into $\Omega_{n-k}(B(G))$ and we have the following known result (cf. [2], 19.1, [7], 19.6).

Lemma 3.1. *The above defined homomorphism $\rho_*\colon F_n(G)\to\Omega_{n-k}(B(G))$ is an isomorphism and $\rho_*\colon F_*(G)\to\Omega_*(B(G))$ is an isomorphism of degree $-k$ as a Ω_*-module homomorphism.*

Here we consider the cobordism groups $SF_n(S^1)$ and $SF_n(S^3)$. Let $\mathfrak{M}_n(U) = \sum_{k=0}^{[n/2]} \Omega_{n-2k}(BU(k))$ and $\mathfrak{M}_n(Sp) = \sum_{k=0}^{[n/4]} \Omega_{n-4k}(BSp(k))$, where $\Omega_n(BU(0)) = \Omega_n = \Omega_n(BSp(0))$. In the following we consider the cobordism group $SF_n(S^1)$, since the cobordism group $SF_n(S^3)$ can be considered similarly.

We define $\beta_*\colon SF_n(S^1)\to\mathfrak{M}_n(U)$ as follows. For each semi-free S^1-action (M^n, f), let F^{n-2k} denote the union of the $(n-2k)$-dimensional components of the fixed point set and let ν^k denote the normal bundle to F^{n-2k} with natural complex vector bundle structure. Then ν^k is canonically oriented and let F^{n-2k} be so oriented that the bundle map $\tau F \oplus \nu^k \to \tau M$ is orientation preserving. Define $\beta_*([M^n, f]) = \sum_k [\nu^k] \in \mathfrak{M}_n(U)$. Then β_* is a well-defined homomorphism.

We also consider the homomorphism $\partial_*\colon \mathfrak{M}_n(U) \to \Omega_{n-2}(CP^\infty)$, the sum of the homomorphisms $\partial_*\colon \Omega_{n-2k}(BU(k)) \to \Omega_{n-2}(CP^\infty)$ as follows. For each complex vector bundle ξ^k, let $S(\xi^k)$ and $CP(\xi^k)$ denote the associated sphere bundle and projective bundle respectively and let $\pi\colon S(\xi^k) \to CP(\xi^k)$ be a projection. Define $\partial_*([\xi^k]) = [\pi]$, the class of principal S^1-bundle π. By definition $\partial_*(\Omega_n) = 0$.

Theorem 3.2. *The sequences*

$$0 \to SF_n(S^1) \xrightarrow{\beta_*} \mathfrak{M}_n(U) \xrightarrow{\partial_*} \Omega_{n-2}(CP^\infty) \to 0$$

and

$$0 \to SF_n(S^3) \xrightarrow{\beta_*} \mathfrak{M}_n(Sp) \xrightarrow{\partial_*} \Omega_{n-4}(HP^\infty) \to 0$$

are split exact.

Proof. Let $\alpha_*\colon F_n(S^1) \to SF_n(S^1)$ be a canonical forgetting homomorphism and $S_*\colon \mathfrak{M}_n(U) \to F_{n-1}(S^1)$ a homomorphism as follows. For each complex vector bundle ξ^k, let $\varphi\colon S^1 \times S(\xi^k) \to S(\xi^k)$ be the scalar multiplication. Define $S_*([\xi^k]) = [S(\xi^k), \varphi]$. Then we have the following sequence:

$$\cdots \to F_n(S^1) \xrightarrow{\alpha_*} SF_n(S^1) \xrightarrow{\beta_*} \mathfrak{M}_n(U) \xrightarrow{S_*} F_{n-1}(S^1) \xrightarrow{\alpha_*} SF_{n-1}(S^1) \to \cdots$$

which is exact by a standard argument (cf. [9], Th. A). Moreover the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{M}_n(U) & \xrightarrow{S_*} & F_{n-1}(S^1) \\ \partial_* \searrow & & \swarrow \rho_* \\ & \Omega_{n-2}(CP^\infty) & \end{array}$$

where ρ_* is the isomorphism in Lemma 3.1.

Next we define $D_*\colon \Omega_{n-2}(CP^\infty) \to \mathfrak{M}_n(U)$ and show $\partial_* D_* =$ identity. For each principal S^1-bundle ξ, let $D_*([\xi])$ be the class of the associated complex line bundle. Clearly D_* is well-defined and $\partial_* D_* = id$. These complete the proof.

Corollary 3.3. *Each fixed point free differentiable S^a-action on an oriented closed differentiable n-manifold is cobordant to zero in $SF_n(S^a)$, for $a=1$ and $a=3$.*

4. Examples of semi-free S^1-actions

Let ξ be a complex k-plane bundle over an oriented closed differentiable manifold V^n and θ^1 a trivial complex line bundle over V^n. Let $CP(\xi \oplus \theta^1)$

be a total space of complex projective bundle over V^n associated with the Whitney sum $\xi\oplus\theta^1$. Then $CP(\xi)$ and $CP(\xi\oplus\theta^1)$ have canonical orientations defined by the complex structure of ξ. Moreover the normal bundle ν of the embedding $CP(\xi)\subset CP(\xi\oplus\theta^1)$ has a complex structure conjugate equivalent to the complex line bundle $\hat{\xi}$ associated with the principal S^1-bundle $\pi: S(\xi)\to CP(\xi)$.

Define

$$\mu: S^1\times CP(\xi\oplus\theta^1)\to CP(\xi\oplus\theta^1)$$

by $\mu(\lambda, \langle u, v\rangle)=\langle\lambda u, v\rangle$ where $\lambda\in S^1$, u and v are vectors of ξ and θ^1 respectively. Then μ is a semi-free differentiable S^1-action with the fixed point set $V^n\cup CP(\xi)$. And ξ becomes the normal bundle of the embedding $V^n\subset CP(\xi\oplus\theta^1)$ with canonical complex structure induced from the action μ. On the other hand, S^1-action on the normal bundle ν of the embedding $CP(\xi)\subset CP(\xi\oplus\theta^1)$ is the conjugate scalar multiplication by definition of the action μ. Therefore we have

$$\beta_n[CP(\xi\oplus\theta^1), \mu] = [\xi] - [\hat{\xi}] \text{ in } \mathfrak{M}_{n+2k}(U). \tag{4.1}$$

Theorem 4.2. *Let (M^n, f) be a semi-free S^1-action, F^{n-2k} a $(n-2k)$-dimensional components of the fixed point set and ν^k a complex normal bundle of the embedding $F^{n-2k}\subset M^n$. Then we have*

$$[M^n, f] = \sum_k [CP(\nu^k\oplus\theta^1), \mu] \text{ in } SF_n(S^1).$$

Proof. By (4.1) and the definition of β_*,

$$\begin{aligned}
&\beta_*[M^n, f]-\sum_k \beta_*[CP(\nu^k\oplus\theta^1), \mu]\\
&=\sum_k ([\nu^k]-([\nu^k]-[\hat{\nu}^k]))\\
&=\sum_k [\hat{\nu}^k].
\end{aligned}$$

where $\hat{\nu}^k$ is a complex line bundle associated with the principal S^1-bundle $\pi: S(\nu^k)\to CP(\nu^k)$. Then $[\hat{\nu}^k]=D_*\partial_*[\nu^k]$ by definition. Therefore

$$\begin{aligned}
\sum_k [\hat{\nu}^k] &= D_*\partial_*(\sum_k [\nu^k])\\
&= D_*\partial_*\beta_*[M^n, f]\\
&= 0
\end{aligned}$$

since $\partial_*\beta_*=0$. Since β_* is injective, this implies $[M^n, f]=\sum_k [CP(\nu^k\oplus\theta^1), \mu]$ in $SF_n(S^1)$. q.e.d.

Let CP_* assign to $[\xi]$ the cobordism class $[CP(\xi\oplus\theta^1), \mu]$ in $SF_*(S^1)$. Then we have well-defined homomorphisms

$$CP_*: \Omega_n(BU(k)) \rightarrow SF_{n+2k}(S^1)$$

and

$$CP_*: \mathfrak{M}_n(U) \rightarrow SF_n(S^1).$$

Corollary 4.3. *The homomorphism* $CP_*: \mathfrak{M}_n(U) \rightarrow SF_n(S^1)$ *is a left inverse of* β_*, i.e. $CP_*\beta_* = $ *identity.*

Corollary 4.4. *The homomorphism*

$$CP_*: \sum_{k \neq 1} \Omega_{n-2k}(BU(k)) \rightarrow SF_n(S^1)$$

is an isomorphism.

Theorem 4.5. *Let* (M^n, f) *be a semi-free* S^a*-action with the fixed point set* F^{n-a-1}, *where* $a=1$ *or* $a=3$. *Then*

$$[M^n, f] = 0 \textit{ in } SF_n(S^a).$$

Proof. Let ν be a normal complex (for $a=1$ and quaternionic for $a=3$) line bundle of the embedding $F \subset M^n$. Let $\hat{\nu}$ be a principal S^a-bundle associated with ν. Then $[\nu] = D_*[\hat{\nu}]$ and

$$[\hat{\nu}] = \partial_* D_*[\hat{\nu}] = \partial_*[\nu] = \partial_*\beta_*[M^n, f] = 0.$$

Thus

$$\beta_*[M^n, f] = [\nu] = D_*[\hat{\nu}] = 0.$$

Therefore $[M^n, f]=0$, since β_* is injective. q.e.d.

Corollary 4.6. *Let* M^n *be an oriented closed manifold. Then the following conditions are equivalent:*

(i) $[M^n] = 0$ *in* Ω_n,
(ii) *there exists a semi-free* S^1*-action* (V^{n+2}, f) *with the fixed point set* M^n,
(iii) *there exists a semi-free* S^3*-action* (W^{n+4}, g) *with the fixed point set* M^n.

Proof. Under the condition (ii), $[V^{n+2}, f]=0$ in $SF_{n+2}(S^1)$ from the above theorem. Then $\beta_*[V^{n+2}, f]=0$ and in particular $[M^n]=0$. Similarly (iii) implies (i). We shall show (i) implies (ii) and (iii).

Let S^a and D^{a+1} be the unit sphere and the unit disk in the field of complex numbers for $a=1$ and of quaternions for $a=3$. There exists an oriented compact differentiable manifold W^{n+1} for which $M^n = \partial W$ since $[M^n]=0$ in Ω_n. Then

$$\partial (W \times S^a) = M^n \times S^a = \partial (M^n \times D^{a+1}).$$

Therefore we obtain an oriented closed differentiable manifold

$$V^{n+a+1} = W^{n+1} \times S^a \bigcup_{id} M^n \times D^{a+1}.$$

Let f: $S^a \times V^{n+a+1} \rightarrow V^{n+a+1}$ be a map defined by $f(\lambda, (x, z)) = (x, \lambda z)$ for $(x, z) \in W^{n+1} \times S^a$ or $(x, z) \in M^n \times D^{a+1}$.

Then f is a semi-free S^a-action with the fixed point set M^n. q.e.d.

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