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ON CATEGORIES OF INDECOMPOSABLE MODULES I

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One of the authors had defined a regular additive category and studied some structures of it in [15]. We shall give, in this note, several applications of [15], Theorem 2.

In the first section, we take an injective module M over a ring R and consider the full sub-category $C(M)$ of the category of right R-modules $\mathfrak{M}_R$, whose object consists of all direct summands of any product of M. By $\mathfrak{J}$ we denote the (Jacobson) radical of $C(M)$, (see the definition in [15]). Then we shall show in Theorem 1 that $C(M)/\mathfrak{J}$ is a spectral C_3-category with generator. In this note we make great use of this theorem.

Especially, we study, in the section 2, the direct decomposition of injective module in the category $\mathfrak{A}=\tilde{\mathfrak{A}}/\mathfrak{J}$, where $\tilde{\mathfrak{A}}$ is the full sub-category of all injective modules in $\mathfrak{M}_R$. Following to [11], we shall give a condition that $\mathfrak{A}$ is completely reducible, and give general type of decompositions of injective modules (Theorem 6). Furthermore, we shall give a different proof of [4], Theorem 6.5 by making use of some structure of $\mathfrak{A}$.

In the sections 3 and 4 we shall study the Krull-Remak-Schmidt-Azumaya's theorem for R-modules. In those sections, we take the full sub-category $\mathfrak{A}'$ of $\mathfrak{M}_R$ whose objects are coproducts of a given family $\{M_\alpha\}$ of completely indecomposable modules. Let $\mathfrak{J}'$ be the ideal of $\mathfrak{A}'$ whose morphisms are all roots-elements, (see the definition in [1]), then we shall show in Theorem 7 that $\mathfrak{A}'/\mathfrak{J}'$ is a completely reducible C_3-abelian category. We prove Azumaya's theorem as a collorary of Theorem 7. Furthermore, we shall give a condition that $\mathfrak{J}'$ is the radical of $\mathfrak{A}'$, from which we study further properties of direct decomposition of modules in Theorem 9.

In the last section, we shall give some remarks to generalize the above results to a case of a C_3-abelian category with generator.

We always assume, in this paper, that a ring R has the identity element and all R-modules are unitary (right) R-modules. We make use of terminologies concerning with category in [12].

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1. Categories of injective modules

First we recall the definitions given in [15]. Let $\mathfrak{A}$ be an additive category. $\mathfrak{A}$ is called *regular* if the endomorphism ring $[A, A]$ of any object A in $\mathfrak{A}$ is regular in the sense of Von Neumann and $\mathfrak{A}$ is called *spectral* if $\mathfrak{A}$ is abelian and every morphism splits.

In this paper, except the last section, we always consider a sub-category $\mathfrak{A}$ of the category $\mathfrak{M}_R$ of right modules over a ring R and its quotient category with respect to an ideal in $\mathfrak{A}$, (see the definition of ideals of $\mathfrak{A}$ in [15]).

Let R be a ring with identity and M an injective right R-module. By $C(M)$ we denote the full sub-category in $\mathfrak{M}_R$ whose object consists of all direct summands of every product ΠM of M. It is clear that $C(M)$ is an additive category with finite coproduct and every object in $C(M)$ is an injective module in $\mathfrak{M}_R$. It is well known that there exists an injective cogenerator A in $\mathfrak{M}_R$, then $C(A)$ means the full sub-category of all injective modules in $\mathfrak{M}_R$, since every injective module Q is a direct summand of $\prod_{f\in[Q,A]} A_f$; $A_f\approx A$. Furthermore, if we take a self-injective regular ring T instead of M, then $C(T)$ coincides with the spectral category in [7], Satz 2.2. We shall generalize this theorem as the next theorem, which is a first application of [15], Theorem 2.

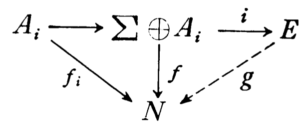
We shall make use of the notion of Jacobson radical in an additive category $\mathfrak{A}$, defined in [15] and denote it by $\mathfrak{J}$ or $\mathfrak{J}(\mathfrak{A})$. By $E(K)$ we denote an injective hull of a right R-module K and by $[N, N']_R$ we denote the set of R-homomrphisms of N to N' for any objects N, N' in $\mathfrak{M}_R$.

Theorem 1. *Let M be an injective right R-module and $C(M)$ an additive category defined above. Then the quotient category $C(M)/\mathfrak{J}$ with respect to the Jacobson radical $\mathfrak{J}$ is a spectral C_3-category with generator.*[1]

Proof. Let N be an object in $C(M)$, and $R_N=[N, N]_R$ with radical $\mathfrak{J}_N$. Then it is known that $R_N/\mathfrak{J}_N$ is a regular ring in the sense of Von Neumann and every idempotent in $R_N/\mathfrak{J}_N$ is lifted to R_N (see [3], §5 or [16]). Hence, $C(M)/\mathfrak{J}$ is a regular category with finite coproduct. Let $\bar{e}$ be an idempotent in $R_N/\mathfrak{J}_N$, then we may assume that e is also idempotent in R_N. Hence, $1_N=e+(1-e)$ and $e\perp(1-e)$. Furthermore, eN is a direct summand of N and hence, eN and $(1-e)N$ are objects in $C(M)$. Since $\bar{1}_N=\bar{e}+(\overline{1-e})$ and $\bar{e}\perp(\overline{1-e})$ in $R_N/\mathfrak{J}_N$, $(1-e)N$ is the kernel of $\bar{e}$. Hence, $C(M)/\mathfrak{J}$ is a spectral category from [15], Theorem 2. We shall show that $C(M)/\mathfrak{J}$ has any coproduct. Let $\{A_i\}_{i\in I}$ be any family of objects in $C(M)$. Then $A_i<\oplus \prod_{I_i} M$ and hence, $\sum_I \oplus A_i<\oplus \sum_I (\oplus \prod_{I_i} M_i)\subset \prod_J M$ in $\mathfrak{M}_R$. Let $E(\sum \oplus A_i)=E$ be an injective

1) Added in proof. It is obtained substantially in [19].

hull of $\sum_I \oplus A_i$ in $\prod_J M$. Then E is an object in $C(M)/\mathfrak{J}$. Let N be any object in $C(M)/\mathfrak{J}$ and $\bar{f}_i$ any morphism in $[N, A_i]$ in $C(M)/\mathfrak{J}$, where f_i is in $[N, A]$. Then there exists a morphism f in $\mathfrak{M}_R$ such that



(1)

is commutative. Since N is injective, there exists $g \in [E, N]_R$ such that $gi=f$. We shall show that $\bar{g}$ is unique in $C(M)/\mathfrak{J}$. We note that every morphisms in the diagram (1) preserve the additiveness and that if $g'i=f=gi$ for some $g' \in [E, N]_R$, then $(g-g')i=0$, and hence $g-g' \in \mathfrak{J}([E, N]_R)$ since ker $(g-g') \supseteq \sum \oplus A_i$ and $\sum \oplus A_i$ is essential in E, (see [3], §5). Therefore, in order to show the uniqueness of $\bar{g}$, we may assume that all f_i are in $\mathfrak{J}([A_i, N]_R)$, which means that ker f_i is essential in A_i. Hence, $\sum \oplus$ Ker f_i is also essential in $\sum \oplus A_i$. Since $gi=f$, ker $g \supseteq \sum \oplus$ Ker f_i. Therefore $g \in \mathfrak{J}([E, N]_R)$. Conversely, let g be in $[E, N]_R$, then put $f_i = g | A_i$. If $\bar{g}=0$ in $C(M)/\mathfrak{J}$, then ker g is essential in E and hence, ker f_i=ker $g \cap A_i$ is essential in A_i. It is clear that those f_i induce g by the above method. Hence, E is a coproduct of $\{A_i\}$ in $C(M)/\mathfrak{J}$. Since $C(M)/\mathfrak{J}$ is abelian, it is cocomplete. Furthermore, $C(M)/\mathfrak{J}$ is spectral and hence, every colimit is exact preserve, since every morphism splits. Hence, $C(M)/\mathfrak{J}$ is a C_3-abelian category. Finally, we shall show that $C(M)/\mathfrak{J}$ has a generator. Let $\mathfrak{R}$ be a right ideal in R and put $E_{\mathfrak{R}}=E(R/\mathfrak{R})$. Let F be the sub-set of right ideals $\mathfrak{R}$ such that $E_{\mathfrak{R}} \in C(M)/\mathfrak{J}$ and $U=E(\sum \oplus E_{\mathfrak{R}}) \in C(M)/\mathfrak{J}$. Let T be a non zero object in $C(M)/\mathfrak{J}$. then $T \supseteq tR \neq 0$ for some $t \in T$ in $\mathfrak{M}_R$. Since T is injective, there exists an R-monomorphism f' of $E_{\mathfrak{R}}$ to T, where $\mathfrak{R}=(0: t)_r$. Hence, there exists f in $[U, T]_R$ such that $f|E_{\mathfrak{R}}=f'$. Since $f' \notin \mathfrak{J}, f \notin \mathfrak{J}([U, T]_R)$. Therefore, U is a generator in $C(M)/\mathfrak{J}$.

REMARK 1. We note from the proof that the coproduct of $\{A_i\}$ in $C(M)/\mathfrak{J}$ is equal to $E(\sum \oplus A_i)$.

Corollary 1. *Let M be an injective right R-module with singular submodule $Z(M)=0$. Then $C(M)$ is a C_3-spectral full sub-category in $\mathfrak{M}_R$. Furthermore, M is a generator in $C(M)$ and the kernel and image of every morphisms in $C(M)$ conicide with them in $\mathfrak{M}_R$.*

Proof. Since $Z(M)=0$, $Z(N)=0$ for every object N in $C(M)$. Hence, the radical $\mathfrak{J}$ of $C(M)$ is equal ot zero, and $C(M)$ is a C_3-spectral category form Theorem 1. It is clear that M is a cogenerator in $C(M)$ and hence, M is a generator, since $C(M)$ is spectral. Let f be any R-homomorphism of N to N' ($N, N' \in C(M)$). Then there exists an idempotent e in $[N, N]_R$ such that $f=xe$ and $x \in [N, N']_R$. We know from the proof of [15], Theorem 2 that ker $f=eN$

in $\mathfrak{M}_R$. However eN is an object in $C(M)$ and hence, ker f in $\mathfrak{M}_R$ is equal to ker f in $C(M)$. Similary we have the same result for im f.

Especially, if we replace M in Corollary 1 by a self-injcetive regular ring R, then $Z(R)=0$, and hence this corollary coincides with [7], Satz 2.2.

We shall give another application of Theorem 1 which gives a different approach of [3], Corollary 9 in p. 62.

Corollary 2. *Let N be a right R-module with $Z(N)=0$. Then for any injective sub-modules Q_1, Q_2 in N, $Q_1 \cap Q_2$ and Q_1+Q_2 are injective. Let f be an R-homomorphism of Q_1 to an R-module such that $Z(f(Q_1))=0$, then im f and ker f are injective.*

Proof. Put $E=E(N)$ and $E'=E(f(Q_1))$, then $Z(E)=Z(E')=0$. If we take $M=E\oplus E'$, then Q_i's are objects in $C(M)$. It is clear that $Q_1 \cap Q_2$ and Q_1+Q_2 are the kernel and image of some morphisms in $C(M)$, respecitvely. Hence, they are injective from Corollary 1. The last statement is also clear.

Lemma 1. *Let $\mathfrak{A}$ be a full sub-category of $\mathfrak{M}_R$. If $\mathfrak{A}$ contains a generator (resp. cogenerator) in $\mathfrak{M}_R$, then every monomorphism (resp. epimorphism) in $\mathfrak{A}$ is mono-(resp. epi-) morphic in $\mathfrak{M}_R$.*

Proof. Let U be a generator in $\mathfrak{M}_R$ contained in $\mathfrak{A}$, and f: $A\to B$ monomorphic in $\mathfrak{A}$. We assume that ker $f\neq 0$ in $\mathfrak{M}_R$. Then there exists $g\neq 0$ in $[U, \ker f]_R$ and $fig=0$, where i is the inclusion of ker f into A. However, $ig\in [U, A]_{\mathfrak{A}}$ and hence, $ig=0$, which is a contradiction.

We note that if a ring R is quasi-Frobeniusean, then R is a self-injective, generator and cogenerator (or see example 1 in [13]).

Theorem 2. *Let M be an injective generator and cogenerator in $\mathfrak{M}_R$. Then $C(M)$ is an abelian category if and only if R is an artinian semi-simple ring.*

Proof. We assume that $C(M)$ is abelian. First, we shall show for any morphism f: $N\to N'$ in $C(M)$ that ker f in $C(M)$ is equal to ker f in $\mathfrak{M}_R$. Let f: $N \xrightarrow{f'} \operatorname{im} f \xrightarrow{i} N'$ be a decomosition in $C(M)$ with f' epimorphic and i monomorphic. Since f' (resp. i) is epi-(resp. mono-) morhic in $\mathfrak{M}_R$ by Lemma 1, $f=if'$ is also a decomposition of f in $\mathfrak{M}_R$. Therefore, im f is also the image of f in $\mathfrak{M}_R$. It is clear that ker f in $\mathfrak{A}$ is contained in ker f in M_R. Since every object in $C(M)$ is injective in $\mathfrak{M}_R$, $N=\ker f\oplus N''$ in $C(M)$ and $\mathfrak{M}_R$. Furthermore, $f\,|\,N''$ is isomorphic to im f, hence ker f in $C(M)=$ker f in $\mathfrak{M}_R$. Since M is a cogenerator in $\mathfrak{M}_R$, we have an exact sequence $0\to A\to \prod_I M \xrightarrow{f} \prod_{I'} M$ for any A in M_R. Since $f\in\mathfrak{A}$, ker $f=A$ is an object in $C(M)$ and hence, A is injective. Therefore, R is an artinian semi-simple ring. The converse is clear.

2. Completely reducible category of injective modules

In this section we shall study a completely reducibility of $C(M)/\mathfrak{J}$ for any injective module M.

Lemma 2. *Let $\mathfrak{B}$ be a full sub-category of $\mathfrak{M}_R$. Then any finite coproduct in $\mathfrak{B}/\mathfrak{J}$ is lifted from a coproduct in $\mathfrak{M}_R$, and every finite coproduct in $\mathfrak{M}_R$ of objects in $\mathfrak{B}$ is reduced to a coproduct in $\mathfrak{B}/\mathfrak{J}$, where $\mathfrak{J}$ is the radical of $\mathfrak{B}$.*

Proof. If $N=N_1\oplus N_2$ in $\mathfrak{M}_R$ for objects in $\mathfrak{B}$, then $1_N=e_1+e_2$, $e_i^2=e_i$ and $e_1\perp e_2$. Hence, $\bar{1}_N=\bar{e}_1+\bar{e}_2$ in $\mathfrak{B}/\mathfrak{J}$ and $N_1\oplus N_2$ in $\mathfrak{B}/\mathfrak{J}$. Conversely, we assume $N=N_1\oplus N_2$ in $\mathfrak{B}/\mathfrak{J}$. From the definition of $\mathfrak{B}/\mathfrak{J}$, there exist R-homomorphisms i, p such that $pi\equiv 1_{N_1}$ (mod $\mathfrak{J}_{N_1}$) and $\bar{i}$ is the inclusion of N_1 to N and $\bar{p}$ is the projection of N to N_1, where $i\in[N_1, N]_R$ and $p\in[N, N_1]_R$. Since $\mathfrak{J}_N$ is the radical, $pi=\alpha$ is isomorphic in $\mathfrak{M}_R$. Let N_1 be a sub-object of N via $i\alpha^{-1}$, then $N=N_1\oplus\ker p$ in $\mathfrak{M}_R$. It is clear from the first argument that $N=N_1\oplus N_2$ in $\mathfrak{B}/\mathfrak{J}$ is reduced from $N=N_1\oplus\ker p$ in $\mathfrak{M}_R$ and $\ker p\approx N_2$ in $\mathfrak{M}_R$.

Let T be an R-module. We call T a *completely (directly) indecomposable* module if $R_T/\mathfrak{J}_T$ is a division ring, where $R_T=[T, T]_R$ and $\mathfrak{J}_T$ is its radical, (cf. the condition (*) in [1]). It is clear that T is directely indecomposable in this case.

Proposition 3. *Let M be an injective module in $\mathfrak{M}_R$. An object N in $C(M)/\mathfrak{J}$ is minimal if and only if N is completely indecomposable in $\mathfrak{M}_R$.*

Proof. Since N is injective, "*only if*" part is clear. The converse is also clear from Lemma 2, since $C(M)/\mathfrak{J}$ is spectral.

Proposition 4. *Let R be a left perfect ring.*[2] *Then $C(M)/\mathfrak{J}$ is a completely reducible abelian C_3-category for any injective module M.*

Proof. Let N be any non-zero object in $C(M)/\mathfrak{J}$. Then N has the non-zero socle $S(N)$ by [2], Theorem P, say $S(N)=\sum\oplus I_\alpha$, I_α's are minimal modules. Since N is injective and R is left perfect, $N=E(\sum\oplus T_\alpha)$. Put $E_i=E(I_i)$, then E_i is completely indecomposable. Hence, N is a coproduct of minimal objects E_i by Proposition 3 and Remark 1.

Next, we shall consider a condition under which $\mathfrak{A}=\tilde{\mathfrak{A}}/\mathfrak{J}$ is completely reducible, where $\tilde{\mathfrak{A}}$ is the full sub-category of all injective modules in $\mathfrak{M}_R$. The essential part in the following argument is due to [11], Remark 2 in p. 516. However, we shall give a proof for the sake of completeness.

DEFINITION Let $\mathfrak{R}$ be a right ideal in R. $\mathfrak{R}$ is called *reducible* if $\mathfrak{R}=\mathfrak{N}_1\cap\mathfrak{N}_2$

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2) See the definition in [2]

for some right ideals $\mathfrak{N}_i$ in R and $\mathfrak{N}_i \neq \mathfrak{R}$ $(i=1, 2)$. If $\mathfrak{R}$ is not reducible, then $\mathfrak{R}$ is called *irreducible*.

Lemma 3 [11]. *Let $\mathfrak{A}$ be as above. Then $E_{\mathfrak{R}}$ is completely reducible in $\mathfrak{A}$ for every right ideal $\mathfrak{R}$ if and only if*

$$\mathfrak{R} = \mathfrak{N}_1 \cap \mathfrak{N}_2 \text{ and } \mathfrak{N}_1 \text{ is irreducible and } \mathfrak{N}_2 \neq \mathfrak{R} \quad \cdots\cdots\cdots\cdots\cdots (*)$$

Proof. We assume that $E_{\mathfrak{R}}$ is completely reducible and $E_{\mathfrak{R}} = E_1 \oplus E_2$ in $\mathfrak{A}$, where E_1 is minimal in $\mathfrak{A}$. Then we may assume from Lemma 2 that $E_{\mathfrak{R}} = E_1 \oplus E_2$ in $\mathfrak{M}_R$. Since E_1 is directly indecomposable, $E_1 = E_{\mathfrak{R}}$ and $\mathfrak{R}$ is irreducible by [11], Theorem 2.4. Let p_1, p_2 be projections of $E_{\mathfrak{R}}$ to E_1 and E_2, respectively. We put $\mathfrak{R}_i = \ker (p_i | R/\mathfrak{R})$. Then $\mathfrak{R} = \mathfrak{R}_1 \cap \mathfrak{R}_2$ and $\mathfrak{R}_i$'s are not equal to R, since $R/\mathfrak{R}$ is essential in $E_{\mathfrak{R}}$. Furthermore, $R/\mathfrak{R}_1 \approx p_1(R/\mathfrak{R}) \subseteq E_1$ and hence, $\mathfrak{R}_1$ is irreducible by [11], Theorem 2.4. If $\mathfrak{R} = \mathfrak{R}_2$, p_2 is monomorphic over $R/\mathfrak{R}$ in $\mathfrak{M}_R$ and hence, p_2 is a monomorphism of $E_{\mathfrak{R}}$ to $E_{\mathfrak{R}}$ which is a contradiction. Thus, we have the condition $(*)$. If the condition $(*)$ is satisfied, then $E_{\mathfrak{R}}$ has a minimal direct summand by Proposition 3 and [11], Theorems 2.3 and 2.4. Now let $S(E_{\mathfrak{R}})$ be the socle of $E_{\mathfrak{R}}$ in $\mathfrak{A}$, (since $\mathfrak{A}$ is a C_3-category, $S(E_{\mathfrak{R}})$ exists), then $E_{\mathfrak{R}} = S(E_1) \oplus E'$. It is clear that E' contains some $E_{\mathfrak{R}}$ if $E' \neq 0$. Hence, $E_{\mathfrak{R}} = S(E_{\mathfrak{R}})$.

Theorem 5. *Let $\mathfrak{A}$ be as in Lemma 3. Then $\mathfrak{A}$ is completely reducible if and only if the condition $(*)$ is satisfied for every right ideal $\mathfrak{R}$ in R.*[3)]

Proof. We know from the proof of Theorem 1 that $U = \sum \oplus E_{\mathfrak{R}}$ is a generator in $\mathfrak{A}$. Hence, $\mathfrak{A}$ is completely reducible if and only if every $E_{\mathfrak{R}}$ is completely reducible, since $\mathfrak{A}$ is a C_3-category,

Corollary 1. *If $\mathfrak{A}$ is completely reducible, then so is $C(M)/\mathfrak{J}$ for every injective module M.*

Proof. Every direct summand in $\mathfrak{A}$ of an object in $C(M)/\mathfrak{J}$ is an object in $C(M)/\mathfrak{J}$.

Corollary 2 ([11]). *Let R be a right noetherian, then every injective modules is a directsum of completely indecomposable modules.*

Proof. It is clear that R satisfies the condition $(*)$ if R is right noetherian. Let Q be an injective module, then Q is a coproduct of minimal object Q_α in $C(Q)/\mathfrak{J}$. Hence $Q \approx E(\sum_I \oplus Q_\alpha)$ in $\tilde{\mathfrak{A}}/\mathfrak{J}$. Since $\mathfrak{J}$ is the radical, $Q \approx E(\sum \oplus Q_\alpha)$ in $\mathfrak{M}_R$. Furthermore, $\sum \oplus Q_\alpha$ is injective, since R is right noetherian. Therefore, $Q = \sum \oplus Q_\alpha$ and Q_α is completely indecomposable from Proposition 3.

3) Added in proof. It is obtained in [19].

REMARK 2. The category $C(M)/\mathfrak{J}$ is a cocomplete C_3-abelian category with generator for any injective modules M by Theorem 1. If M is an essential extension of a sub-module which is a directsum of indecomposable injective sub-modules, M is completely reducible in $C(M)/\mathfrak{J}$ by Proposition 3 and Remark 1. In this case, since $C(M)/\mathfrak{J}$ is locally small, we can apply results in completely reducible modules in $\mathfrak{M}_R$ to M in $C(M)/\mathfrak{J}$, which we shall use freely in the following.

DEFINITION. An R-module M is called *uniform* if $M_1 \cap M_2 \neq 0$ for any non zero sub-modules M_1, M_2 in M. We consider sub-modules N which is a directsum of unifrom sub-modules M_α over an index I; $N=\sum_{\alpha\in I}\oplus M_\alpha$. We define $\dim M=\max_N \text{card } I$ if it exists.

Theorem 6. *Let E be an injective module in $\mathfrak{M}_R$. Then E has dim E and is a directsum of sub-modules E_1 and E_2 such that E_1 is a minimal module with dim E $=$dim E_1 and dim $E_2=0$, Furthermore, this decomposition is unique up to isomorphism.*

Proof. We note that an injective module Q is unifrom if and only if Q is directly indecomposable. First we consider every modules in $C(E)/\mathfrak{J}$ or in $\mathfrak{A}$. It is clear that $\dim E=0$ if and only if E contains no minimal objects in $C(E)/\mathfrak{J}$. Assume $\dim E \neq 0$, then E has the non zero socle $S(E)$ in $C(E)/\mathfrak{J}$, say $S(E)=E(\sum_{\alpha\in I}\oplus E_\alpha)$ $(=\sum\oplus E_\alpha$ in $C(E)/\mathfrak{J})$. Hence, $E=S(E)\oplus E_2$ and $\dim E_2=0$. Let N be a submodule such that $N=\sum_{\beta\in J}\oplus E'_\beta$. where E'_β is directly indecomposable, then $E(N)$ is contained in $S(E)$ in $C(M)/\mathfrak{J}$. Hence, card $J \leqslant$ card I. Therefore, $\dim S(E)=\dim E$. Next we assume $E=E_1\oplus E_2$ $=E_1'\oplus E_2'$ such that $\dim E_1=\dim E_1'=\dim E$ and $\dim E_2=\dim E_2'=0$. It is clear that $E_1=E_1'=S(E)$ and $E_2\approx E_2'$ in $C(M)/\mathfrak{J}$ and hence, $E_1\approx E_1'$ and $E_2\approx E_2'$ in $\mathfrak{M}_R$.

Corollary 1. *Let N be an R-module. Then N has the dimension and N is an essential extension of $N_1\oplus N_2$, where dim $N_2=0$ and N_1 contains a sub-module T such that $T=\sum_{\alpha\in I}\oplus T_\alpha$, T_α is unifrom and card I$=$dim N. If N is a quasi-injective and N_i's are closed in N*[4]*, then this decomposition is unique up to isomorphism.*

Proof. Put $E=E(N)$. Then $E=E_1\oplus E_2$ as in the theorem. We put $N_i=E_i\cap N$. It is clear that N is an essential extension of $N_1\oplus N_2$ and dim $N_2=0$. If $E_1=E(\sum_{\alpha\in I}\oplus E_\alpha)$, $E_\alpha\cap N_1=N_\alpha\neq 0$ and $T=\sum_\alpha \oplus N_\alpha$ is essential in N_1. If $T'=\sum_J \oplus N'_\alpha$, N_α's are unifrom, then E_1 contains an isomorphic

4) See the definition in [3], p. 15.

image of $E(T')=E(\sum_J \oplus E(N'_\alpha))$. Hence, card $J \leqslant$ card I. Therefore, $\dim N_1 = \dim N$. We assume that N is quasi-injective and N_1, N_2 and N_1', N_2' are as in the corollary. Since N_i's are closed, $N_i=N \cap E(N_i)$. Furthermore, $E(N_i)$ and $E(N'_i)$ are isomorphic each other by Theorem 6. There exists an R-automorphism φ of E such that $\varphi(E_i)=E'_i$, where $E_i=E(N_i)$ and $E_i'=E(N_i)$. Hence, $\varphi(N_i)=\varphi(N \cap E_i)=N \cap E'_i=N'_i$ by [3], §5 or [9].

We shall give a slight generalization of [4], Theorem 6, 5. However, the proof is much simpler than them. We shall study further the problem of this type in the section 3.

Corollary 2 ([4]). *Let Q be an R-module which is a directsum of directly indecomposable injective modules Q_α; $Q=\sum \oplus Q_\alpha$. Then we have*

i. *S is a sub-module of Q such that $S=\sum_J \oplus P_\beta$ and P_β's are directly indecomposable, then card $I \geq$ card J. Furthermore, if Q is injective then S is injective.*

ii. *If Q is quasi-injective and S is injective, then $S=\sum_J \oplus P_\beta$.*

Proof. We consider all objects in $\mathfrak{A}$; category of injective modules modulo $\mathfrak{J}$. Let $E=E(Q)$ and $F=E(S)$ $(\subset E)$, then F is contained in the socle $S(E)$ of E in $\mathfrak{A}$. Hence, card $J \leqslant$ card I. Furthermore, every P_β is isomorphic to some $Q_{\pi(\beta)}$ in $\mathfrak{A}$ and hence in $\mathfrak{M}_R$, where π is a one-to-one mapping from J to I. If Q is injective, then S is isomorphic to $\sum_{\beta \in J} \oplus Q_{\pi(\beta)}$ in $\mathfrak{A}$ and hence, in $\mathfrak{M}_R$. Therefore, S is injective. ii. If S is injective, then $F=S$. $S=E(\sum_{\beta \in J} \oplus P_\beta)$ and $P_\beta \overset{\rho_\beta}{\approx} Q_{\pi(\beta)}$. We define an R-monomorphism f of $\sum_J \oplus P_\beta$ to $\sum_J \oplus Q_{\pi(\beta)}$ via ρ_β. Then we have a diagram

$$\begin{array}{c} 0 \rightarrow \sum \oplus P_\beta \rightarrow S \\ \downarrow f \\ \sum \oplus Q_{\pi(\beta)} \oplus \sum_{\alpha \notin \pi(J)} Q_\alpha = Q \end{array}$$

Since Q is quasi-injective, we have an extension $g \in [Q, Q]_R$ of f. Since $\sum \oplus P_j$ is essential in S, g is monomorphic. Hence, $\sum_J \oplus Q_{\pi(\beta)}$ is essential in $g(S)$. Therefore, $g(S)=\sum_J \oplus Q_{\pi(\beta)}$.

Finally, we shall give some remarks and examples concerned with the category $\mathfrak{A}$ of injective modules modulo $\mathfrak{J}$.

Let $\mathfrak{A}_i$ be a full sub-category of $\mathfrak{A}$ whose object consists of all E_i in Theorem 6. Then $\mathfrak{A}_1$ is a completely reducible C_3-category and every objects in $\mathfrak{A}_2$ has zero-socle. Hence, $\mathfrak{A}=\mathfrak{A}_1 \times \mathfrak{A}_2$ (cf. [7].) Furthermore, we have another decomposition of $\mathfrak{A}$. Let $\mathfrak{A}_1^*$ (resp. $\mathfrak{A}_2^*$) be the full sub-category of $\mathfrak{A}$ whose object consists of all A in $\mathfrak{A}$ such that $A=E(S_R(A))$ (resp. $S_R(A)=0$), where $S_R(A)$ is the socle of A in $\mathfrak{M}_R$. It is clear that we have for any object N in $\mathfrak{A}$ $N=N_1\oplus N_2$, $N_i\in\mathfrak{A}_i^*$. For any $f\in[N_1, N_2]_R$ we have $\ker f\supseteq S(N_1)$ and hence, $f\in\mathfrak{J}$. Thus, we have $\mathfrak{A}=\mathfrak{A}_1^*\times\mathfrak{A}_2^*$. It is clear that $\mathfrak{A}_1^*\subseteq\mathfrak{A}_1$.

EXAMPLE 1. We shall use the same example given in [13], p. 378. Let $Z_{(p)}$ be the p-adic integers for some prime p and $R=Z_{(p)}\oplus Z_{p^\infty}$ with multiplication $(\lambda, x)(\mu, y)=(\lambda\mu, \lambda y+\mu x)$, where $\lambda\mu,\in Z_{(p)}$, x, $y\in Z_{p^\infty}$. Then $E_{\mathfrak{R}}=R$ or $Q_{(p)}$: the quotient field of $Z_{(p)}$, where $\mathfrak{R}$ is an ideal in R. Hence, $\mathfrak{A}$ is completely reducible in this case, however R is neither noetherian nor perfect. Furthermore, $\mathfrak{A}_1=\mathfrak{A}$ and $\mathfrak{A}_1^*$ is a category with generator R and $\mathfrak{A}_2^*$ with $Q_{(p)}$. This fact shows that Corollaty 1 is not true if $Z(M)\neq 0$. Since $U=R\oplus Q_{(p)}$ is a samll generator, $\mathfrak{A}$ is equivalent to the category of $(Z/p\oplus Q_{(p)})$-modules.

EXAMPLE 2. We shall give an example in which $\mathfrak{A}$ is not completely reducible.

Let $J=[0, 1]$ be the close interval in the real numbers K. Let R be the ring of continuous functions from J to K. By $I_{\mathfrak{r}}$ we denote $\bigcap_{f\in\mathfrak{r}} f^{-1}(0)$ for an ideal $\mathfrak{r}$ in R. We assume that $I_{\mathfrak{r}}$ containes a closed interval I which is not equal to a point. Let $I=I_1\cup I_2$ and $I_i=\bar{I}_i$, $\bar{I}_1\cap\bar{I}_2=(x)$. We put $\mathfrak{r}_i'=\{f\mid\in R$, $f\mid\bar{I}_i^c=0\}$, and $\mathfrak{r}_i=\mathfrak{r}+\mathfrak{r}_i'$. Then $\mathfrak{r}_i\supsetneqq\mathfrak{r}$ and $\mathfrak{r}=\mathfrak{r}_1\cap\mathfrak{r}_2$. Hence $\mathfrak{r}$ is reducible. It is clear that the zero ideal (0) is reducible. We assume $(0)=\mathfrak{r}\cap\mathfrak{r}'$ and $\mathfrak{r}$ is irreducible. There exists, for any $f\neq 0\in\mathfrak{r}'$, a not point colsed interval L such that $f(l)\neq 0$ for all $l\in L$. Hence, $L\cap I_{\mathfrak{r}'}=\phi$. If $I_{\mathfrak{r}}\cup I_{\mathfrak{r}'}=J$, then $I_{\mathfrak{r}}\supseteq L$, which contradicts to the fact that $\mathfrak{r}$ is irreducible. Hence, there exists a point a not in $I_{\mathfrak{r}}\cup I_{\mathfrak{r}'}$. Then there exist $f\in I_{\mathfrak{r}}$ and $g\in I_{\mathfrak{r}'}$ such that $f(a)\neq 0$, $g(a)\neq 0$. On the other hand, $fg\in\mathfrak{r}\cap\mathfrak{r}'=(0)$, which is a contradiction. Thus, the category of injective R-modules modulo $\mathfrak{J}$ is not completely reducible.

3. Krull-Remak-Schmidt-Azumaya's theorem

We shall study the Krull-Remak-Schmidt-Azumaya's theorem for R-modules. Our proof will be somewhat different from ordinal ones. We shall make use of the same argument in the previous sections, however our method will be substantially analogous to that in [1].

Let M be a right R-module and we assume

$$M=\sum_{\alpha\in I}\oplus M_\alpha \quad \cdots(1) \quad \text{and} \quad M=\sum_{\beta\in J}\oplus N_\beta \qquad \cdots(1'),$$

where M_α's and N_β's are completely indecomposable.
We consider the following statements.

I. card $I=$ card J, *and there exists a one-to-one mapping* φ *of* I *to* J *such that* $M_\alpha\approx N_{\varphi(\alpha)}$ *for all* $\alpha\in I$.

II. *For any sub-set I' in I (resp. J' in J) there exists a one-to one mapping ψ of I' into J (resp. J' into I) such that $M_\alpha \approx N_{\psi(\alpha)}$ for all $\alpha \in I'$ (resp. $N_\beta \approx M_{\psi(\beta)}$ for all $\beta \in J'$) and*

$$M = \sum_{\alpha' \in I'} \oplus N_{\psi(\alpha')} \oplus \sum_{\beta \in I - I'} \oplus M_\beta \text{ (resp. } M = \sum_{J \in J'} \oplus N_\beta \oplus \sum_{\beta' \in I - \psi(I')} \oplus M_{\beta'})$$

III. *Every direct summand of M is also a directsum of completely indecomposable modules, which are isomorphic to some M_α.*

It is well known as the Krull-Remak-Schmidt-Azumaya's theorem that II and III for any finite set I' and condition I for any set I and J are satisfied for any decomposition (1) and (1′), (cf. [1]). Corollary 2 is a special case for the condition III. It is clear that if M_β's and N_β's are all minimal modules, then all conditions are satisfied, and we note that those arguments for completely reducible modules are valid for a completely reducible object in a C_3-abelian locally small category.

Some parts in the following will overlap with results in [1], however, we shall give prooves for the sake of completeness.

We assume a right R–module M has a decomposition as in (1) and (1′). We take a set $\{a_{\sigma\tau}\}_\sigma$ of R-homomorphisms $a_{\sigma\tau}$ of M_τ to M_σ. We call $\{a_{\sigma\tau}\}_\sigma$ *summable* if for any non-zero element m in M_τ, $a_{\sigma\tau}(m)=0$ for almost all σ. In this case $\sum_\sigma a_{\sigma\tau}$ is an R-homomorphism of M_τ to M. It is well known that $[M, M]_R$ is isomorphic to the ring of matrices whose (σ, τ)-component consists of all elements of $[M_\tau, M_\sigma]_R$ and every family of components in any column is summable (we call it simply *column summable*).

Let $M = \sum_I \oplus M_\alpha$ and $N = \sum_J \oplus N_\tau$ as in (1). Then $[M, N]_R$ is isomorphic to the module of matrices as above. By $\mathfrak{J}^{(\tau,\sigma)}$ denote the sub-set of those matrices whose each components are not isomorphic. It is clear that $\mathfrak{J}^{(\tau,\sigma)}$ is a module since M_σ's and N_τ'a are completely indecomposable and $\mathfrak{J}^{(\tau,\sigma)}$ may depend on a decomposition (1)

The following lemma is well known

Lemma 4. *Let M_i $(i=1, 2, 3)$ be completely indecomposable and α_i $(i=1, 2)$ R-homomorphisms of M_i to M_{i+1}. If $\alpha_2\alpha_1$ is isomorphic, then α_1 and α_2 are isomorpihc.*

Lemma 5. *Let $M = \sum_I \oplus M_\alpha$, $N = \sum_J \oplus N_\sigma$ and $T = \sum_K \oplus T_\rho$ be as in (1). Then $[N, T]_R \mathfrak{J}^{(\sigma,\alpha)} \subseteq \mathfrak{J}^{(\rho,\alpha)}$ and $\mathfrak{J}^{(\rho,\sigma)}[M, N]_R \in \mathfrak{J}^{(\rho,\alpha)}$.*

Proof. Let $f=(a_{ij})$ be in $\mathfrak{J}^{(\alpha,\sigma)}$ and $h=(b_{lk})$ in $[N, T]_R$. Put $hf=(x_{ts})$, $x_{ts} = \sum_k b_{tk}a_{ks}$. If $M_s \not\approx M_t$, then x_{ts} is not isomorphic. We assume $M_s \approx M_t$ and $a_{ks}(m)=0$ for $k \in J-(k_1, \cdots k_n)=J'$ and $m \neq 0 \in M_s$. Then we put

$x_{ts}=\sum_{i=1}^{n} b_{tk_i}a_{k_is}+\sum_{k\in J} b_{tk}a_{ks}$. Since $b_{tk_i}a_{k_is}$'s are not isomorphic from Lemma 4, x_{ts} is not isomorphic from the assumtion of M_α. Hence, $hf\in\mathfrak{J}^{(\rho,\alpha)}$. The last part is similar.

Corollary. *$\mathfrak{J}^{(\sigma,\alpha)}$ does not depend on the decomposition* (1). *Furthermore, if $M=N$, $\mathfrak{J}^{(\sigma,\alpha)}$ is a two-sided ideal of $[M, M]_R$, (cf.* [1], *Theorem* 2.3).

Proof. Let $M=\sum\oplus M_\alpha$ and $N=\sum\oplus N_\sigma=\sum\oplus N'_{\sigma'}$. We put $N=T=\sum\oplus N'_{\sigma'}$ in Lemma 5. For any f in $\mathfrak{J}^{(\sigma,\alpha)}$ we have $f=1_N f\in\mathfrak{J}^{(\sigma',\alpha)}$. Hence, $\mathfrak{J}^{(\sigma,\alpha)}\subseteq\mathfrak{J}^{(\sigma',\alpha)}$. Similarly $\mathfrak{J}^{(\sigma',\alpha)}\subseteq\mathfrak{J}^{(\sigma,\alpha)}$. The last part is clear.

We shall denote $\mathfrak{J}^{(\sigma,\alpha)}$ by $\mathfrak{J}'[M, N]$.

Let $\mathfrak{A}$ (resp. $\mathfrak{A}_f$) be the full sub-category of $\mathfrak{M}_R$ whose object consists of all modules which are coproduct (resp. finite coproduct) of a given family $\{M_\alpha\}$ of completely indecomposable modules M_α. We define a two-dised ideal $\mathfrak{J}'$ in $\mathfrak{A}$ (resp. in $\mathfrak{A}_f$) by setting: $\mathfrak{J}'\cap[M, N]=\mathfrak{J}'[M, N]$ for every M, N in $\mathfrak{A}$. It is clear from Corollary to Lemma 5 that $\mathfrak{J}'$ is an ideal in $\mathfrak{A}$.

Theorem 7. *Let $\mathfrak{A}$ and $\mathfrak{A}_f$ be as above for a given family $\{M_\alpha\}$ of completely indecomposable modules M_α. Then $\mathfrak{A}/\mathfrak{J}'$ (resp. $\mathfrak{A}_f/\mathfrak{J}'$) is a C_3-completely reducible (resp. completely reducible) abelian category.*

We need some well known results for the proof.

Lemma 6. *Let R be a ring and e, f be idempotents such that $eR\approx fR$ and $(1-e)R\approx(1-f)R$. Then there exists a regular element a in R such that $f=a^{-1}ea$.*

Proof. $R=eR\oplus(1-e)R=fR\oplus(1-f)R$. Let φ_1, φ_2 be given isomorphisms. $R=[R, R]_R\ni\varphi=\varphi_1+\varphi_2$. Hence, $\varphi=a_l$ for some regular element a and $a_le_l=f_la_l$.

Corollary. *Let Δ be a division ring and P a right Δ-module. We put $R=[P, P]_\Delta$ and $P=\sum_I\oplus u_\alpha\Delta$. Let e be an idempotent in R. Then there eixst a subset J of I and a regular element a in R such that $e=a^{-1}fa$, where f is a projection of P to $\sum_J\oplus u_\beta\Delta$.*

Proof. $P=eP\oplus(1-e)P$ as a Δ-module and $eP=\sum_J\oplus v_\beta\Delta$. Then $eR\approx fR$ and $(1-e)R\approx(1-f)R$. Hence, the corollary is true from Lemma 6.

Proof of Theorem 7. It is clear that $\mathfrak{A}/\mathfrak{J}'$ (resp. $\mathfrak{A}_f/\mathfrak{J}'$) has any (resp. finite) coproduct from Corollary to Lemma 5. We shall denote every morphisms in $\mathfrak{A}$ by column summable matrices. Let $(a_{\sigma\tau})$ be any morphism in $[M, M]_R$. Since $a_{\sigma\tau}(m_\tau)=0$ for almost all σ and m_τ, $a_{\sigma\tau}\in\mathfrak{J}'[M_\tau, M_\sigma]$ for alsmost all σ. Hence, $[M, N]/\mathfrak{J}'[M, M]$ is isomrphic to the ring of column finite matrices.

Let $M=\sum_{\alpha}\sum_{\rho\in I_\alpha}\oplus M_{\alpha\rho}$ and $M_{\alpha\rho}\approx M_{\alpha\rho'}$, $M_{\alpha\rho}\not\approx M_{\alpha'\rho'}$ if $\alpha\neq\alpha'$. Then $[M, M]_{\mathfrak{A}/\mathfrak{J}'}$ $=\prod_{\alpha}[\sum_{\rho\in I_\alpha}\oplus M_{\alpha\rho}, \sum_{\rho\in I_\alpha}\oplus M_{\alpha\rho}]_{\mathfrak{A}/\mathfrak{J}'}$ Furthermore, $[\sum_{\rho\in I_\alpha}\oplus M_{\alpha\rho}, \sum_{\rho\in I_\alpha}\oplus M_{\alpha\rho}]_{\mathfrak{A}/\mathfrak{J}'}$ is isomorphic to the ring of column finite matrices over the division ring $[M_{\alpha 1}, M_{\alpha 1}]/\mathfrak{J}'[M_{\alpha 1}, M_{\alpha 1}]$. Hence, $\mathfrak{A}/\mathfrak{J}'$ is regular category defined in [15]. We denote $[\sum\oplus M_{\alpha\rho}, \sum\oplus M_{\alpha\rho}]_{\mathfrak{A}/\mathfrak{J}'}$ and $[\sum\oplus M_{\alpha\rho}, \sum\oplus M_{\alpha\rho}]_R$ by $\bar{R}_\alpha$ and R_α, respectively. Let $\bar{e}$ be an idempotent in $[M, M]_{\mathfrak{A}/\mathfrak{J}'}$. Then $\bar{e}=\prod_{\alpha}\bar{e}_\alpha$, where $\bar{e}_\alpha$'s are idempotents in $\bar{R}_\alpha$. There exist an idempotent $\bar{f}_\alpha$ and a regular element $\bar{a}_\alpha$ in $\bar{R}_\alpha$ such that $\bar{e}_\alpha=\bar{a}_\alpha{}^{-1}\bar{f}_\alpha\bar{a}_\alpha$ and f_α is a projection of $\sum_{\rho\in I_\alpha}\oplus M_{\alpha\rho}$ to a direct summand $\sum_{\beta\in J_\alpha}\oplus M_{\alpha\beta}$ by Corollary to Lemma 6. Hence, ker $\bar{f}_\alpha$ exists and is equal to $\sum_{\beta'\in I_\alpha-J_\alpha}\oplus M_{\alpha\beta'}$ in $\mathfrak{A}/\mathfrak{J}'$. Therefore, ker $\prod\bar{f}_\alpha$ exists and, since $\bar{a}=\prod\bar{a}_\alpha$ is regular in $\prod\bar{R}_\alpha$, $\bar{e}=\bar{a}^{-1}(\prod\bar{f}_\alpha)\bar{a}$ has the kernel in $\mathfrak{A}/\mathfrak{J}'$. Thus, $\mathfrak{A}/\mathfrak{J}'$ (resp. $\mathfrak{A}_f/\mathfrak{J}'$) is a C_3-spectral (resp. spectral) abelian category by [15], Theorem 2. Since $\mathfrak{A}/\mathfrak{J}'$ is semi-simple and $[M_{\alpha 1}, M_{\alpha 1}]_{\mathfrak{A}/\mathfrak{J}'}$ is a division ring, $M_{\alpha 1}$ is a minimal object by [8], Lemma 1.3. Therefore, $\mathfrak{A}/\mathfrak{J}'$ and $\mathfrak{A}_f/\mathfrak{J}'$ are completely reducible.

Lemma 7. *Let M be an object in $\mathfrak{A}$. Then $\mathfrak{J}'[M, M]$ does not contain non zero idempotents* ([1], Theorem 3).

Proof. $M=\sum\oplus M_\alpha$ and e_α is the projection of M to M_α. For an idempotent e we have $e_\alpha=e_\alpha ee_\alpha+e_\alpha(1-e)e_\alpha$. Hence, $e_\alpha ee_\alpha$ or $e_\alpha(1-e)e_\alpha$ is isomorphic. If $e\neq 0$, there exists a finite set $\{\alpha_i\}$ such that $eM\cap\sum_{i=1}^{n}\oplus M_{\alpha_i}\neq(0)$. Hence, $e_{\alpha_i}ee_{\alpha_i}$ is isomorphic for some i, which implies $e\notin\mathfrak{J}'[M, M]$.

Lemma 8. *The ideal $\mathfrak{J}'$ in $\mathfrak{A}_f$ is the Jacobson radical of $\mathfrak{A}_f$.*

Proof. First, we assume that $M=\sum_{i=1}^{n}\oplus M_i$ and $M_i\approx M_1$ for all i. Let $X=(x_{ij})$ be in $\mathfrak{J}'=\mathfrak{J}'[M, M]$. Then $1_{M_i}-x_{ii}$ is regular in $[M_i, M_i]_R$. Hence, by Lemma 4 and taking fundamental transformations, we know that there exists regular matrices P and Q in $[M, M]_R$ such that $P(I-X)Q=I$. Hence, X is quasi-regular, and $\mathfrak{J}'$ is contained in the radical of $[M, M]_R$. Since $\mathfrak{A}_f/\mathfrak{J}$ is semi-simple, $\mathfrak{J}'$ is the radical of $[M, M]_R$. In general case, we can use the same argument by Lemmas 4 and 5 and hence, $\mathfrak{J}'$ is the radical of $\mathfrak{A}_f$.

Corollary 1. (K-R-S-A Theorem) ([1], Theorem 1) *Let M be a right R-module which is a directsum of completely indecomposable modules M_α and N_β as in* (1) *and* (1′). *Then Condition* I *and Condition* II *for any finite sub-set I' and J' are satisfied and for any direct summand M' of M M' is either isomorphic to $\sum_{i=1}^{n}\oplus M_{\alpha_i}$ for some n or M' contains a direct summand which is isomorphic to $\sum_{i=1}^{n}\oplus M_{\alpha_i}$ for any n.*

Proof. We take that $\{M_\alpha, N_\beta\}$ is a given family and consider the additive category $\mathfrak{A}$ as before. Condition I is clear from Theorem 7 and Lemma 8. Put $M'=\sum_{i=1}^{n}\oplus M_{\alpha_i}$ and p the projection of M to M'. Then $M=\ker p\oplus(N_{\psi(\alpha_1)}\oplus\cdots\oplus N_{\psi(\alpha_n)})$ in $\mathfrak{A}/\mathfrak{J}'$, which means that pi is isomorphic in $[N', M']/\mathfrak{J}'[N', M']$, where $N'=\sum\oplus N_{\psi(\alpha_i)}$ and i is the inclusion of N' to M. Since M' and N' are in $\mathfrak{A}_f$, $\mathfrak{J}'[N', M']$ is the radical of $[N', M']_R$. Hence, pi is isomorphic in $\mathfrak{M}_R$. Therefore, $M=N'\oplus\ker p=N'\oplus\sum_{I-I'} M$, where $I'=\{1, 2, \cdots, n\}$. Conversely, we take a finite family $\{N_i\}_{i\in J'}$, then $M=N''\oplus\sum_{I-\psi(J')}\oplus M_{\beta'}$ in $\mathfrak{A}/\mathfrak{J}'$, where $N''=\sum_{J'}\oplus N_i$ and $\psi'\colon J'\to I$. Let p' be a projection of M to $\sum\oplus M_{\psi'(\alpha)}$. Then it is clear that $p\,|\,N''$ in $\mathfrak{A}/\mathfrak{J}'$ is isomorphic. Hence, $M=N''\oplus\sum_{I-\psi(J')}\oplus M_{\beta'}$ in $\mathfrak{M}_R$. Thus, we have proved Condition II for finite sub-set I' and J'. Finally, let $M'=eM$ be a direct summand of M and $e^2=e\neq 0$. Then $e\notin\mathfrak{J}'$ by Lemma 7. Hence, $\bar{e}$ has the image $\bar{e}M=\sum_{\alpha'\in I'}\oplus M'_{\alpha'}$ in $\mathfrak{A}/\mathfrak{J}'$. $\sum_{I'}\oplus M'_{\alpha'}$ contains a direct summand $M''=\sum_{i=1}^{t}\oplus M'_{\alpha}$. Let p and i be R-homomorphisms of M to M' and M'' to M such that $\bar{p}$ and $\bar{i}$ in $\mathfrak{A}/\mathfrak{J}'$ are projection and injection, respectively. Then pei is isomorphic in $\mathfrak{A}/\mathfrak{J}$.' Since M'' is in $\mathfrak{A}_f$, pei is isomorphic in $\mathfrak{M}_R$ by Lemma 8. Therefore, eM contains a direct summand which is isomorphic to M''. If $\bar{e}M=M''$ in $\mathfrak{A}/\mathfrak{J}'$, then $eM=M''\oplus M_0$. Hence, $e=e''+e_0$, $e''^2=e''$ and $e_0^2=e_0$. Furthermore, $\bar{e}=\bar{e}''$. Therefore, $\bar{e}_0=0$, which implies $e=0$ by Lemma 7. It is clear that M_{α_i} is isomorphic to some M_α.

Corollary 2. *Let M be an in* (1). *We assume that $\mathfrak{J}'[M\ M]$ is the radical of $[M, M]_R$, then Conditions* II *and* III *are satisfied for any sub-set I' and J'.*

Proof. Let M' be a direct summand of M. Then $R_{M'}=[M', M']=eR_Me$ for some idempotent e. If $M'\approx\sum_{\alpha'\in I'}\oplus M_{\alpha'}$ and $M/M'\approx\sum_{I''}\oplus M_{\beta'}$, then $\mathfrak{J}'R_{M'}=R_{M'}\cap\mathfrak{J}'=e\mathfrak{J}'e$. Hence, $\mathfrak{J}'R_{M'}$ is the radical of $R_{M'}$. In this case we can replace a finite subset I' by any sub-set in I in the proof of the corollary 1. Hence, Condition II is satisfied for any sub-set in I and J. For Condition III we consider $M=M_1\oplus M_2$ in $\mathfrak{M}_R$. Then $\bar{e}_iM=\sum_{I_i\in\alpha_i}\oplus M_{\alpha_i}$ in $\mathfrak{A}/\mathfrak{J}'$ and hence, $e_iM\approx\sum_{I_i\in\alpha_i}\oplus M_{\alpha_i}$ by the above and the proof of the corollary 1.

We shall give a converse of [15], Theorem 7.

Proposition 8. *Let $\mathfrak{B}$ be a full sub-category of $\mathfrak{M}_R$. Then $\mathfrak{B}/\mathfrak{J}$ is an artinian completely reducible category if and only if $\mathfrak{B}=\mathfrak{A}_f$ for some family $\{M_\alpha\}$ of completely indecomposable modules M_α, where $\mathfrak{J}$ is the radical of $\mathfrak{B}$.*

Proof. "*If part*" is clear from Lemma 8. We assume that $\mathfrak{B}/\mathfrak{J}$ is artinian and completely reducible. Let M be an object in $\mathfrak{B}$. Then $M=\sum_{i=1}^{n}\oplus M_i$ and M_i's are minimal in $\mathfrak{B}/\mathfrak{J}$. We may assume $M=\sum_{i=1}^{n}\oplus M_i$ in $\mathfrak{M}_R$ by Lemma 2. It is clear from the definition of $\mathfrak{J}$ that M_i's are completely indecomposable.

We shall consider the converse of Corollary 2.

We take a decomposition (1) for an R-module M with infinit set I, and consider a linear ordered sub-set I_0 with card $I_0=\aleph_0$ in I. Let f_i be an R-homomorphism of M_i to M_{i+1}, which is not isomorphic for any i, $i+1\in I_0$. By $\theta(j, i)$ we denote the composition of $f_j, f_{j-1}, \cdots, f_{i+1}, f_i$ for $j>i$.

Lemma 9. *Let M be as above and $\{f_k\}$ a family of non-isomorphisms of M_k to M_{k+1}. If Condition* II *is satisfied for any sub-set I_0, then for any i and any element m_i in M_i, there exists j ($j>i$) such that $\theta(j, i)(m_i)=0$. Especially, in this case there exist only finite many non-isomorphic monomorphisms f_k.*

Proof. First, we assume that all f_i are monomorphic. We put $M'_n=\{x+f_n(x)\,|\,x\in M_n\}$ and $M_0=\sum_{I-I_0}\oplus M_\alpha$. Then it is clear that

$$M=M_1\oplus M_2'\oplus M_3\oplus M_4'\oplus\cdots\oplus M_0=M_1'\oplus M_2\oplus M_3'\oplus M_4\oplus\cdots\oplus M_0 \quad\cdots(2)$$

We apply Condition II for $I'=(2, 4, \cdots, 2n, \cdots)$, then we have $M=M_1'\oplus M_3'\oplus\cdots\oplus M_0\oplus\psi_2(M_2)\oplus\psi_4(M_4)\oplus\cdots$, where ψ_{2n} are isomorphisms of M_{2n} to some direct summand of the left side in the above. From the assumption, any f_{2n-1} are not epimorphic and hence, $\sum\psi_{2n}(M_{2n})\supseteq\sum\oplus M'_{2n}$. It is clear that $(\sum\psi_{2n}(M_{2n}))\cap M_0=(0)$. We assume $\psi_{2n}(M_{2n})=M_{2i+1}$ and $\psi_{2n'}(M_{2n'})=M_{2j+1}$ and $j>i$. In this case we have for $x\neq 0$ in M_{2i+1}

$$\begin{aligned} x &= (x+f_{2i+1}x)-(f_{2i+1}x+f_{2i+2}f_{2i+1}x)+\cdots \\ &\quad \pm(\theta(2j-1, 2i+1)x+\theta(2j, 2i+1)x)\mp\theta(2j, 2i+1)x \\ &\in M_{2i+1}\cap M'_{2i+1}\oplus M'_{2i+2}\oplus\cdots\oplus M'_{2j}\oplus M_{2j+1} \qquad\cdots(3) \end{aligned}$$

This is a contradiction. Hence, we may assume that $\psi_{2n}(M_{2n})$ is equal to some M'_{2m} for all $n\geqslant$ some n_0. Then if we consider a non-zero element in $M_{2n'+1}$ for some large n' as the expression (3), we have that $M\not\supset M_{2n'+1}$, since f_i's are monomorphic, which is a contradiction. Thus, we have proved the last part. Therefore, there exists infinite many of non-monomorphisms $\{f_{i_t}\}$ in $\{f_k\}$. We put $g_t=\theta(i_{t+1}-1, i_t)$, then any g_t are not monomorphic. It is clear that we may assume $M_{i_t}=M_t$ in the lemma. We shall use the same argument for the new non-monomorphism f_i. Let x be a non-zero element in ker f_{2i+1}, then $x\in M_{2i+1}\cap M'_{2i+1}$. Hence, we know that $\psi_{2n}(M_{2n})$ is not equal to any M_{2i+1} and $\psi_{2n}(M_{2n})=M'_{2m}$ for some m. Now we take any non-zero element x in M_1 and consider an expression of x as in (3). Then we know that $\theta(m, 1)(x)=0$ for some m.

We call a family $\{f_i\}$ of R-homomorphisms an *elementwise T-nilpotent system* (or left *vanishing*) if $\{f_i\}$ satisfies the consequence of Lemma 9.

Lemma 10. *Let M be as in* (1) *and $\mathfrak{J}'$ the ideal in Theorem* 7. *If any family of components in $\mathfrak{J}'$ is an elementwise T-nilpotent system, then $\mathfrak{J}'$ is the radical in $\mathfrak{A}$.*

Proof. Let A be any element in $\mathfrak{J}'$ and m_σ any element in M_σ. Put $A=(a_{\sigma\tau})$. Since A is column summable, there exists a finite set F_1 of indeces $\tau_i^{(1)}$ such that $a_{\tau_i^{(1)}\sigma}(m_\sigma)=m_{\tau_i^{(1)}}\neq 0$ and $a_{\tau\sigma}(m_\sigma)=0$ if $\tau\in I-F_1$. Similarly we have a finite set F_2 of $\tau_j^{(2)}$ such that $a_{\tau_j^{(2)}\tau_i^{(1)}}(m_{\tau_i^{(1)}})=m_{\tau_j^{(2)}}\neq 0$ and $a_{\tau\tau_i^{(1)}}(m_{\tau_i^{(1)}})=0$ if $\tau\in I-F_2$ for any $\tau_i^{(1)}$ in F_1. Repeating this argument, we have a family of finite sub-set $\{F_i\}$. Then $A^n(m_\sigma)=\sum a_{\tau_{i_n}^{(n)}\tau_{i_{n-1}}^{(n-1)}}a_{\tau_{i_{n-1}}^{(n-1)}\tau_{i_{n-2}}^{(n-2)}}\cdots a_{\tau_{i_1}^{(1)}\sigma}(m_\sigma)$, where $\tau^{(i)}$ runs through elements in F_i. From the assumption there exists n_σ such that $A^{n_\sigma}(m_\sigma)=0$ by the Konig Graph theorem, (cf. [11], p.42). Hence, if we put $B=\sum_{n=1}^{\infty}A^n$, then B is an element in $[M, M]_R$ since $\{A^n\}$ is a summable system. Furthermore, $A(-B)-A=-B$. Hence, A is quasi-regular in $[M, M]_R$. Therefore, $\mathfrak{J}'$ is the radical in $\mathfrak{A}$ by Theorem 7.

Theorem 9. *Let M be a directsum of completely indecomposable sub-modules M_α and N_β as in* (1) *and* (1′). *Then the following three statements are equivalent.*

i. *Condition* II *is satisfied for any objects in $\mathfrak{A}$ defined in Theorem* 7 *and any sub-set I' and J'*

ii. *$\mathfrak{J}'$ defined in Theorem 7 is the radical in $\mathfrak{A}$.*

iii. *Every family of non-isomorphic R-homomorphisms of M_α to M_{α}' (not necessarily $\alpha\neq\alpha'$) is an elementwise T-nilpotent system.*

Furthermore, the fact that Condition III *is satisfied for any direct summand of M is equivalent to*

iv. *Let e, f be idempotents in $R_M=[M, M]_R$. If $eR_M/\mathfrak{J}'_M\approx fR_M/\mathfrak{J}'_M$, then $eR_M\approx fR_M$.*

And i *implies* iv.

Proof. i implies iii by Lemma 9. iii implies ii by Lemma 10 and Theorem 7. ii implies i by Corollary 2 to Theorem 7. We assume Condition III. Then $M=eM\oplus(1-e)M$ and eM and $(1-e)M$ are objects in $\mathfrak{A}$. Hence, eM is equal to the the image $\bar{e}$ in $\mathfrak{A}/\mathfrak{J}'$ by Lemma 2. If $\bar{e}\approx\bar{f}$ in $\mathfrak{A}/\mathfrak{J}'$. then im $\bar{e}=eM$ is isomorphic to im$\bar{f}=fM$ in $\mathfrak{A}/\mathfrak{J}'$. Since eM and fM are objects in $\mathfrak{A}$, $eM\approx fM$ in $\mathfrak{M}_R$ by Condition I in Theorem 7. Therefore, $eR_M\approx fR_M$. Next, we assume iv. Let M' be a direct summand of M, say $M'=eM$, $e^2=e$. Then we know in the proof of Theorem 7 that there exists an idempotent f in R_M such that $\bar{e}\bar{R}_M\approx\bar{f}\bar{R}_M$ and $fM=\sum_{\alpha}\sum_{I'_\alpha}\oplus M_{\alpha\rho}$. Since $eR_M\approx fR_M$ by iv, $eM\approx fM$ in M_R. The last part is also clear from Corollary 2.

REMARK 3. If we replace i in Theorem 9 by i′; *Condition* II *is satisfied for M and any sub-set I′ and J′*, then i′ does not imply ii and iii. For example, let $\{p, p_i\}$ be a family of distinct primes and $M=Z_p\oplus \sum_i \oplus Z/p_i\oplus \sum \oplus Z/p_i \oplus\cdots$. Then M satisfies condition II for any sub-set I' and J', however, iii is not satisfied. We shall show later that iv does not imply i.

We slightly generalize Carollary 2, ii to Theorem 6.

Proposition 10. *Let M be an R-module as in* (1). *If M is a quasi-injective, then* $\mathfrak{J}'[M, M]_R$ *is the radical of* R_M. *Hence, Conditions* II *and* III *are satisfied for M.*

Proof. The first half of the following is due to [1]. Theorem 2. Let e_α be a projection of M to M_α and x be in $\mathfrak{J}'[M, M]$. Then $e_\alpha=e_\alpha x e_\alpha+e_\alpha(1-x)e_\alpha$ and $e_\alpha x e_\alpha$ is not isomprphic by the definition of $\mathfrak{J}'$. Hence, $e_\alpha(1-x)e_\alpha$ is isomorphic for all α. If $\ker(1-x)\neq 0$, $\ker(1-x)\cap \sum_{i=1}^n M_{\alpha_i}\neq 0$ for some $\{\alpha_i\}$ and $\sum_i e_{\alpha_i}(1-x)e_{\alpha_i}$ is an automorphism of $\sum_{i=1}^n M_{\alpha_i}$, which is a contradiction. Therefore, $1-x=a$ is monomorphic. Since M is quasi-injcetive, aM is a direct summand; $aM=eM$, $e^2=e$. Hence, $0=(1-e)a\equiv 1-e \,(\mathrm{mod}\, \mathfrak{J}')$. Therefore, $e=1$ by Lemma 7 and x is quasi-regular.

Corollary. *Let* $\{M_\alpha\}_{\alpha\in I}$ *be a family of infinite many of injective indecomposable R-modules. If* $\sum_I \oplus M_\alpha$ *is injective, then any family* $\{f_\alpha\}$ *of non-iso homomorphisms of* M_α *to* M_β $(\alpha\neq\beta)$ *is an elementwise T-nilpotent system. If R is right noetherian, then any family of R-non-iso homomorphisms of injective indecomposable modules is an elementwise T-nilpotent system.*

Proof. It is clear from Proposition 10, Corollary 2 to Theorem 7 and Lemma 9.

4. Special cases

In this section we shall consider special modules. Let R be a commutative Dedekind domain, which is not local. Then a finitely generated and completely indecomposable R-module is isomorphic to R/p^n for some prime p (cf. [10], Theorems 1 and 9).

Proposition 11. *Let R be a not local Dedekind domain and* $R/p_\alpha^{n(p_\alpha)}$ *be a family of completely indecomposable modules. Then* $M=\sum \oplus R/p_\alpha^{n(p_\alpha)}$ *satisfies Condition* II *for any sub-set I′ and J′ if and only if* $n(p_\alpha)$ *is bounded for each* p_α. *In this case, Condition* III *is satisfied.*

Proof. It is clear that each R/p^n has a composition series. If $n(p_\alpha)$ is not bounded for some p. Then we have a family $\{f_i\}$ of non-iso monomorphisms f_i of R/p^{n_i} to $R/p^{n_{i+1}}$, which is not elementwise T-nilpotent. Hence, Condition II is not satistied. We assume that $n(p_\alpha)$ is bounded for all p_α. Since $[R/p^n, R/q^m]_R=0$ if $p\neq q$, $[M, M]_R=\prod_P [\sum \oplus R/p^{n_i}, \sum \oplus R/p^{n_i}]_R$. The radical of latter rings are nilpotent by the well known theorem (see Corollary to Lemma 12 below). Hence, $\mathfrak{J}'[M, M]_R$ is the radical of $[M, M]_R$. Therefore, we have the proposition by Corollary 2 to Theorem 7.

The following lemmas may be well known, however we shall give prooves for the sake of completeness. By $|M|$ we denote the composition length of R-module M.

Lemma 11. *Let $\{M_i\}$ be a family of indecomposable R-modules with $|M_i|=n<\infty$ for all i. Let f_i be a non-iso homomorphism of M_i to M_{i+1}(M_{i+1} may be equal to M_i). Then $|\theta(2^m, 1)(M_1)|\leqslant n-m-1$ for any m.*

Proof. Since $|M_i|=n$ for all i, each f_i is neither monomorphic nor epimorphic. We shall prove it by the induction on m. 1) If $m=0$, $|f_1(M_1)|\leqslant n-1=n-0-1$. 2) We assume $|\theta(2^m, 1)(M_1)|\leqslant n-\text{m}-1$ and $n-m-1\neq 0$. 3) It is clear that $|\theta(2^{m+1}, 1)(M_1)|\leqslant m-m-1$. If $|\theta(2^{m+1}, 1)(M_1)|=n-m-1$, then $|\theta(2^m, 1)(M_1)|=n-m-1$. Hence, $(\theta(2^{m+1}, 2^m+1)|\ \theta(2^m, 1)(M_1))$ is monomorphic. Furthermore, we have $|\theta(2^{m+1}, 2^m+1)(M_{2^m+1})|\leqslant n-m-1$ from the assumption 2). Since $|\theta(2^{m+1}, 2^m+1)(M_{2^m+1})|\geqslant|\theta(2^{m+1}, 1)(M_1)|=n-m-1$, $\theta(2^{m+1}, 2^m+1)(M_{2^m+1})=\theta(2^{m+1}, 1)(M_1)$. Hence, $M_{2m+1}=\theta(2^m, 1)(M_1)\oplus\ker\theta(2^{m+1}, 2^m+1)$ and $\theta(2^m, 1)(M_1)\neq 0$, $\ker\theta(2^{m+1}, 2^m+1)\neq 0$, which is a contradiction. Therefore, $|\theta(2^{m+1}, 1)(M_1)|\leqslant n-m-2$.

Lemma 12. *Let $\{M_i\}$, $\{f_i\}$ be as above with $|M_i|\leqslant n<\infty$ for all i. Then $\theta(n_0, 1)=0$ for some n_0.*

Proof. It is clear from the assumption that at least one f_{i+j} among f_{i+k} $k=0, 1, \cdots, n$ is not monomorphic. Let $f_{i_1}, f_{i_2}, \cdots$ be not monomorphic, then $g_1=\theta(i_2-1, i_1)$, $g_2=\theta(i_3-1, i_2)$, ... are not monomorphic. We take a family $\{M_{i_j}\}$. Since $|M_{i_j}|<n$, there exist some $r\leqslant n$ and an infinite sub-system $\{M_{k_i}\}$ such that $|M_k|=r$ for all k. We put $h_1=g_{k_2-1}g_{k_2-2}\cdots g_{k_1}$, $h_2=g_{k_3-1}\cdots g_{k_2}, \cdots$ and apply Lemma 11 for a system $\{h_i\}$, then we have a fixed large k_0 such that $\theta(n_0, 1)=0$. It is clear that we can find such n_0 independently on a choice of M_i and f_i.

Corollary. *Let $\{M_\alpha\}$ be as above and $M=\sum_I \oplus M_\alpha$. Then $\mathfrak{J}'[M, M]_R$ is nilpotent.*

Proof. Since each M_α is finitely generated R-module, $[M, M]_R$ is isomorphic to the ring of column finite matrices. Hence, $\mathfrak{J}'[M, M]_R$ is nilpotent.

Finally we shall show that Condition III does not imply Condition II.

Proposition 12. *Let $\{M_\alpha\}_1^\infty$ be a family of finitely generated and completely indecomposable R-modules such that $[M_\alpha, M_\beta]_R=0$ for $\alpha>\beta$. Then $M=\sum_1^\infty \oplus M_\alpha$ satisfies Condition* III.[5]

Corollary. *Let Δ be a division ring and R be a ring of lower tri-angular matrices over Δ with dimension $\aleph_0$. Let $\{e_{ij}\}$ be a system of matrix units in R. Then $M=\sum_{i=1}^\infty \oplus e_{ii}R$ satisfies Condition* III, *however some family of elements in $\mathfrak{J}'[M, M]_R$ is not T-nilpotent.*

Proof. It is clear that $\{M_i=e_{ii}R\}$ satisfies the condition of the proposition and that $[M, M]_R$ is isomorphic to the ring of column finite lower tri-angular matrices over Δ with dimension $\aleph_0$. Then $\{e_{i+1i}\}$ is not T-nilpotent. Hence, Condition II is not satisfied by Lemma 9.

Proof of Proposition 12. *We put* $R_{\sigma\tau}=[M_\tau, M_\sigma]_R$. Then $R_M=[M, M]_R$ is isomorphic to the ring of column finite lower tri-angular matrices whose component consists of all elements in $R_{\sigma\tau}$. Furthermore, $\bar{R}_M=R_M/\mathfrak{J}'R_M = \prod_\sigma (R_{\sigma\sigma}/\mathfrak{J}'R_{\sigma\sigma})$. Let E be an idempotent in R_M, then $\bar{E}=\prod_{\sigma_i} \bar{e}_{\sigma_i}$, where e_σ's are identities in R_σ. We put $F=\prod_{\sigma_i} e_{\sigma_i}$ and show that $ER_M\approx FR_M$. Let σ be an integer and A in R_M. We shall divide A into four parts:

$$A=\left(\begin{array}{c|c} A_{11}^{(\sigma)} & 0 \\ \hline A_{21}^{(\sigma)} & A_{22}^{(\sigma)} \end{array}\right)$$

and $A_{11}^{(\sigma)}$ is a $(\sigma\times\sigma)$-matrix. Then $E=\begin{pmatrix} E_{11}^{(\sigma)} & 0 \\ E_{21}^{(\sigma)} & E_{22}^{(\sigma)} \end{pmatrix}$, $F=\begin{pmatrix} F_{11}^{(\sigma)} & 0 \\ F_{21}^{(\sigma)} & F_{22}^{(\sigma)} \end{pmatrix}$ and $E_{11}^{(\sigma)}R_{M11}^{(\sigma)}/E_{11}^{(\sigma)}\mathfrak{J}'R_{M11}^{(\sigma)}\approx F_{11}^{(\sigma)}R_{M11}^{(\sigma)}/F_{11}^{(\sigma)}\mathfrak{J}'R_{M11}^{(\sigma)}$ and $(E_{11}^{(\sigma)})^2=E_{11}^{(\sigma)}$, $(F_{11}^{(\sigma)})^2=F_{11}^{(\sigma)}$ in $R_{M11}^{(\sigma)}$. If $\sigma=1$, $E_{11}^{(1)}=F_{11}^{(1)}$ and hence, there exists an $R_{M11}^{(1)}$-isomorphism $f^{(1)}$ of $F_{11}^{(1)}R_{M11}^{(1)}$ to $E_{11}^{(1)}R_{M11}^{(1)}$. We assume there exist $R_{M11}^{(\tau)}$-isomorphisms $f^{(\tau)}$ of $F_{11}^{(\tau)}R_{M11}^{(\tau)}$ to $E_{11}^{(\tau)}R_{M11}^{(\tau)}$ for all $\tau<\sigma$ such that $f^{(\tau_1)}|F_{11}^{(\tau)}R_{M11}^{(\tau_1)}=f^{(\tau)}$ for all $\tau<\tau_1$. 1). Let $E_{11}^{(\sigma)}=\begin{pmatrix} E'_{11} & 0 \\ x_{21} & 0 \end{pmatrix}$, where $E'_{11}=E_{11}^{(\sigma-1)}$. Put $f'=f^{(\sigma-1)}$. Since $(E_{11}^{(\sigma)})^2=E_{11}^{(\sigma)}$, we have $x_{21}E_{11}^{(\sigma)}=x_{21}$ and $E_{11}^{(\sigma)}R_{M11}^{(\sigma)}=\left\{\begin{pmatrix} E'_{11}x_{11} & 0 \\ x_{21}x_{11} & 0 \end{pmatrix}\middle| x_{11}\in R'_M\right\}$. On the other hand $F_{11}^{(\sigma)}$

5) Added in proof. This proposition is true without the assumption $[M_\alpha, M_\beta]=0$ for $\alpha>\beta$ by [18]. However, we can apply our proof to more general case under certain assumptions without finitely generatedness for any infinite family of M_α.

$=\begin{pmatrix} F'_{11} & 0 \\ 0 & 0 \end{pmatrix}$ and $F_{11}^{(\sigma)}R_{M11}^{(\sigma)}=\begin{pmatrix} F'_{11}R'_M & 0 \\ 0 & 0 \end{pmatrix}$. We define $f^{(\sigma)}$ by setting $f^{(\sigma)}(\begin{pmatrix} F'_{11}x_{11} & 0 \\ 0 & 0 \end{pmatrix})$ $=\begin{pmatrix} f'(F'_{11}x_{11}) & 0 \\ x_{21}x'_{11} & 0 \end{pmatrix}$, where $f'(F'_{11}x_{21})=E'_{11}x'_{11}$. Since $x_{21}E'_{11}=x_{21}$, $f^{(\sigma)}$ is well defined and $f^{(\sigma)}$ is $R_M^{(\sigma)}$-isomorphic. 2). Let $E_{11}^{(\sigma)}=\begin{pmatrix} E'_{11} & 0 \\ x_{21} & e_\sigma \end{pmatrix}$, then $E_{11}^{(\sigma)}R_M^{(\sigma)}=$ $\left\{\begin{pmatrix} E'_{11}y_{11} & 0 \\ x_{21}y_{11}+y_{21} & e_\sigma y_{22} \end{pmatrix} \middle| y_{11}\in R'_M,\ y_{22}\in R_{\sigma\sigma}\right\}$ and $F_{11}^{(\sigma)}R_{M11}^{(\sigma)}=\left\{\begin{pmatrix} F'_{11}y_{11} & 0 \\ y_{21} & e_\sigma y_{22} \end{pmatrix} \middle| y_{11}\in R'_M,\ y_{22}\in R_{\sigma\sigma}\right\}$. It is clear that $\{x_{21}y_{11}+y_{21}\}$ is equal to $\{y_{21}\}$ when y_{21} runs through all elements. We define $f^{(\sigma)}$ by setting $f^{(\sigma)}(\begin{pmatrix} F'_{11}y_{11} & e_\sigma y_{22} \\ y_{21} & 0 \end{pmatrix})$ $=\begin{pmatrix} f'(F'_{11}y_{11}) & e_\sigma y_{22} \\ y_{21} & 0 \end{pmatrix}$. Then $f^{(\sigma)}$ is $R_M^{(\sigma)}$-isormorphic. In either case, it is clear that $f^{(\sigma)}|F_{11}^{(\tau)}R_M^{(\tau)}=f^{(\tau)}$ for all $\tau<\sigma$. Thus, we have defined a system $\{f^{(\sigma)}\}$. Since R_M is column finite, each column of elements in FR_M is contained in some $F_{11}^{(\sigma)}R_{M11}^{(\sigma)}$. Hence, we can define an R_M-isomorphism f of FR_M to ER_M via $\{f^{(\sigma)}\}$. Therefore, M satisfies Condition III by iv in Theorem 9.

Finally, we shall consider a contrary case of Proposition 12.

Proposition 13. *Let $\{M_\alpha\}_1^\infty$ be as above. We assume $[M_\alpha, M_\beta]_R=0$ if $\alpha<\beta$. Then $M=\sum_1^\infty \oplus M_\alpha$ satisfies Conditions* I, II *and* III.

Proof. In this case, $[M, M]_R$ is isomorphic to the ring of upper tri-angular matrices with components in $[M_\sigma, M_\tau]_R$. We shall show that $\mathfrak{J}'[M, M]_R$ is the radical. It is clear that if I is finite, then $\mathfrak{J}'[M, M]_R$ is the radical. As before, we divide matrices A into four parts; $A=\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$. Let A be in $\mathfrak{J}'[M, M]_R$ and assume A has the quasi-inverse $B=(B_{kj})$. Then A_{ii} has the quasi-inverse B_{ii}. Conversely, if A_{ii} has the quasi-inverse, then we put $B_{12}=-(1-A_{11})^{-1}$ $A_{22}(1-B_{12})^{-1}$. Hence, A has the quasi-inverse (B_{ij}). From those facts and the induction as in the proof of Proposition 12, we can prove A is quasi-regular. Hence, the proposition is clear from Corollary 2 to Theorem 7.

5. In case of C_3-abelian category

In this section we always assume $\mathfrak{C}$ a C_3-abelian category with generator U. In this case $\mathfrak{C}$ has any products by [6], and any object has an injective envelope by [11], p. 87, Theorem 3.2. Let $T=[U,\ ]$ be the functor of $\mathfrak{C}$ to $\mathfrak{M}_R$ and S a coadjoint for T, where $R=[U, U]$. Then TM is injective if and only if so is M in $\mathfrak{C}$ by [17]. Furthermore, $T(\Pi M)=\Pi (TM)$ and if $TN=A_1\oplus A_2$

for $N\in\mathfrak{C}$, then $STN=SA_1\oplus SA_2$ and $STN\overset{\psi_N}{\approx}N$ by [6]. $TSTN=TSA_1\oplus TSA_2$ and φ_{TN}: $TN\approx TSTN$; φ_{A_i}: $A_i\to TSA_i$ are homomorphic. Therefore, φ_{A_i} are isomorphic. Thus if we want to study the full sub-category $C(M)$ in $\mathfrak{C}$ as in the section 1, then it is equivalent to study the full sub-category $C(TM)$ in $\mathfrak{M}_R$. Therefore, we have the following from Theorem 1:

Theorem 1′. *Let $\mathfrak{C}$ be a C_3-abelian category with generator and M an injective object in $\mathfrak{C}$. Then $C(M)/\mathfrak{J}$ is a spectral abelian category with generator.*

Since $\mathfrak{C}$ has an injective cogenerator by [12], p. 90, Corollary 3.4, we can obtain the similar results as in the section 2.

Finally, we shall consider results in the section 3 for $\mathfrak{C}$. P. Gabriel has already generalized Azumaya's theorem to $\mathfrak{C}$ in [5]. He proved it by replacing Azumaya's method by categorical terms. We shall make use of Gabriel's method.

We use the same definitions and notations in the section 3.

Lemma 5′. *Let $M=\sum_I\oplus M_\alpha$, $N=\sum_{I'}\oplus N_\beta$ and $T=\sum_{I''}\oplus T_\gamma$ be coproducts of completely indecomposable objects. Let $\mathfrak{J}^{(\beta,\alpha)}$ be the sub-set of $[M, N]$ whose element consists of all f in $[M, N]$ such that $p_\beta fi_\alpha$ is not isomorphic for all α and β. Then $[N, T]\mathfrak{J}^{(\beta,\alpha)}\subseteq\mathfrak{J}^{(\gamma,\alpha)}$, $\mathfrak{J}^{(\gamma,\beta)}[M, N]\subseteq\mathfrak{J}^{(\gamma,\alpha)}$ and $\mathfrak{J}^{(\beta,\alpha)}$ is an abelian group, where p_β, i_α mean projection to N_β and injection of M_α, respectively.*

Proof. It is clear from the definition that $\mathfrak{J}^{(\beta,\alpha)}$ is abelian. Let $f\in\mathfrak{J}^{(\beta,\alpha)}$ and $g\in[N, T]$. Since $\mathfrak{C}$ is a C_3-category, there exists a finite set K such that $f(M_\alpha)\cap\sum_{\beta\in K}N_\beta\neq(0)$ if $f(M_\alpha)\neq 0$. Hence, $p_\gamma gfi_\alpha=p_\gamma g(1_K+1_{I'-K})fi_\alpha=p_r g1_K i_\alpha+p_\gamma g1_{I'-K}fi_\alpha$. Since $f^{-1}(\sum_{\beta'\in I'-K}N_{\beta'})\neq(0)$, $p_\gamma g1_{I'-K}fi_\alpha$ is not isomorphic and $p_\gamma g1_K fi_\alpha$ is not isomorphic by the assumption. Hence, $gf\in\mathfrak{J}^{(\gamma,\alpha)}$. Similarly, we have $\mathfrak{J}^{(\gamma,\beta)}[M, N]\subseteq\mathfrak{J}^{(\gamma,\alpha)}$.

If we put $M=N=T$ in Lemma 5′, we know that $\mathfrak{J}^{(\beta,\alpha)}=\mathfrak{J}^{(\gamma,\alpha)}$ and $\mathfrak{J}^{(\beta,\alpha)}$ is an ideal in $\mathfrak{C}$. We shall denote it by $\mathfrak{J}'$.

Lemma 13. *Let M and $\mathfrak{J}'$ be as above. Then $[M, M]/\mathfrak{J}'$ is isomorphic to a product of rings of linear transformations of vector spaces over division ring. Furthermore, we may regard $[M, M]/\mathfrak{J}'$ as the ring of endomorphisms of M considering M_α minimal, (cf. [1], Theorem 3).*

Proof. Put $S=[M, M]$ and $\bar{S}=S/\mathfrak{J}'$. Since $S=\prod_\alpha[M_\alpha, M]$, we put $f=\Pi f_\alpha$ for $f\in S$ and $f_\alpha\in[M_\alpha, M]$. Let $K=\{\gamma\}$ be a sub-set of I such that $p_\gamma f_\alpha$ is isomorphic. If we put $f'_\alpha=f_\alpha-(1-1_{K_\alpha})f_\alpha$ and $f'=\Pi f'_\alpha$, then $f'-f=\Pi(1-1_{K_\alpha})f_\alpha\in\mathfrak{J}'$. Hence, we can chose a representative f in $\bar{f}$ such that each $p_\beta f i_\alpha$ is isomorphic or zero. If $f(M_\alpha)\neq(0)$ for α, then there exists a finite set K in I such that $f(M_\alpha)\cap\sum_K\oplus M_\beta\neq(0)$. Put $f^{-1}(\sum_K\oplus M_\beta)=M'_\alpha\neq(0)$. Then $p_t f i_\alpha|M'_\alpha=0$ for all $t\in I-K$. Hence, $p_t f i_\alpha(M_\alpha)=(0)$ by the choice of f. Therefore, $f(M_\alpha)\subseteq\sum_K\oplus M_\beta$. From this fact we have an isomorphism of S to the ring of column finite matrices with components $p_\beta f i_\alpha$: $M_\alpha\to M_\beta$. Since $[M_\alpha, M_\alpha]/\mathfrak{J}'\cap[M_\alpha, M_\alpha]$ is a division ring, we may regard M_α as a minimal object by the above isomorphism. Hence, we have the lemma.

From those lemmas and the proof of Theorem 7 we have

Theorem 7′. *Let $\mathfrak{D}$ and $\mathfrak{D}_f$ be the full sub-category in $\mathfrak{C}$ whose object consists of any (resp. finite) coproducts of a given family $\{M_\alpha\}$ of completely indecomposable objects in $\mathfrak{C}$. Then $\mathfrak{D}/\mathfrak{J}'$ (resp. $\mathfrak{D}_f/\mathfrak{J}'$) is a C_3-completely reducible (resp. completely reducible) abelain category.*

From this theorem we have the K-R-S-A theorem for a C_3-abelian category $\mathfrak{C}$ as in Corollary 1 to Theorem 7, (cf. [5]).

REMARK 4. We replace the argument in the proof of Lemma 9 by categorical terms. The relation (2) is true, since they are images of automorphisms f_i of M such that

$$F_1=\begin{pmatrix} 1 & 0\cdots & & & \\ f_1 & 1 & 0\cdots & & \\ & 0 & 1 & 0\cdots & \\ & & f_3 & 1 & \\ 0 & & & & \ddots \end{pmatrix},\quad F_2=\begin{pmatrix} 1 & & & & & 0 \\ 0 & 1 & & & & \\ & f_2 & 1 & & & \\ & & 0 & 1 & & \\ & & & f_4 & 1 & \\ 0 & & & & 0 & \ddots \end{pmatrix}$$

and we obtain $M_1\cap(M'_1\oplus M'_2\oplus\cdots\oplus M'_n)=\ker(f_{n-1}f_{n-2}\cdots f_1)$ by pullback. Hence, if M satisfies Condition II then $M_n=\bigcup_m\ker\theta(m,n)$ for all n. However, it may be necessary some assumption on M_α to obtain Theorem 9.

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Added in proof.

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