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THE STRUCTURE OF THE COBORDISM GROUPS $B(n, k)$ OF BUNDLES OVER MANIFOLDS WITH INVOLUTION

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Introduction

In the previous paper [4] we have considered the cobordism groups of generic immersions and introduced the cobordism groups $\boldsymbol{B}(n, k)$ of bundles over manifolds with involution as follows. The basic object is a triple (W, T, ξ) where T is a fixed point free differentiable involution on a compact differentiable manifold W and ξ is a k-plane bundle over W. If M_1 and M_2 are closed n-manifolds then (M_1, T_1, ξ_1) is cobordant to (M_2, T_2, ξ_2) if and only if there exists a triple (W, T, ξ) for which $\partial(W, T, \xi)=(M_1, T_1, \xi_1)+(M_2, T_2, \xi_2)$. Then this is an equivalence relation and the set of all cobordism classes $\boldsymbol{B}(n, k)$ becomes an abelian group by disjoint union.

These groups $\boldsymbol{B}(n, k)$ play an important role in the study of the cobordism groups of generic immersions. And there is an exact sequence [4]:

$$\cdots \xrightarrow{\psi_*} \boldsymbol{B}(n, k) \xrightarrow{\rho_*} \mathfrak{N}_n(BO(k)\times BO(k)) \xrightarrow{\varphi_*} \boldsymbol{B}(n, k) \xrightarrow{\psi_*} \boldsymbol{B}(n-1, k) \xrightarrow{\rho_*} \cdots$$

where the homomorphism ψ_* is a modified Smith homomorphism.

The object of this paper is to determine the structure of these groups. Let $\{x_\omega\}$ be the basis of the free $\mathfrak{N}_*$-module $\mathfrak{N}_*(BO(k))$, then $\mathfrak{N}_*(BO(k)\times BO(k)) = A^{(k)}\oplus R^{(k)}\oplus S^{(k)}$ where $A^{(k)}$, $R^{(k)}$ and $S^{(k)}$ are the free $\mathfrak{N}_*$-modules with basis $\{x_\omega\times x_\omega\}$, $\{x_\omega\times x_{\omega'}\mid \omega<\omega'\}$ and $\{x_\omega\times x_{\omega'}+x_{\omega'}\times x_\omega \mid \omega\neq\omega'\}$ respectively. Let $\boldsymbol{B}^{(k)}=\sum_n \boldsymbol{B}(n, k)$ and $C^{(k)}=\psi_*(\boldsymbol{B}^{(k)})$. Then we will prove $\boldsymbol{B}^{(k)}\cong C^{(k)}\oplus S^{(k)}$ and $C^{(k)}\cong A^{(k)}\otimes Z[t]$.

Next we consider the objects $(W, T, \xi, \tilde{T})$ where ξ is a k-plane bundle over a compact differentiable manifold W, $T\colon W\to W$ is a fixed point free differentiable involution and $\tilde{T}\colon \xi\to\xi$ is a bundle map covering T such that $\tilde{T}^2=$ identity, then the cobordism group $\tilde{\boldsymbol{B}}(n, k)$ analogous to $\boldsymbol{B}(n, k)$ is obtained. And there is a short exact sequence:

$$0 \to \mathfrak{N}_n(BO(k)) \xrightarrow{\varphi_*} \tilde{\boldsymbol{B}}(n, k) \xrightarrow{\psi_*} \tilde{\boldsymbol{B}}(n-1, k) \to 0\,.$$

Now let $\sigma: \tilde{\boldsymbol{B}}(n, k) \to \boldsymbol{B}(n, k)$ be the canonical forgetting homomorphism and $d: BO(k) \to BO(k) \times BO(K)$ be the diagonal map, then the following diagram is commutative:

$$\begin{array}{ccccc} \mathfrak{N}_n(BO(k)) & \xrightarrow{\varphi_*} & \tilde{\boldsymbol{B}}(n, k) & \xrightarrow{\psi_*} & \tilde{\boldsymbol{B}}(n-1, k) \\ \downarrow d_* & & \downarrow \sigma & & \downarrow \sigma \\ \mathfrak{N}_n(BO(k) \times BO(k)) & \xrightarrow{\varphi_*} & \boldsymbol{B}(n, k) & \xrightarrow{\psi_*} & \boldsymbol{B}(n-1, k) . \end{array}$$

Clearly $\sigma(\tilde{\boldsymbol{B}}(n, k))$ is contained in $\psi_*(\boldsymbol{B}(n+1, k))$ and in fact we will prove $\sigma(\tilde{\boldsymbol{B}}(n, k)) = \psi_*(\boldsymbol{B}(n+1, k))$.

In the last section we will determine the rank of the oriented cobordism groups $\boldsymbol{B}^{\pm}(n, k)$ which are also defined in the previous paper [4].

1. The structure of $\boldsymbol{B}(n, k)$

Let $\pi(n, k)$ be the set of partitions of n into integers each of which is $\leqq k$. Let $\pi^{(k)}$ be the disjoint union of $\pi(n, k)$ for all $n \geqq 0$. Denote $n(\omega) = n$ for $\omega \in \pi^{(k)}$ if $\omega \in \pi(n, k)$. Throughout this paper, suppose any fixed order is given in $\pi^{(k)}$.

One may choose $\{x_\omega \mid \omega \in \pi^{(k)}, x_\omega \in \mathfrak{N}_{n(\omega)}(BO(k))\}$ as the basis of the free $\mathfrak{N}_*$-module $\mathfrak{N}_*(BO(k))$ such that $e(x_\omega)$ is the dual of $W_{i_1} \cdots W_{i_r}$ if $\omega = (i_1, \cdots, i_r)$ where $e: \mathfrak{N}_*(\ ) \to H_*(\ , Z_2)$ is the evaluation homomorphism and W_i is the i-th universal Stiefel-Whitney class. Suppose (M_ω, ξ_ω) represents the class x_ω, where ξ_ω is a k-plane bundle over the closed $n(\omega)$-dimensional differentiable manifold M_ω. Then $\{x_\omega \times x_{\omega'} \mid \omega, \omega' \in \pi^{(k)}\}$ becomes the basis of the free $\mathfrak{N}_*$-module $\mathfrak{N}_*(BO(k) \times BO(k))$, where $x_\omega \times x_{\omega'}$ is represented by $(M_\omega \times M_{\omega'}, \xi_\omega \times 0, 0 \times \xi_{\omega'})$.

Let $A^{(k)} = \sum_n A_n^{(k)}$, $R^{(k)} = \sum_n R_n^{(k)}$ and $S^{(k)} = \sum_n S_n^{(k)}$ be the free $\mathfrak{N}_*$-modules with basis $\{x_\omega \times x_\omega \mid \omega \in \pi^{(k)}\}$, $\{x_\omega \times x_{\omega'} \mid \omega, \omega' \in \pi^{(k)}, \omega < \omega'\}$ and $\{x_\omega \times x_{\omega'} + x_{\omega'} \times x_\omega \mid \omega, \omega' \in \pi^{(k)}, \omega \neq \omega'\}$ respectively, where $A_n^{(k)}$, $R_n^{(k)}$ and $S_n^{(k)}$ are the factors of degree n. Then

$$(1.1) \qquad \mathfrak{N}_*(BO(k) \times BO(k)) = A^{(k)} \oplus R^{(k)} \oplus S^{(k)} \qquad \text{(direct sum).}$$

Lemma 1.2. *$\rho_* \varphi_* \mid R^{(k)}: R^{(k)} \to S^{(k)}$ is an isomorphism of $\mathfrak{N}_*$-modules.*

Proof. $\rho_* \varphi_*(x_\omega \times x_{\omega'}) = x_\omega \times x_{\omega'} + x_{\omega'} \times x_\omega$, since $\rho_* \varphi_* = 1 + \tau_*$, where τ_* is induced from the map $\tau: BO(k) \times BO(k) \to BO(k) \times BO(k)$ switching factors.

q.e.d.

Lemma 1.3. *For any $\omega \in \pi^{(k)}$ and any $l = 0, 1, 2, \cdots$, there exists an element $y_\omega^l \in \boldsymbol{B}(2n(\omega) + l, k)$ such that $\psi_*(y_\omega^l) = y_\omega^{l-1}$ and $y_\omega^0 = \varphi_*(x_\omega \times x_\omega)$.*

Proof. Let y_ω^l be the class of $(S^l \times M_\omega \times M_\omega, A \times T, 0 \times \xi_\omega \times 0)$, where

$A: S^l \to S^l$ is the antipodal map on the sphere, (M_ω, ξ_ω) represents x_ω and T is the map switching factors on $M_\omega \times M_\omega$. Then this is the desired element.

q.e.d.

Let $C^{(k,l)}=\sum_n C_n^{(k,l)}$, $C^{(k)}=\sum_n C_n^{(k)}$ and $\bar{C}^{(k)}=\sum_n \bar{C}_n^{(k)}$ be the $\mathfrak{N}_*$-submodules of $\boldsymbol{B}^{(k)}=\sum_n \boldsymbol{B}(n, k)$ generated by $\{y_\omega^l \mid \omega \in \pi^{(k)}\}$, $\{y_\omega^l \mid \omega \in \pi^{(k)},\ l \geqq 0\}$ and $\{y_\omega^l \mid \omega \in \pi^{(k)},\ l > 0\}$ respectively, where $C_n^{(k,l)}$, $C_n^{(k)}$ and $\bar{C}_n^{(k)}$ are the factors of degree n. From Lemma 1.3, if we define $\varphi_*^{(l)}(x_\omega \times x_\omega)=y_\omega^l$, then we obtain the following result.

Lemma 1.4. *There exist $\mathfrak{N}_*$-module homomorphisms $\varphi_*^{(l)}: A^{(k)} \to C^{(k,l)}$ of degree l for any $l \geqq 0$ such that $\varphi_*^{(0)}=\varphi_*$ and $\varphi_*^{(l)}$ are surjective for any $l \geq 0$, and the following diagram is commutative:*

$$\begin{array}{ccc} & A^{(k)} & \\ {\scriptstyle \varphi_*^{(l)}} \swarrow & & \searrow {\scriptstyle \varphi_*^{(l-1)}} \\ C^{(k,l)} & \xrightarrow{\ \psi_*\ } & C^{(k,l-1)} \end{array}.$$

Lemma 1.5. *For any integer $n \geqq 0$, the following statements are true:*

(a_n) *the homomorphism $\varphi_*: A_n^{(k)} \to C_n^{(k,0)}$ is an isomorphism,*

(b_n) *$C_n^{(k)}=\sum_{l \geqq 0} C_n^{(k,l)}$ (direct sum) and the homomorphism $\psi_*: \bar{C}_{n+1}^{(k)} \to C_n^{(k)}$ is an isomorphism,*

(c_n) *$\boldsymbol{B}(n, k)=\varphi_*(A_n^{(k)}) \oplus \varphi_*(R_n^{(k)}) \oplus \bar{C}_n^{(k)}$.*

This lemma will be proved in the next section. As a corollary of this lemma we obtain the following results.

Theorem 1.6. *$\boldsymbol{B}^{(k)}=\sum_n \boldsymbol{B}(n, k)$ is the direct sum $C^{(k)} \oplus \varphi_*(R^{(k)})$ and $C^{(k)}=\psi_*(B^{(k)})$ where $C^{(k)}$ is the free $\mathfrak{N}_*$-module with basis $\{y_\omega^l \mid \omega \in \pi^{(k)},\ l \geqq 0\}$ and the degree of y_ω^l is $2n(\omega)+l$. In particular $\boldsymbol{B}^{(k)}$ is a free $\mathfrak{N}_*$-module.*

Corollary 1.7. *$\boldsymbol{B}^{(k)} \cong (A^{(k)} \otimes Z[t]) \oplus S^{(k)}$ as $\mathfrak{N}_*$-modules where $Z[t]$ is the polynomial ring with one generator t of degree* 1.

2. Proof of Lemma 1.5

Consider the following exact sequence:

$$\boldsymbol{B}(0, k) \xrightarrow{\rho_*} \mathfrak{N}_0(BO(k) \times BO(k)) \xrightarrow{\varphi_*} \boldsymbol{B}(0, k) \to 0\,.$$

Then φ_* is isomorphic since $\mathfrak{N}_0(BO(k) \times BO(k))=A_0^{(k)} \cong Z_2$. Therefore (a_0) and (c_0) are true. In general we will prove the statements by induction on n.

(i) Suppose "(a_r) is true for $r \leqq n$". Then the homomorphisms

$$\varphi_*^{(l)}: A_r^{(k)} \to C_{r+l}^{(k,l)} \quad \text{and} \quad (\psi_*)^l: C_{r+l}^{(k,l)} \to C_r^{(k,0)}$$

are isomorphic for $r \leqq n$ and $l \geqq 0$ by Lemma 1.4. Suppose

$$x^{(0)} + x^{(1)} + \cdots + x^{(l_0)} = 0$$

where $x^{(l)} \in C_n^{(k,l)}$, then

$$(\psi_*)^{l_0}(x^{(l_0)}) = (\psi_*)^{l_0}(x^{(0)} + x^{(1)} + \cdots + x^{(l_0)}) = 0 \qquad \text{thus } x^{(l_0)} = 0\,.$$

Therefore $x^{(l)}=0$ for all $0 \leqq l \leqq l_0$ and $C_n^{(k)}$ is the direct sum of $C_n^{(k,l)}$, $l \geqq 0$. On the other hand, the homomorphisms

$$\psi_*: C_{n+1}^{(k,l+1)} \to C_n^{(k,l)}$$

are isomorphic for $l \geqq 0$. Therefore the homomorphism

$$\psi_*: \bar{C}_{n+1}^{(k)} \to C_n^{(k)}$$

is isomorphic. Consequently "(a_r) is true for $r \leqq n$" implies "(b_n) is true".

(ii) Suppose "(b_n) and (c_n) are true." Then

$$\boldsymbol{B}(n, k) = C_n^{(k)} \oplus \varphi_*(R_n^{(k)})$$

and the homomorphisms

$$\psi_*: \bar{C}_{n+1}^{(k)} \to C_n^{(k)} \quad \text{and} \quad \rho_*: \varphi_*(R_n^{(k)}) \to S_n^{(k)}$$

are isomorphic. Thus $C_n^{(k)} \subset \psi_*(\boldsymbol{B}(n+1, k))$ and $\psi_*(\boldsymbol{B}(n+1, k)) \cap \varphi_*(R_n^{(k)})=0$. Therefore $C_n^{(k)}=\psi_*(\boldsymbol{B}(n+1, k))$. Then the following is an exact sequence of Z_2-modules:

$$0 \to (\textit{kernel of } \psi_*) \to \boldsymbol{B}(n+1, k) \xrightarrow{\psi_*} C_n^{(k)} \to 0\,.$$

Therefore

$$\begin{aligned} \boldsymbol{B}(n+1, k) &= \bar{C}_{n+1}^{(k)} \oplus (\textit{kernel of } \psi_*) \\ &= \bar{C}_{n+1}^{(k)} \oplus \varphi_*(\mathfrak{N}_{n+1}(BO(k) \times BO(k))) \\ &= \bar{C}_{n+1}^{(k)} \oplus \varphi_*(A_{n+1}^{(k)} \oplus R_{n+1}^{(k)})\,, \end{aligned}$$

since $\varphi_*(S_{n+1}^{(k)})=0$. Suppose $\varphi_*(x+y)=0$ for $x \in A_{n+1}^{(k)}$ and $y \in R_{n+1}^{(k)}$, then $\rho_*\varphi_*(y)=\rho_*\varphi_*(x+y)=0$ since $\rho_*\varphi_*(A_{n+1}^{(k)})=0$, and $y=0$ since $\rho_*\varphi_* | R^{(k)}$ is isomorphic. Thus $\varphi_*(A_{n+1}^{(k)}) \cap \varphi_*(R_{n+1}^{(k)})=0$. Therefore

$$\boldsymbol{B}(n+1, k) = \bar{C}_{n+1}^{(k)} \oplus \varphi_*(A_{n+1}^{(k)}) \oplus \varphi_*(R_{n+1}^{(k)})\,.$$

Consequently "(b_n) and (c_n) are true" implies "(C_{n+1}) is true."

(iii) Suppose "(c_n) is true". Then

$$\rho_*(\boldsymbol{B}(n, k)) = \rho_*\varphi_*(R_n^{(k)}) = S_n^{(k)},$$

since $\varphi_*(A_n^{(k)})\oplus\bar{C}_n^{(k)}\subset\psi_*(\boldsymbol{B}(n+1, k))$. Thus the restriction of φ_* on $A_n^{(k)}\oplus R_n^{(k)}$ is injective by the following exact sequence:

$$\boldsymbol{B}(n, k) \xrightarrow{\rho_*} A_n^{(k)}\oplus R_n^{(k)}\oplus S_n^{(k)} \xrightarrow{\varphi_*} \boldsymbol{B}(n, k).$$

In particular, $\varphi_*: A_n^{(k)}\to C_n^{(k,0)}$ is isomorphic. Consequently "(c_n) is true" implies "(a_n) is true".

These complete the proof of Lemma 1.5.

3. Forgetting homomorphisms

Clearly the following diagram of the forgetting homomorphisms is commutative:

$$\begin{array}{ccc} \boldsymbol{B}(n, k) & \xrightarrow{i_1} & \mathfrak{N}_n(BO(k)) \\ \downarrow b_1 & & \downarrow b_2 \\ \boldsymbol{B}(n, 0) & \xrightarrow{i_2} & \mathfrak{N}_n \end{array}$$

where $b_1([M, T, \xi])=[M, T]$, $i_1([M, T, \xi])=[M, \xi]$, $i_2([M, T])=[M]$ and $b_2([M, \xi])=[M]$. Moreover the homomorphism i_2 is the zero map and the homomorphism b_2 is surjective. Then we obtain the following result by the above diagram.

Lemma 3.1. *The forgetting homomorphism* $i_1: \boldsymbol{B}(n, k)\to\mathfrak{N}_n(BO(k))$ *is not surjective if* $\mathfrak{N}_n\neq 0$.

Lemma 3.2. *The restriction of the forgetting homomorphism* $i_1: \boldsymbol{B}(n, k)\to \mathfrak{N}_n(BO(k))$ *on* $C^{(k,0)}=\varphi_*(A^{(k)})$ *is the zero map.*

Proof. Let $\{[M_\omega\times M_\omega, \xi_\omega\times 0, 0\times\xi_\omega]\}$ be the generating elements of $A^{(k)}$, then $i_1\varphi_*([M_\omega\times M_\omega, \xi_\omega\times 0, 0\times\xi_\omega])=[M_\omega\times M_\omega, \xi_\omega\times 0]\cup[M_\omega\times M_\omega, 0\times\xi_\omega]$. But $[M_\omega\times M_\omega, \xi_\omega\times 0]=[M_\omega\times M_\omega, 0\times\xi_\omega]$ by the map switching factors on $M_\omega\times M_\omega$. Therefore $i_1\varphi_*=0$ on $A^{(k)}$. q.e.d.

Theorem 3.3. *In general, the forgetting homomorphism*

$$F: \boldsymbol{B}(n, k) \to \boldsymbol{B}(n, 0)\oplus\mathfrak{N}_n(BO(k))$$

is not injective, where $F(x)=b_1(x)+i_1(x)$.

Proof. Let $f: A^{(k)}\to\mathfrak{N}_*$ be the restriction of the forgetting homomorphism $f': \mathfrak{N}_*(BO(k)\times BO(k))\to\mathfrak{N}_*$ defined by $f'([M, \xi, \eta])=[M]$. Then $f: A_n^{(k)}\to\mathfrak{N}_n$ is not injective in general, by comparing the rank of $A_n^{(k)}$ and $\mathfrak{N}_n$ over Z_2. Let x be an element of $A_n^{(k)}$ such that $x\neq 0$ and $f(x)=0$, then $b_1\varphi_*(x)=0$

by definition of the homomorphisms φ_* and b_1. Moreover $\varphi_*(x)\neq 0$, since $\varphi_*: A^{(k)}\to \boldsymbol{B}(n, k)$ is injective. On the other hand, $i_1\varphi_*(A^{(k)})=0$ by Lemma 3.2. Thus $F(\varphi_*(x))=0$. Therefore F is not injective in general. q.e.d.

4. Cobordism groups $\tilde{\boldsymbol{B}}(n, k)$

Now we consider the objects $(W, T, \xi, \tilde{T})$ where ξ is a k-plane bundle over a compact differentiable manifold W, $T: W\to W$ is a fixed point free differentiable involution and $\tilde{T}: \xi\to\xi$ is a bundle map covering T such that $\tilde{T}^2$=identity, then the cobordism group $\tilde{\boldsymbol{B}}(n, k)$ analogous to $\boldsymbol{B}(n, k)$ is obtained. This group $\tilde{\boldsymbol{B}}(n, k)$ is canonically isomorphic to the bordism group $\mathfrak{N}_n(BO(k)\times B(Z_2))$ where $B(Z_2)$ is the classifying space for the double covering spaces. And we obtain an exact sequence by the similar argument as the case of $\boldsymbol{B}(n, k)$:

$$\cdots \xrightarrow{\psi_*} \tilde{\boldsymbol{B}}(n, k) \xrightarrow{\rho_*} \mathfrak{N}_n(BO(k)) \xrightarrow{\varphi_*} \tilde{\boldsymbol{B}}(n, k) \xrightarrow{\psi_*} \tilde{\boldsymbol{B}}(n-1, k) \xrightarrow{\rho_*} \cdots$$

where ψ_* is the modified Smith homomorphism similarly defined as the case of $\boldsymbol{B}(n, k)$, ρ_* is the forgetting homomorphism $\rho_*([M, T, \xi, \tilde{T}])=[M, \xi]$ and φ_* is defined by $\varphi_*([M, \xi])=[M\times S^0, id\times A, \xi\times 0, id\times A]$ where $A: S^k\to S^k$ is the antipodal map and 0 is the 0-plane bundle.

Lemma 4.1. *The homomorphism ρ_* is the zero map.*

Proof. Let $[M, T, \xi, \tilde{T}]$ be any class of $\tilde{\boldsymbol{B}}(n, k)$. Let W be the quotient space of $M\times[0, 1]$ by identifying $(x, 1)$ with $(T(x), 1)$ for any $x\in M$, then W becomes a differentiable manifold with boundary M such that the quotient map $p: M\times[0, 1]\to W$ is differentiable. By the similar method there exists a k-plane bundle ζ over W satisfying $p^*\zeta=\xi\times 0$. Thus $(M, \xi)=\partial(W, \zeta)$. Therefore ρ_* is the zero map. q.e.d.

Thus we obtain a short exact sequence:

$$0 \to \mathfrak{N}_n(BO(k)) \xrightarrow{\varphi_*} \tilde{\boldsymbol{B}}(n, k) \xrightarrow{\psi_*} \tilde{\boldsymbol{B}}(n-1, k) \to 0 . \tag{4.2}$$

Now let $\sigma: \tilde{\boldsymbol{B}}(n, k)\to \boldsymbol{B}(n, k)$ be the canonical forgetting homomorphism defined by $\sigma([M, T, \xi, \tilde{T}])=[M, T, \xi]$ and $d: BO(k)\to BO(k)\times BO(k)$ be the diagonal map. Then the following diagram is commutative by the definition of the homomorphisms:

$$\begin{array}{ccccc} \mathfrak{N}_n(BO(k)) & \xrightarrow{\varphi_*} & \tilde{\boldsymbol{B}}(n, k) & \xrightarrow{\psi_*} & \tilde{\boldsymbol{B}}(n-1, k) \\ \downarrow d_* & & \downarrow \sigma & & \downarrow \sigma \\ \mathfrak{N}_n(BO(k)\times BO(k)) & \xrightarrow{\varphi_*} & \boldsymbol{B}(n, k) & \xrightarrow{\psi_*} & \boldsymbol{B}(n-1, k) . \end{array}$$

Since $\tau_*d_*=d_*$, $d_*(\mathfrak{N}_*(BO(k)))$ is contained in $A^{(k)}\oplus S^{(k)}$. And $\sigma(\tilde{\boldsymbol{B}}(n, k))$

is contained in $C_n^{(k)}=\psi_*(\boldsymbol{B}(n+1, k))$, because $\psi_*(\tilde{\boldsymbol{B}}(n+1, k))=\tilde{\boldsymbol{B}}(n, k)$. Let π be the projection of $\mathfrak{N}_*(BO(k)\times BO(k))=A^{(k)}\oplus R^{(k)}\oplus S^{(k)}$ onto $A^{(k)}$. Then the following diagram is commutative:

$$
\text{(4.3)}\qquad
\begin{array}{ccccc}
\mathfrak{N}_n(BO(k)) & \xrightarrow{\varphi_*} & \tilde{\boldsymbol{B}}(n, k) & \xrightarrow{\psi_*} & \tilde{\boldsymbol{B}}(n-1, k) \\
\downarrow{\scriptstyle \pi d_*} & & \downarrow{\scriptstyle \sigma} & & \downarrow{\scriptstyle \sigma} \\
A_n^{(k)} & \xrightarrow{\varphi_*} & C_n^{(k)} & \xrightarrow{\psi_*} & C_{n-1}^{(k)}
\end{array}
$$

and the lower horizontal line is exact by Theorem 1.6.

Let F^l be the $\mathfrak{N}_*$-submodule of $\mathfrak{N}_*(BO(k)\times BO(k))$ generated by $\{x_\omega\times x_{\omega'}\mid n(\omega)+n(\omega')\leqq l\}$. Then

$$
\text{(4.4)}\qquad d_*(x_\omega)-\sum_{\omega_1\omega_2=\omega} x_{\omega_1}\times x_{\omega_2}\in F^{n(\omega)-1}
$$

where $\omega_1\omega_2=(i_1, \cdots, i_r, j_1, \cdots, j_s)$ if $\omega_1=(i_1, \cdots, i_r)$ and $\omega_2=(j_1, \cdots, j_s)$, since $d^*(W_{i_1}\cdots W_{i_r}\otimes W_{j_1}\cdots W_{j_s})=W_{i_1}\cdots W_{i_r}W_{j_1}\cdots W_{j_s}$ in $H^*(BO(k); Z_2)$.

We will use the following known result.

Lemma 4.5. *Suppose the following diagram of the homomorphisms is commutative*:

$$
\begin{array}{ccccc}
A & \longrightarrow & B & & \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & \searrow{\scriptstyle \gamma} & \\
A' & \longrightarrow & B' & \longrightarrow & C'
\end{array}
$$

and the lower horizontal line is exact. Then β is surjective if α and γ are surjective.

Lemma 4.6. *The homomorphism πd_*: $\mathfrak{N}_*(BO(k))\rightarrow A^{(k)}$ is surjective.*

Proof. Let $F^{1,l}=\sum_n F_n^{1,l}$ be the $\mathfrak{N}_*$-submodule of $\mathfrak{N}_*(BO(k))$ generated by $\{x_\omega\mid n(\omega)\leqq l\}$ and $F^{2,l}=\sum_n F_n^{2,l}$ be the $\mathfrak{N}_*$-submodule of $A^{(k)}$ generated by $\left\{x_\omega\times x_\omega\mid n(\omega)\leqq\frac{l}{2}\right\}$ where $F_n^{1,l}$ and $F_n^{2,l}$ are the factors of degree n. Then

$$0=F_n^{1,-1}\subset F_n^{1,0}\subset\cdots\subset F_n^{1,n}=\mathfrak{N}_n(BO(k))$$

and

$$0=F_n^{2,-1}\subset F_n^{2,0}\subset\cdots\subset F_n^{2,n}=A_n^{(k)}\,.$$

Moreover $F^{1,l}/F^{1,l-1}$ is isomorphic to the free $\mathfrak{N}_*$-module generated by $\{x_\omega\mid n(\omega)=l\}$ and $F^{2,l}/F^{2,l-1}$ is isomorphic to the free $\mathfrak{N}_*$-module generated by $\{x_\omega\times x_\omega\mid n(\omega)=l/2\}$. Then $\pi d_*(F^{1,l})\subset F^{2,l}$ by (4.4), thus πd_* induces $\bar{d}_*$: $F^{1,l}/F^{1,l-1}\rightarrow F^{2,l}/F^{2,l-1}$ and $\bar{d}_*(x_{\omega\omega})=x_\omega\times x_\omega$ in $F^{2,2n(\omega)}/F^{2,2n(\omega)-1}$. Thus $\bar{d}_*$ is surjective and therefore πd_* is surjective by Lemma 4.5. q.e.d.

Theorem 4.7. $\sigma(\tilde{\boldsymbol{B}}(n, k))=C_n^{(k)}=\psi_*(\boldsymbol{B}(n+1, k))$.

Proof. This is an easy consequence of (4.2), (4.3), Lemma 4.5 and Lemma 4.6. q.e.d.

5. The rank of $\boldsymbol{B}^{\pm}(n, k)$

In the last section of the previous paper [4] we have considered the oriented cobordism groups of generic immersions and introduced the cobordism groups $\boldsymbol{B}^{+}(n, k)$ and $\boldsymbol{B}^{-}(n, k)$ of oriented k-plane bundles over oriented n-manifolds with orientation preserving involution and with orientation reversing involution respectively. These groups play an important role in the study of the oriented cobordism group of generic immersions and there exist following exact sequences [4]:

$$\cdots \to \boldsymbol{B}^{-}(n, k) \xrightarrow{\rho_*} \Omega_n(BSO(k)\times BSO(k)) \xrightarrow{\varphi_*^+} \boldsymbol{B}^{+}(n, k) \xrightarrow{\psi_*} \boldsymbol{B}^{-}(n-1, k) \to \cdots,$$

$$\cdots \to \boldsymbol{B}^{+}(n, k) \xrightarrow{\rho_*} \Omega_n(BSO(k)\times BSO(k)) \xrightarrow{\varphi_*^-} \boldsymbol{B}^{-}(n, k) \xrightarrow{\psi_*} \boldsymbol{B}^{+}(n-1, k) \to \cdots.$$

In this section we will consider the rank of $\boldsymbol{B}^{\pm}(n, k)$. Let Q be the field of rational numbers. Then the following sequences are exact:

$$(1)\quad \cdots \to \boldsymbol{B}^{-}(n, k)\otimes Q \xrightarrow{\rho_*} \Omega_n(BSO(k)\times BSO(k))\otimes Q \xrightarrow{\varphi_*^+} \boldsymbol{B}^{+}(n, k)\otimes Q \xrightarrow{\psi_*} \boldsymbol{B}^{-}(n-1, k)\otimes Q \to \cdots,$$

$$(2)\quad \cdots \to \boldsymbol{B}^{+}(n, k)\otimes Q \xrightarrow{\rho_*} \Omega_n(BSO(k)\times BSO(k))\otimes Q \xrightarrow{\varphi_*^-} \boldsymbol{B}^{-}(n, k)\otimes Q \xrightarrow{\psi_*} \boldsymbol{B}^{+}(n-1, k)\otimes Q \to \cdots.$$

We will use the following fact (cf. [1], [2]).

(5.1) *Let (X, A) be a CW-pair, then $\Omega_*(X, A)\otimes Q$ is a free $\Omega_*\otimes Q$-module isomorphic to $H_*(X, A; Q)\otimes_Q(\Omega_*\otimes Q)$.*

Now let $\{t_\alpha \mid \alpha\in\Lambda\}$ be a homogeneous basis of $\Omega_*(BSO(k))\otimes Q$ over $\Omega_*\otimes Q$. Then $\{t_\alpha\times t_\beta \mid \alpha, \beta\in\Lambda\}$ be a basis of $\Omega_*(BSO(k)\times BSO(k))\otimes Q$. Let $A^{(k)}=\sum_n A_n^{(k)}$, $S^{(k)}=\sum_n S_n^{(k)}$ and $T^{(k)}=\sum_n T_n^{(k)}$ be the free $\Omega_*\otimes Q$-module with basis $\{t_\alpha\times t_\alpha \mid \alpha\in\Lambda\}$, $\{t_\alpha\times t_\beta+t_\beta\times t_\alpha \mid \alpha, \beta\in\Lambda, \alpha\neq\beta\}$ and $\{t_\alpha\times t_\beta-t_\beta\times t_\alpha \mid \alpha, \beta\in\Lambda, \alpha\neq\beta\}$ respectively, where $A_n^{(k)}$, $S_n^{(k)}$ and $T_n^{(k)}$ are the factors of degree n. Then $\Omega_*(BSO(k)\times BSO(k))\otimes Q=A^{(k)}\oplus S^{(k)}\oplus T^{(k)}$ (direct sum).

Lemma 5.2. *The homomorphisms*

$$\rho_*\varphi_*^+\colon A^{(k)}\oplus S^{(k)} \to A^{(k)}\oplus S^{(k)}$$

and

$$\rho_*\varphi_*^-\colon T^{(k)} \to T^{(k)}$$

are the multiplication by 2.

Proof. $\rho_*\varphi_*^{\pm}(t_\alpha\times t_\beta)=t_\alpha\times t_\beta\pm t_\beta\times t_\alpha$ since $\rho_*\varphi_*^{\pm}=1\pm\tau_*$, where τ_* is induced from the map $\tau: BSO(k)\times BSO(k)\to BSO(k)\times BSO(k)$ switching factors. q.e.d.

Let $P_n^{(k)}=\varphi_*^+(A_n^{(k)}\oplus S_n^{(k)})$ and $M_n^{(k)}=\varphi_*^-(T_n^{(k)})$. Then

$$(5.3)\qquad \begin{aligned} &\varphi_*^+: A_n^{(k)}\oplus S_n^{(k)} \cong P_n^{(k)}, && \rho_*: P_n^{(k)} \cong A_n^{(k)}\oplus S_n^{(k)},\\ &\varphi_*^-: T_n^{(k)} \cong M_n^{(k)}, && \rho_*: M_n^{(k)} \cong T_n^{(k)} \end{aligned}$$

by Lemma 5.2.

Lemma 5.4. *$\boldsymbol{B}^+(n, k)\otimes Q=P_n^{(k)}$ and $\boldsymbol{B}^-(n, k)\otimes Q=M_n^{(k)}$.*

Proof. Since $BSO(k)$ is simply connected, $\Omega_0(BSO(k)\times BSO(k))\cong Z$ and $\Omega_1(BSO(k)\times BSO(k))=0$. Therefore $\boldsymbol{B}^+(0, k)\cong Z$, $\boldsymbol{B}^-(0, k)\cong Z_2$, $\boldsymbol{B}^+(1, k)\cong Z_2$ and $\boldsymbol{B}^-(1, k)=0$ by direct calculation. On the other hand $P_0^{(k)}\cong Q$, $M_0^{(k)}=0$ and $P_1^{(k)}=M_1^{(k)}=0$. Therefore Lemma 5.4 is true for $n=0, 1$. In general we will prove the lemma by induction on n.

Suppose $\boldsymbol{B}^+(n-1, k)\otimes Q=P_{n-1}^{(k)}$, then the homomorphism

$$\rho_*: \boldsymbol{B}^+(n-1, k)\otimes Q \to \Omega_{n-1}(BSO(k)\times BSO(k))\otimes Q$$

is injective by (5.3). Thus the homomorphism

$$\psi_*: \boldsymbol{B}^-(n, k)\otimes Q \to \boldsymbol{B}^+(n-1, k)\otimes Q$$

is the zero map by the exact sequence (2). Therefore

$$\begin{aligned} \boldsymbol{B}^-(n, k)\otimes Q &= \varphi_*^-(\Omega_n(BSO(k)\times BSO(k))\otimes Q)\\ &= \varphi_*^-(A_n^{(k)}\oplus S_n^{(k)}\oplus T_n^{(k)})\\ &= \varphi_*^-(\rho_*(P_n^{(k)})\oplus T_n^{(k)})\\ &= \varphi_*^-(T_n^{(k)})\\ &= M_n^{(k)} \end{aligned}$$

by (5.3) and the exact sequence (2). Similarly $\boldsymbol{B}^-(n-1, k)\otimes Q=M_{n-1}^{(k)}$ implies $\boldsymbol{B}^+(n, k)\otimes Q=P_n^{(k)}$. q.e.d.

Corollary 5.5. *$\psi_*(\boldsymbol{B}^{\pm}(n, k))$ is contained in the torsion subgroup of $\boldsymbol{B}^{\mp}(n-1, k)$.*

Corollary 5.6. *$\boldsymbol{B}^+(n, k)$ and $\boldsymbol{B}^-(n, k)$ are torsion groups if $\Omega_n(BSO(k)\times BSO(k))$ is a torsion group.*

Now $\Omega_*\otimes Q\cong Q[x_1, x_2, \cdots, x_n, \cdots]$ where the degree of $x_n=4n$, $H^*(BSO(k); Q)\cong Q[p_1, p_2, \cdots, p_r]$ for $k=2r+1$ and $H^*(BSO(k); Q)\cong Q[p_1, p_2, \cdots, p_{r-1}, x_r]$ for $k=2r$ where the degree of $p_i=4i$ and the degree of $x_r=2r$. Therefore the rank of $\boldsymbol{B}^{\pm}(n, k)$ is determined by (5.3) and Lemma 5.4.

REMARK. Recently R. Stong [3] studied the equivariant bordism groups. The cobordism group $\tilde{\boldsymbol{B}}(n, k)$ is $\hat{\mathfrak{N}}_n(BO(k), \tau)$ in his notation.

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