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ON HOMOTOPY SPHERES WHICH ADMIT DIFFERENTIABLE ACTIONS II

KATSUO KAWAKUBO*

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1. Introduction

A differentiable action (M^m, φ, G) is called semi-free if it is free outside the fixed point set, i.e., there are two types of orbits, fixed points and G. We shall study the situation where (Σ^m, φ, S^1) is a semi-free differentiable action of S^1 on a homotopy sphere Σ^m, and the fixed point set F^p is a homotopy sphere. Concerning semi-free differentiable actions, Browder has studied in [5] and has posed the following problem.

"What are the homotopy spheres which are being operated on in our constructions?"

On this problem we shall prove some theorems (see Theorems 2.1–2.5), generalizing a theorem stated in [11]. They give a partial answer to this problem of Browder. As corollaries we shall give non existence theorems of semi-free S^1-actions on some homotopy spheres (see Corollaries 2.6, 2.7). They give an answer to a problem of Bredon (see [19, problem 4, page 235]) and a partial answer to a problem of Hirzebruch (see [19. problem 12, page 236]).

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2. Definitions, notations and statement of results

Let us denote by (M^m, φ, G) a differentiable action of the Lie group G on the smooth manifold M, *i. e.*, φ: $G \times M \rightarrow M$ such that, if $m \in M$, $x, y \in G$,

(i) $\varphi\big(x, \varphi(y, m)\big) = \varphi(xy, m)$,

(ii) $\varphi(e, m) = m$, $\quad e$=identity of G,

(iii) φ is a C^∞-map.

A smooth submanifold $N \subset M^m$ is called *invariant* if $\varphi(G \times N) \subset N \subset M^m$. An

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action is called *semi-free* if it is free off of fixed point set, i. e., there are two types of orbits, fixed points and G. All manifolds, with or without boundary, are to be compact, oriented and differentiable of class C^∞. The boundary of M will be denoted by ∂M. We write $M_1 = M_2$ for manifolds M_1, M_2, if there is an orientation preserving diffeomorphism f: $M_1 \to M_2$. Let Θ_n be the group of homotopy n-spheres and $\Theta_n(\partial\pi)$ be the subgroup consisting of those homotopy spheres which bound parallelizable manifolds. The inertia group of an oriented closed differentiable manifold M^n is defined to be the group $\{\Sigma \in \Theta_n \mid M^n \# \Sigma = M^n\}$ which is denoted by $I(M)$. Let Σ_M^n be the generator of $\Theta_n(\partial\pi)$ due to Kervaire and Milnor [14]. D^n and S^{n-1} denote, respectively, the unit disk and the unit sphere in euclidean n-space and $\boldsymbol{CP}^n$ denotes the complex projective n-space. Denote by $S(\xi)$, $B(\xi)$, $\boldsymbol{CP}(\xi)$, the total space of the sphere bundle, the total space of the disk bundle, the total space of the projective space bundle, respectively, associated to a complex vector bundle ξ. Let (Σ^m, φ, S^1) be a semi-free action on a homotopy sphere Σ^m, with fixed point set F a homotopy p-sphere. Let η be the normal complex q-plane bundle of F in Σ^m, $2q = m - p$. The fixed point set F^p is called *untwisted* when η is the trivial complex q-plane bundle.

Then we shall have

Theorem 2.1. *If a homotopy sphere Σ^{p+2q} admits a semi-free S^1-action with some $F^p \in \Theta_p$ as fixed point set for $p+2q \geqq 7$, then*

$$\boldsymbol{CP}(\eta \oplus \boldsymbol{C}) = (S^p \times \boldsymbol{CP}^q) \# \Sigma^{p+2q},$$

where η is the normal complex q-plane bundle of F^p in Σ^{p+2q} and $\boldsymbol{C}$ denotes the trivial complex line bundle.

Theorem 2.2. *If a homotopy sphere Σ^{p+2q} admits a semi-free S^1-action with some $F^p \in \Theta_p$ as fixed point set, for $p+2q \geqq 7$, $p \leqq 2q-1$, then*

$$F^p \times \boldsymbol{CP}^q = (S^p \times \boldsymbol{CP}^q) \# \Sigma^{p+2q}.$$

Theorem 2.3. *If a homotopy sphere Σ^{p+2q} admits a semi-free S^1-action with $F^p \in \Theta_p(\partial\pi)$ as untwisted fixed point set for $p+2q \geqq 7$ and q: odd,, then*

$$\Sigma^{p+2q} \in I(S^p \times \boldsymbol{CP}^q).$$

Theorem 2.4. *If a homotopy sphere $\Sigma^{4p-1+4q}$ admits a semi-free S^1-action with $F^{4p-1} \in \Theta_{4p-1}(\partial\pi)$ as untwisted fixed point set for $4p-1+4q \geqq 7$, then*

$$\Sigma \# (-\partial U) \in I(S^{4p-1} \times \boldsymbol{CP}^{2q}),$$

where U is a manifold constructed as follows. Let W^{4p} be a parallelizable manifold with $\partial W = F$. Then U is a parallelizable $(4p+4q)$–manifold such that Index U $=$Index W *and ∂U is a homotopy sphere.*

Theorem 2.5. *If a homotopy sphere $\Sigma^{4p+1+4q}$ admits a semi-free S^1-action with $F^{4p+1}\in\Theta_{4p+1}(\partial\pi)$ as untwisted fixed point set for $4p+1+4q$ $(\neq 13)\geqq 7$, then*

$$\Sigma \# (-\partial U)\in I(S^{4p+1}\times \boldsymbol{CP}^{2q}),$$

where U is a manifold constructed as follows. Let W^{4p+2} be a parallelizable manifold with $\partial W=F$. Then U is a parallelizable $(4p+2+4q)$-manifold such that Arf $U=$Arf W *and ∂U is a homotopy sphere. When $4p+1+4q=13$ or 29, $\Sigma\in I(S^{4p+1}\times \boldsymbol{CP}^{2q})$.*

Corollary 2.6. *Any homotopy sphere Σ^{p+2q} which is not a spin boundary, does not admit any semi-free S^1-action with $F^p\in\Theta_p(\partial\pi)$ as untwisted fixed point set for $p\neq 1$, q: odd and $p+2q\geqq 7$.*

Milnor [17] and Anderson, Brown and Peterson [1] have proved that there exist homotopy spheres Σ_0^{8k+1}, Σ_0^{8k+2} not bounding spin-manifolds for any $k\geqq 1$. Hence Corollary 2.6 brings about the following

Corollary 2.7. *The homotopy sphere Σ_0^{8k+1}* (resp. Σ_0^{8k+2}) *does not admit any semi-free S^1-action with $F^p\in\Theta_p(\partial\pi)$ as untwisted fixed point set, if $p\neq 1$ and $(8l+1-p)/2$* $\big($resp. $(8k+2-p)/2\big)$ *is odd.*

REMARK 2.8. When $(8k+2-p)/2$ is even, G.E. Bredon has constructed some examples in [2]. For example, the homotopy sphere Σ_0^{10} (resp. Σ_0^{18}) admits a semi-free S^1-action with the natural sphere as untwisted fixed point set of any codimension divisible by 4.

On the other hand we can construct some semi-free S^1-actions on homotopy spheres by making use of the results of Brieskorn and Hirzebruch [4], [8].

Proposition 2.9. *For any $k\in \boldsymbol{Z}$, $k\,\Sigma_M^{4p-1+4q}$ admits a semi-free S^1-action with $k\,\Sigma_M^{4p-1}$ as fixed point set.*

Proposition 2.10. *For any $k\in \boldsymbol{Z}$, $k\,\Sigma_M^{4p+1+4q}$ admits a semi-free S^1-action with $k\,\Sigma_M^{4p+1}$ as fixed point set.*

REMARK 2.11. Theorem 2.1 is a generalization of H. Maehara [15].

3. Preliminaries

In this section we shall, for the benefit of the reader, prove a lemma of Browder [5] which will be necessary afterward. For a general discussion of semi-free S^1-actions we refer to [3] and [5].

Let (Σ^m, φ, S^1) be a semi-free action, with fixed point set $F^p\subset\Sigma^m$, F^p a homotopy p-sphere. According to Uchida [23], the normal bundle of F^p has a complex structure such that the induced action of S^1 on it, is the scalar multiplication when we regard S^1 as $\{z\in C\,\big|\,|z|=1\}$. In particular the codimension

$m-p=2q$. Let η be the complex bundle over F defined by the action. It is shown by Hsiang [9] and Montgomery-Yang [18] that if $q=1$ and $m>6$, then $\Sigma^m=S^m$ and $F=S^{m-2}$ embedded as usual, and the action is linear. Therefore we may restrict ourselves to $q>1$. Let $B(\eta)$ be an invariant tubular neighbourhood of F in Σ^m (see [7, page 57]) (here we identified an invariant tubular neighbourhood with the total space of the normal disk bundle), and let S^{2q-1} be the boundary of a fibre of $B(\eta)$. When $q>1$, it follows from a general position argument that $\pi_1(\Sigma-F)\cong\{1\}$. By making use of the Alexander duality theorem, we can prove that the inclusion $S^{2q-1}\subset\Sigma-F$ induces isomorphisms $H_*(S^{2q-1})\cong H_*(\Sigma-F)$ of homology groups. It follows from J.H.C. Whitehead [24] that if $q>1$, then $S^{2q-1}\subset\Sigma-F$ is a homotopy equivalence. Now let $N=\Sigma-B_0(\eta)$ where $B_0(\eta)$ is the interior of an invariant tubular neighbourhood of F, with $\overline{B_0(\eta)}\subset \operatorname{Int} B(\eta)$. Then S^1 acts freely on N, and on $S^{2q-1}\subset N$, and S^{2q-1} is homotopy equivalent to N. It follows from the exact homotopy sequence of the fibre maps, using the diagram

$$\begin{tikzcd} S^{2q-1} \arrow[r] \arrow[d] & N \arrow[d] \\ S^{2q-1}/S^1 \arrow[r] & N/S^1 \end{tikzcd}$$

that $S^{2q-1}/S^1\to N/S^1$ is a homotopy equivalence. Set $\overline{N}=N/S^1$. Since the action of S^1 on S^{2q-1} is standard, $S^{2q-1}/S^1=\boldsymbol{CP}^{q-1}$, and since S^{2q-1} is the fibre of $B(\eta)$ over F it follows that its normal bundle is equivariantly trivial, so that we get an embedding $D^{p+1}\times\boldsymbol{CP}^{q-1}\subset\overline{N}^{m-1}$, and it is a homotopy equivalence. Similarly it is easy to prove that the region between $\partial\overline{N}$ and $S^p\times\boldsymbol{CP}^{q-1}$ is an h-cobordism, so if $m>6$, by the h-cobordism theorem of Smale if $p>1$ [22], or its generalization, the s-cobordism theorem if $p=1$ [13], it is diffeomorphic to the product $S^p\times\boldsymbol{CP}^{q-1}\times I$, and hence $\overline{N}$ is diffeomorphic to $D^{p+1}\times\boldsymbol{CP}^{q-1}$, and $N\to\overline{N}$ is equivalent to

$$id\times h\colon D^{p+1}\times S^{2q-1}\to D^{p+1}\times\boldsymbol{CP}^{q-1}$$

where $h\colon S^{2q-1}\to\boldsymbol{CP}^{q-1}$ is the Hopf map, i.e. the principal bundle $N\to\overline{N}$ is induced by the map $\overline{N}\to\boldsymbol{CP}^{q-1}$ of the homotopy equivalence.

Hence we have shown the following

Lemma 3.1. *Let (Σ^m, φ, S^1) be a semi-free action on a homotopy sphere Σ^m, with fixed point set F a homotopy p-sphere. Then the normal bundle of F in Σ has a complex structure such that the induced action of S^1 on it, is the scalar multiplication when we regard S^1 as $\{z\in C \mid |z|=1\}$. In particular $m-p=2q$. Let N be the complement of an invariant open tubular neighbourhood of F in Σ^m. If $q>1$ and $m>6$, then N is equivariantly diffeomorphic to $D^{p+1}\times S^{2q-1}$,* *with the standard action on* S^{2q-1}, *trivial action on* D^{p+1}. *In particular* Σ^m *is diffeomorphic to* $B(\eta)\underset{f}{\cup}D^{p+1}\times S^{2q-1}$ *where* f *is an equivariant diffeomorphism* $f\colon \partial B(\eta)\to S^p\times S^{2q-1}$ *and* $\underset{f}{\cup}$ *means we identify* $\partial B(\eta)\subset B(\eta)$ *with* $S^p\times S^{2q-1}\subset D^{p+1}\times S^{2q-1}$ *via the diffeomorphism* f.

4. Proof of Theorem 2.1

When $q=1$, Theorem 2.1 trivially holds (see §3). Hence we may assume that $q>1$. Let (Σ^m, φ, S^1) be a semi-free S^1-action on a homotopy sphere Σ^m, with fixed point set F a homotopy p-sphere. Let η be the normal complex q-plane bundle of F in Σ^m, $2q=m-p$. Then we have an equivariant diffeomorphism $f\colon S(\eta)\to S^p\times S^{2q-1}$ such that $B(\eta)\underset{f}{\cup}D^{p+1}\times S^{2q-1}$ is diffeomorphic to the homotopy sphere Σ^m by Lemma 3.1. We write $B(\eta)$ (resp. $S^p\times D^{2q}$) in the form

$$B(\eta)=D_1^p\times D^{2q}\underset{\eta}{\cup}D_2^p\times D^{2q}$$
$$(\text{resp. } S^p\times D^{2q}=D_3^p\times D^{2q}\underset{id}{\cup}D_4^p\times D^{2q})$$

where $\underset{\eta}{\cup}$ means we identify $(\partial D_1^p)\times D^{2q}$ with $(\partial D_2^p)\times D^{2q}$ *via* the diffeomorphism h obtained as follows. Let $l\in\pi_{p-1}(U_q)$ be the characteristic map of the bundle η. Then the diffeomorphism

$$h\colon (\partial D_1^p)\times D^{2q}\longrightarrow(\partial D_2^p)\times D^{2q}$$

is defined by

$$h(x, y)=\big(x, l(x)y\big)\,.$$

We can assume that

$$f\,|\,D_2^p\times S^{2q-1}\colon\ D_2^p\times S^{2q-1}\longrightarrow D_4^p\times S^{2q-1}$$

and that $f\,|\,D_2^p\times S^{2q-1}=id$ by making use of the relative h-cobordism theorem. Let $B_\varepsilon(\eta)$ be $D_1^p\times D_\varepsilon^{2q}\underset{\eta'}{\cup}D_2^p\times D_\varepsilon^{2q}$ where D_ε^{2q} denotes the disk of radius ε, $0<\varepsilon<1$ and η' denotes the restriction of η. Canonically we can extend the diffeomorphism f to the equivariant diffeomorphism

$$\bar{f}\colon\ B(\eta)-\operatorname{Int}B_\varepsilon(\eta)\longrightarrow S^p\times D^{2q}-S^p\times\operatorname{Int}D_\varepsilon^{2q}\,.$$

Hence we have the following equivariant diffeomorphism

$$\begin{aligned}&D_2^p\times D_\varepsilon^{2q}\underset{id}{\cup}\big(B(\eta)-\operatorname{Int}B_\varepsilon(\eta)\big)\underset{f}{\cup}D^{p+1}\times S^{2q-1}\\ &\xrightarrow[id\,\cup\,\bar{f}\,\cup\,id]{}D_4^p\times D_\varepsilon^{2q}\underset{id}{\cup}(S^p\times D^{2q}-S^p\times\operatorname{Int}D_\varepsilon^{2q})\underset{id}{\cup}D^{p+1}\times S^{2q-1}\,.\end{aligned}$$

It is clear that $D_2^p \times D_\varepsilon^{2q} \underset{id}{\cup} \big(B(\eta) - \mathrm{Int}\, B_\varepsilon(\eta)\big) \underset{f}{\cup} D^{p+1} \times S^{2q-1}$ is diffeomorphic to $\Sigma^m - \mathrm{Int}\,(D_1^p \times D_\varepsilon^{2q})$ and

$$D_4^p \times D_\varepsilon^{2q} \underset{id}{\cup} (S^p \times D^{2q} - S^p \times \mathrm{Int}\, D_\varepsilon^{2q}) \underset{id}{\cup} D^{p+1} \times S^{2q-1}$$

is diffeomorphic to $S^m - \mathrm{Int}\,(D_3^p \times D_\varepsilon^{2q})$. It follows that the obstruction to extending the diffeomorphism

$$id \cup \bar{f} \cup id \colon \Sigma^m - \mathrm{Int}\,(D_3^p \times D_\varepsilon^{2q}) \longrightarrow S^m - \mathrm{Int}\,(D_3^p \times D_\varepsilon^{2q})$$

to $\Sigma^m \to S^m$ is nothing but Σ^m. Here we identified Θ_m with the pseudo isotopy group $\tilde{\pi}_0\,(\mathrm{Diff}\, S^{m-1})$ of diffeomorphisms of S^{m-1} due to Smale [22]. Consequently we have

Lemma 4.1. *The obstruction to extending the diffeomorphism*

$$f\colon S(\eta) \longrightarrow S^p \times S^{2q-1}$$

to $B(\eta) \to S^p \times D^{2q}$ *is nothing but* Σ^m.

Let $\big(S(\eta \oplus \boldsymbol{C}), \varphi_1, S^1\big)$ denote the S^1-action which is given as follows. By making use of a local trivialization, we can represent each point of $S(\eta \oplus \boldsymbol{C})$ by $(x, z_1, \cdots, z_q, z)$ with $\sum_{i=1}^{q} |z_i|^2 + |z|^2 = 1$ where x is a point of F. Then the action

$$\varphi_1\colon S^1 \times S(\eta \oplus \boldsymbol{C}) \longrightarrow S(\eta \oplus \boldsymbol{C})$$

is defined by

$$\varphi_1\big(g, (x, z_1, \cdots, z_q, z)\big) = (x, gz_1, \cdots, gz_q, gz)\,.$$

Since the bundle $\eta \oplus \boldsymbol{C}$ is a complex vector bundle, this operation does not depend on the choice of local trivializations.

Let $\big(S(\eta) \times D^2 \underset{id}{\cup} B(\eta) \times S^1, \varphi_2, S^1\big)$, $(S^p \times S^{2q+1}, \varphi_3, S^1)$, $(S^p \times S^{2q-1} \times D^2 \underset{id}{\cup} S^p \times D^{2q} \times S^1, \varphi_4, S^1)$ denote the S^1-actions which are given in similar ways. Denote by $S_1(\eta \oplus \boldsymbol{C})$ $\big($resp. $S_2(\eta \oplus \boldsymbol{C})\big)$ the following invariant submanifold of $S(\eta \oplus \boldsymbol{C})$ for ε, $0 < \varepsilon < 1$:

$$\big\{(x, z_1, \cdots, z_q, z) \,\big|\, |z_1|^2 + \cdots + |z_q|^2 + |z|^2 = 1,\ |z| \leqq \varepsilon\big\}$$
$$\Big(\text{resp. } \big\{(x, z_1, \cdots, z_q, z) \,\big|\, |z_1|^2 + \cdots + |z_q|^2 + |z|^2 = 1,\ |z| \geqq \varepsilon\big\}\Big)\,.$$

Since the structural group of the fibre bundle $S(\eta \oplus \boldsymbol{C})$ is the unitary group $U(q+1)$, the above set does not depend on trivializations. Let $d_1\colon S_1(\eta \oplus \boldsymbol{C}) \to S(\eta) \times D^2$ $\big($resp. $d_2\colon S_2(\eta \oplus \boldsymbol{C}) \to B(\eta) \times S^1\big)$ be the diffeomorphism defined by

$$d_1(x, z_1, \cdots, z_q, z) = \left(x, \frac{z_1}{a}, \cdots, \frac{z_q}{a}, \frac{z}{\varepsilon}\right)$$

$$\left(\text{resp. } d_2(x, z_1, \cdots, z_q, z) = \left(x, \frac{z_1}{\sqrt{1-\varepsilon^2}}, \cdots, \frac{z_q}{\sqrt{1-\varepsilon^2}}, \frac{z}{|z|}\right)\right)$$

where $$a = \sqrt{|z_1|^2 + \cdots + |z_q|^2}\,.$$

Since for $g \in S^1$, $(x, z_1, \cdots, z_q, z) \in S_1(\eta \oplus \boldsymbol{C})$

$$\begin{aligned}
& d_1 \circ \varphi_1\big(g, (x, z_1, \cdots, z_q, z)\big) \\
&= d_1(x, gz_1, \cdots, gz_q, gz) \\
&= \left(x, \frac{gz_1}{a}, \cdots, \frac{gz_q}{a}, \frac{gz}{\varepsilon}\right) \\
&= \varphi_2\left(g, \left(x, \frac{z_1}{a}, \cdots, \frac{z_q}{a}, \frac{z}{\varepsilon}\right)\right) \\
&= \varphi_2\big(g, d_1(x, z_1, \cdots, z_q, z)\big)\,,
\end{aligned}$$

d_1 is equivariant. Similarly d_2 is equivariant. Hence we have the following equivariant diffeomorphism

$$\begin{aligned}
d = d_1 \cup d_2 \colon S(\eta \oplus \boldsymbol{C}) &= S_1(\eta \oplus \boldsymbol{C}) \cup S_2(\eta \oplus \boldsymbol{C}) \\
&\longrightarrow \big(S(\eta) \times D^2 \underset{id}{\cup} B(\eta) \times S^1, \varphi_2, S^1\big)\,.
\end{aligned}$$

Similar arguments prove that there exists an equivariant diffeomorphism $d' \colon (S^p \times S^{2q+1}, \varphi_3, S^1) \to (S^p \times S^{2q-1} \times D^2 \underset{id}{\cup} S^p \times D^{2q} \times S^1, \varphi_4, S^1)$. Define a map

$$d_3 \colon B(\eta) \times S^1 \longrightarrow B(\eta) \times S^1$$

$$(\text{resp. } d_4 \colon S^p \times D^{2q} \times S^1 \longrightarrow S^p \times D^{2q} \times S^1)$$

by

$$d_3(y, z) = \big(\phi_2(z, y), z\big) \qquad \text{for} \quad y \in B(\eta),\ z \in S^1$$

$$\big(\text{resp. } d_4(y, z) = \big(\phi_4(z, y), z\big) \qquad \text{for} \quad y \in S^p \times D^{2q},\ z \in S^1\big)$$

where ϕ_2 (resp. ϕ_4) denotes the action defined by

$$\phi_2\big(g, (x, z_1, \cdots, z_q)\big) = (x, gz_1, \cdots, gz_q) \quad \text{for} \quad (x, z_1, \cdots, z_q) \in B(\eta)$$

$$\big(\text{resp. } \phi_4\big(g, (x, z_1, \cdots, z_q)\big) = (x, gz_1, \cdots, gz_q) \quad \text{for} \quad (x, z_1, \cdots, z_q) \in S^p \times D^{2q}\big)\,.$$

Let $\big(B(\eta) \times S^1, \varphi_5, S^1\big)\big(\text{resp. } (S^p \times D^{2q} \times S^1, \varphi_6, S^1)\big)$ be the action defined by

$$\varphi_5\big(g, (y, z)\big) = (y, gz) \qquad \text{for} \quad y \in B(\eta),\ z, g \in S^1$$

$$\left(\text{resp. } \varphi_6(g, (y, z)) = (y, gz) \qquad \text{for} \quad y \in S^p \times D^{2q}, \quad z, g \in S^1\right).$$

Then we have

Lemma 4.2. *d_3 (resp. d_4) is an equivariant diffeomorphism*

$$d_3\colon \left(B(\eta)\times S^1, \varphi_2', S^1\right) \longrightarrow \left(B(\eta)\times S^1, \varphi_5, S^1\right)$$

$$\left(\textit{resp. } d_4\colon (S^p\times D^{2q}\times S^1, \varphi_4', S^1) \longrightarrow (S^p\times D^{2q}\times S^1, \varphi_6, S^1)\right)$$

where φ_2' (resp. φ_4') denotes the restriction of φ_2 (resp. φ_4).

Proof

$$\begin{aligned} d_3\circ\varphi_2'\left(g, (y, z)\right) &= d_3\left(\phi_2(g, y), gz\right) \\ &= \left(\phi_2\left(\overline{gz}, \phi_2(g, y)\right), gz\right) = \left(\phi_2(\overline{gz}g, y), gz\right) \\ &= \left(\phi_2(\bar z, y), gz\right) = \varphi_5\left(g, \left(\phi_2(\bar z, y), z\right)\right) \\ &= \varphi_5\left(g, d_3(y, z)\right). \end{aligned}$$

This shows that d_3 is equivariant with respect to φ_2', φ_5. On the other hand, define a map

$$d_5\colon B(\eta)\times S^1 \longrightarrow B(\eta)\times S^1$$

by

$$d_5(y, z) = \left(\phi_2(z, y), z\right).$$

Then we have $d_5\circ d_3(y, z) = d_5\left(\phi_2(\bar z, y), z\right) = \left(\phi_2\left(z, \phi_2(\bar z, y)\right), z\right) = \left(\phi_2(\bar z\cdot z, y), z\right)$ $= (y, z)$ and $d_3\circ d_5(y, z) = d_3\left(\phi_2(z, y), z\right) = \left(\phi_2\left(\bar z, \phi_2(z, y)\right), z\right) = \left(\phi_2(\bar z\cdot z, y), z\right)$ $= (y, z)$, i.e., $d_5\circ d_3 = d_3\circ d_5 =$ identity. Obviously d_3 and d_5 are differentiable, hence d_3 is an equivariant diffeomorphism. As for d_4, the proof is left to the reader.

It follows from Lemma 4.2 that we can construct a semi-free differentiable action

$$\left(S(\eta)\times D^2 \underset{d_3'}{\cup} B(\eta)\times S^1, \varphi_2''\cup\varphi_5, S^1\right)$$

$$(\text{resp. } S^p\times S^{2q-1}\times D^2 \underset{d_4'}{\cup} S^p\times D^{2q}\times S^1, \varphi_4''\cup\varphi_6, S^1)$$

where $\quad d_3' = d_3\,|\,S(\eta)\times S^1$ (resp. $d_4' = d_4\,|\,S^p\times S^{2q-1}\times S^1$)

and $\quad \varphi_2'' = \varphi_2\,|\,S(\eta)\times D^2$ (resp. $\varphi_4'' = \varphi_4\,|\,S^p\times S^{2q-1}\times D^2$).

Then we have

Lemma 4.3. *$id\cup d_3$ (resp. $id\cup d_4$) is an equivariant diffeomorphism*

$$id\cup d_3\colon \big(S(\eta)\times D^2\underset{id}{\cup}B(\eta)\times S^1,\ \varphi_2,\ S^1\big)$$
$$\longrightarrow \big(S(\eta)\times D^2\underset{d_3'}{\cup}B(\eta)\times S^1,\ \varphi_2''\cup\varphi_5,\ S^1\big)$$
$$\big(\text{resp. } id\cup d_4\colon (S^p\times S^{2q-1}\times D^2\underset{id}{\cup}S^p\times D^{2q}\times S^1,\ \varphi_4,\ S^1)$$
$$\longrightarrow (S^p\times S^{2q-1}\times D^2\underset{d_4'}{\cup}S^p\times D^{2q}\times S^1,\ \varphi_4''\cup\varphi_6\ S^1)\big)\,.$$

Proof. Since the map is well-defined, this lemma follows easily from Lemma 4.2.

It is clear that the orbit space $S(\eta\oplus\boldsymbol{C})/\varphi_1$ is diffeomorphic to $\boldsymbol{CP}(\eta\oplus\boldsymbol{C})$ and $S^p\times S^{2q+1}/\varphi_3$ is diffeomorphic to $S^p\times\boldsymbol{CP}^q$.

Lemma 4.4. *The composition $d_4\circ(f\times id)\circ d_3^{-1}|\partial B(\eta)\times S^1$ is equal to $f\times id\,|\,\partial B(\eta)\times S^1$.*

Proof. For $y\in\partial B(\eta)$, $z\in S^1$, we have

$$\begin{aligned} &d_4\circ(f\times id)\circ d_3^{-1}(y,\ z)\\ &= d_4\circ(f\times id)\circ\big(\phi_2(z,\ y),\ z\big)\\ &= d_4\circ\big(\phi_4\big(z,\ f(y)\big),\ z\big)\\ &= \big(\phi_4\big(\bar{z},\ \phi_4\big(z, f(y)\big)\big),\ z\big)\\ &= \big(f(y),\ z\big),\end{aligned}$$

completing the proof of Lemma 4.4.

Lemma 4.5. *The composition $(d_4/\sim)\circ\{(f\times id)/\sim\}\circ(d_3^{-1}/\sim)\,|\,\partial B(\eta)$ of the maps induced by the equivariant maps, is equal to f.*

Proof. Since the action φ_5 (resp. φ_6) is trivial on the first factor $B(\eta)$ of $B(\eta)\times S^1$ (resp. $S^p\times D^{2q}$ of $S^p\times D^{2q}\times S^1$), this lemma follows directly from Lemma 4.4.

Now we prove Theorem 2.1. It is clear that the orbit space $S(\eta\oplus\boldsymbol{C})/\varphi_1$ is diffeomorphic to $\boldsymbol{CP}(\eta\oplus\boldsymbol{C})$, hence $\big(S(\eta)\times D^2\underset{d_3'}{\cup}B(\eta)\times S^1\big)/(\varphi_2''\cup\varphi_5)$ is diffeomorphic to $\boldsymbol{CP}(\eta\oplus\boldsymbol{C})$ by Lemma 4.3. Similarly $(S^p\times S^{2q-1}\times D^2\underset{d_4'}{\cup}S^p\times D^{2q}\times S^1)/(\varphi_4''\cup\varphi_6)$ is diffeomorphic to $S^p\times\boldsymbol{CP}^q$ by Lemma 4.3. Hence the composition

$$T=\big\{(id\cup d_4)/\sim\big\}\circ\big\{(f\times id)/\sim\big\}\circ\big\{(id\cup d_3)^{-1}/\sim\big\}$$

gives a diffeomorphism

$$T\colon \boldsymbol{CP}(\eta\oplus\boldsymbol{C})-\operatorname{Int}B(\eta)\longrightarrow S^p\times\boldsymbol{CP}^q-S^p\times\operatorname{Int}D^{2q}$$

such that $T\,|\,\partial B(\eta)=f$ by Lemma 4.5. It follows from Lemma 4.1 that the obstruction to extending the diffeomorphism

$$T\,|\,\partial B(\eta)\colon \partial B(\eta)\longrightarrow S^p\times\partial D^{2q}$$

to $B(\eta)\to S^p\times D^{2q}$ is nothing but Σ^{p+2q}. Thus we have a diffeomorphism

$$T\cup S\colon \boldsymbol{CP}(\eta\oplus\boldsymbol{C})\longrightarrow(S^p\times\boldsymbol{CP}^q)\,\#\,\Sigma^{p+2q}$$

where S denotes a diffeomorphism obtained by Lemma 4.1. This makes the proof of Theorem 2.1 complete.

5. Proof of Theorems 2.2, 2.3, 2.4 and 2.5

5.1. Proof of Theorem 2.2

According to Theorem 5.5 of Browder [5], the normal complex bundle η of the fixed point set F in Σ^m is stably trivial. Therefore this theorem follows directly from Theorem 2.1.

5.2. Proof of Theorem 2.3

In the proof of theorem 6.1 of Browder [5], it is shown that $F^p\times\boldsymbol{CP}^q$ is diffeomorphic to $S^p\times\boldsymbol{CP}^q$ for $F^p\in\Theta_p(\partial\pi)$ and for q: odd. Applying Theorem 2.1, it follows that $S^p\times\boldsymbol{CP}^q=F^p\times\boldsymbol{CP}^q=(S^p\times\boldsymbol{CP}^q)\,\#\,\Sigma^{p+2q}$, i.e., Σ^{p+2q} belongs to the inertia group $I(S^p\times\boldsymbol{CP}^q)$, completing the proof of Theorem 2.3.

5.3 Proof of Theorem 2.4

Let W^{4p} be a parallelizable manifold with $\partial W=F^{4p-1}$. Let U be a parallelizable $(4p+4q)$-manifold such that Index $W=$ Index U and ∂U is a homotopy sphere. Remark that there always exists such a manifold U (see Milnor [16]). Then it is shown that $F^{4p-1}\times\boldsymbol{CP}^{2q}$ is diffeomorphic to $(S^{4p-1}\times\boldsymbol{CP}^{2q})\,\#\,\partial U$ in the proof of Theorem 6.2 of Browder [5]. Applying theorem 2.1, it follows that $(S^{4p-1}\times\boldsymbol{CP}^{2q})\,\#\,\partial U=(S^{4p-1}\times\boldsymbol{CP}^{2q})\,\#\,\Sigma^{4p-1+4q}$, i.e., $\Sigma\,\#\,(-\partial U)\in I(S^{4p-1}\times\boldsymbol{CP}^{2q})$, completing the proof of Theorem 2.4.

5.4 Proof of Theorem 2.5

We first show the following

Lemma 5.4.1. *There exists a parallelizable $(4k+2)$-manifold M^{4k+2} with boundary a homotopy sphere ∂M^{4k+2} such that* Arf *invariant of M is equal to* 1 *for any integer $k(\neq 1, 3)>0$.*

Proof. Let $\iota\colon \pi_{2k}(SO_{2k+1})\to\pi_{2k}(SO)$ be the natural homomorphism induced by the inclusion $SO_{2k+1} \subset SO$. Let $\nu \in \operatorname{Ker} \iota$ be the unique non trivial element (see Kervaire [12]) and let $(B, S^{2k+1}, D^{2k+1}, p)$ be the disk bundle over sphere with the characteristic map $\nu \in \pi_{2k}(SO_{2k+1})$. Let B_α, B_β be two copies of B. When we regard

$$B_\alpha \quad \text{as} \quad D_1^{2k+1} \times D_2^{2k+1} \underset{\nu}{\cup} D_3^{2k+1} \times D_4^{2k+1}$$

and

$$B_\beta \quad \text{as} \quad D_5^{2k+1} \times D_6^{2k+1} \underset{\nu}{\cup} D_7^{2k+1} \times D_8^{2k+1},$$

the plumbing manifold of B_α and B_β is defined to be the oriented differentiable $(4k+2)$-manifold obtaind as a quotient space of $B_\alpha \cup B_\beta$ by identifying $D_3^{2k+1} \times D_4^{2k+1}$ and $D_5^{2k+1} \times D_6^{2k+1}$ by the relation $(x, y) = (y, x)(x \in D_3^{2k+1} = D_5^{2k+1}$, $y \in D_4^{2k+1} = D_6^{2k+1})$ and is denoted by $B_\alpha \veebar B_\beta$ $(= B \veebar B)$. Let M^{4k+2} be the manifold $B_\alpha \veebar B_\beta$. Since ν belongs to $\operatorname{Ker} \iota$ and $\partial M^{4k+2} \neq \phi$, M^{4k+2} is parallelizable. It is easy to prove that ∂M^{4k+2} is a homotopy sphere. According to Lemma 8.3 of Kervaire and Milnor [14], Arf invariant of M is equal to 1. This completes the proof of Lemma 5.4.1.

Now we prove Theorem 2.5. Let W^{4p+2} denote a parallelizable manifold with $\partial W = F^{4p+1}$. Let $W_0 = W - \operatorname{Int} D^{4p+2}$. Regarding W_0 as a parallelizable cobordism between F^{4p+1} and the natural sphere S^{4p+1}, we can construct a normal map

$$G\colon (W_0;\, F^{4p+1} \cup S^{4p+1}) \longrightarrow (S^{4p+1} \times I;\, S^{4p+1} \times 0 \cup S^{4p+1} \times 1)$$

with $G \mid S^{4p+1} =$ identity. Multiplying by $\boldsymbol{CP}^{2q}$ we get a normal map $G \times 1\colon (W_0;\, F \cup S^{4p+1}) \times \boldsymbol{CP}^{2q} \to (S^{4p+1} \times I;\, S^{4p+1} \times 0 \cup S^{4p+1} \times 1) \times \boldsymbol{CP}^{2q}$ with $G \times 1 \mid S^{4p+1} \times \boldsymbol{CP}^{2q} =$ identity. Then the invariant $\sigma(G \times 1)$ of Theorem 2.6 of Browder [5] is defined. Since the index of $\boldsymbol{CP}^{2q}$ is equal to one, $\sigma(G \times 1)$ is equal to $\sigma(G)$ by Sullivan's product formula (see Rourke [21]). By the definition $\sigma(G)$ is nothing but Arf W. If $4p+2+4q \neq 14$, we can find a parallelizable $(4p+2+4q)$-manifold U such that Arf $U =$ Arf W and ∂U is a homotopy sphere by Lemma 5.4.1. It follows as in the proof of Novikov's Classification Theorem [20] that $F^{4p+1} \times \boldsymbol{CP}^{2q}$ is diffeomorphic to $(S^{4q+1} \times \boldsymbol{CP}^{2q}) \,\#\, \partial U$. Hence $\Sigma \,\#\, (-\partial U)$ belongs to the inertia group $I(S^{4p+1} \times \boldsymbol{CP}^{2q})$ by Theorem 2.1. When $4p+1+4q = 13$ or 29, $\operatorname{Ker}(G \times 1)_*$ can be killed by surgeries (see Theorem 2.10 of Browder [5] and [6]), hence $F^{4p+1} \times \boldsymbol{CP}^{2q}$ is diffeomorphic to $S^{4p+1} \times \boldsymbol{CP}^2$. Therefore the homotopy sphere $\Sigma^{4p+1+4q}$ belongs to the inertia group $I(S^{4p+1} \times \boldsymbol{CP}^{2q})$. This completes the proof of Theorem 2.5.

6. Proof of Corollary 2.6

If a homotopy sphere Σ^{p+2q} admits a semi-free S^1-action with $F^p \in \Theta_p(\partial\pi)$ as untwisted fixed point set for q: odd, then

$$\Sigma^{p+2q} \in I(S^p \times \boldsymbol{CP}^q)$$

by Theorem 2.3. Since the second Stiefel-Whitney class $W_2(S^p\times \boldsymbol{CP}^q)$ is zero for q: odd, $S^p\times \boldsymbol{CP}^q$ is a spin-manifold (see Lemma 1 of Milnor [17]). Clearly $\pi_1(S^p\times \boldsymbol{CP}^q)\cong\{1\}$ for $p\neq 1$. It follows from Lemma 9.1 of Kawakubo [10] that the homotopy sphere Σ^{p+2q} bounds a spin-manifold. This completes the proof of Corollary 2.6.

7. Proofs of Propositions

7.1. Proof of Proposition 2.9

Let us recall the explicit description of homotopy spheres in $\Theta_{4p-1+4q}(\partial\pi)$ given by Brieskorn and Hirzebruch [4], [8]:

$$\Sigma^{4p-1+q}_{3,6k-1}=\Big\{(z_1,\cdots,z_{2p+2q+1})\in C^{2p+2q+1}\,\Big|\, z_1^3+z_2^{6k-1}+z_3^2+\cdots$$
$$\cdots+z^2_{2p+2q+1}=0,\ |z_1|^2+\cdots+|z_{2p+2q+1}|^2=1\Big\}=k\,\Sigma_M^{4p-1+4q}\,.$$

Let $k\,\Sigma_M^{4p-1}\subset k\,\Sigma_M^{4p-1+4q}$ be the imbedding defined by

$$(z_1,\cdots,z_{2p+1})\mapsto(z_1,\cdots,z_{2p+1},0\cdots 0)\,.$$

Consider the action of S^1 on the last $2q$ variables of $\Sigma^{4p-1+4q}_{3,6k-1}$ defined as follows. Let A: $S^1\to SO(2)$ be the representation defined by

$$A(e^{i\theta})=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

and let φ: $S^1\to SO(2q)$ be the representation defined by

$$\varphi(e^{i\theta})=\begin{pmatrix}A(e^{i\theta}) & & 0\\ & A(e^{i\theta}) & \\ & \ddots & \\ 0 & & A(e^{i\theta})\end{pmatrix}.$$

Then S^1 acts on the last $2q$ variables of $\Sigma^{4p-1+4q}_{3,6k-1}$ by means of the representation φ. It is obvious that this action is semi-free and the fixed point set is $\Sigma^{4p-1}_{3,6k-1}$. This completes the proof of Proposition 2.9.

7.2 Proof of Proposition 2.10

Let us reall the explicit description of homotopy spheres in $\Theta_{4p+1+4q}(\partial\pi)$ given by Brieskorn [4];

$$\Sigma_M^{4p+1+4q}=\Big\{(z_1,\cdots,z_{2p+2q+2})\in C^{2p+2q+2}\,\Big|\, z_1^3+z_2^2+\cdots+z^2_{2p+2q+2}=0,$$
$$|z_1|^2+\cdots+|z_{2p+2q+2}|^2=1\Big\}$$

Let $\Sigma_M^{4p+1}\subset\Sigma_M^{4p+1+4q}$ be the imbedding defined by

$$(z_1, \cdots, z_{2p+2}) \mapsto (z_1, \cdots, z_{2p+2}, 0 \cdots 0).$$

Let $\varphi: S^1 \to SO(2q)$ be the representation defined in the proof of Proposition 2.9. Then S^1 acts on the last $2q$ variables of Σ^{p+1+4q} by means of the representation φ. It is obvious that this action is semi-free and the fixed point set is Σ_M^{4p+1}. On the other hand there always exists the natural semi-free S^1-action on $S^{4p+1+4q}$ with S^{4p+1} as fixed point set. This completes the proof of Proposition 2.10.

8. A concluding remark

Concerning semi-free S^3-actions, it is shown in F. Uchida [23] that the normal bundle of the fixed point set becomes the quaternionic vector bundle. Hence similar results are obtained about semi-free S^3-actions.

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Bibliography

[1] D.W. Anderson, E.H. Brown and F.P. Peterson: *The structure of the spin cobordism ring*, Ann. of Math. **86** (1967), 271–298.

[2] G.E. Bredon: *A π^*-module structure for Θ_* and application to transformation groups*, Ann. of Math. **86** (1967), 434–448.

[3] G.E. Bredon: *Exotic actions on spheres*, Proc. of the Conference on Transformation Groups, Springer-Verlag, New York, 1968, 47–76.

[4] E. Brieskorn: *Beispiele zur Differentialtopologie von Singularitäten*, Invent. Math. **2** (1966/67), 1–14.

[5] W. Browder: *Surgery and the theory of differentiable transformation groups*, Proc. of the Conference on Transformation Groups, Springer-Verlag, New York, 1968, 1–46.

[6] W. Browder: *The Kervaire invariant of framed manifolds and its generalization*, Ann. of Math. **90** (1969), 157–186.

[7] P.E. Conner and E.E. Floyd: Differentiable Periodic Maps, Berlin-Göttingen-Heidelberg-New York, Springer 1964.

[8] F. Hirzebruch: *Singularities and exotic spheres*, Séminaire Bourbaki, 1966/67, No. 314.

[9] W.Y. Hsiang: *On the unknottedness of the fixed point set of differentiable circle group actions on spheres-P.A. Smith conjecture*, Bull. Amer. Math. Soc. **70** (1964), 678–680.

[10] K. Kawakubo: *Smooth structures on $S^p \times S^p$*, Osaka J. Math. **6** (1969), 165–196.

[11] K. Kawakubo: *Free and semi-free differentiable actions on homotopy spheres*, Proc. Japan Acad. **45** (1969) 651–655.

[12] M. Kervaire: *Some non-stable homotopy groups of Lie groups*, Illinois J. Math. **4** (1960), 161–169.

[13] M. Kervaire: *Le théorèm de Barden-Mazur-Stallings*, Comment. Math. Helv. **40** (1966), 31–42.

[14] M.A. Kervaire and J. Milnor: *Groups of homotopy spheres*, I, Ann. of Math. **77** (1963), 504–537.

[15] H. Maehara: *On the differentiable involution on homotopy spheres*, (to appear).

[16] J. W. Milnor: Differentiable manifolds which are homotopy spheres, mimeographed notes, Princeton, 1958.

[17] J.W. Milnor: *Remarks concerning spin manifolds*, Differential and Combinatorial Topology, Princeton University Press, 1965, 55–62.

[18] D. Montgomery and C.T. Yang: *Differentiable transformation groups on homotopy spheres*, Michigan Math. J. **14** (1967), 33–46.

[19] P.S. Mostert (ed.): Proceedings of the Conference on Transformation Groups, Springer-Verlag, New York, 1968.

[20] S.P. Novikov: *Homotopically equivalent smooth manifolds* I, Izv. Akad. Nauk. SSSR, Ser. Math. **28** (1964), 365–474.
English transl., Amer. Math. Soc. Transl. (2) **48** (1965), 271–396.

[21] C.P. Rourke: *On the Kervaire obstruction*, (to appear).

[22] S. Smale: *Generalized Poincaré conjecture in dimension greater than four*, Ann. of Math. **74** (1961), 381–406.

[23] F. Uchida: *Cobordism groups of semi-free S^1- and S^3-actions*, (to appear).

[24] J.H.C. Whitehead: *Combinatorial homotopy* I, Bull. Amer. Math. Soc. **55** (1949), 213–245.