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CLASSIFYING BOUNDED 2-MANIFOLDS IN S^4

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Noguchi has shown that if M_1 and M_2 are closed orientable 2–manifolds in S^4 having the same local oriented knot types, then they are *isoneighboring*, that is, for regular neighborhoods N_1 and N_2 in S^4 of M_1 and M_2, respectively, there is a homeomorphism of N_1 onto N_2 carrying M_1 onto M_2 [5]. In a later paper, Noguchi showed that one may replace S^4 by an orientable 4–manifold, if one adds the restriction that M_1 and M_2 have the same Stiefel-Whitney numbers [6]. In this paper we show that if M_1 definitely has nonempty boundary (in each of its components), one may drop the orientability requirement (the Stiefel-Whitney numbers are, of course, zero), and obtain the much stronger conclusion that one may ambient isotope M_2 onto M_1. The starting point for our proof is the case $N=S^4$ and M a 2–cell, proved by Gugenheim in 1953 [2]. We work throughout in the piecewise linear (*PL*) category, and assume the reader familiar with the elements of *PL* topology. We will also use results of Hudson and Zeeman: the isotopy extension theorem [3], and the theory of relative regular neighborhoods [4], [1].

A *PL* imbedding f of a *PL* manifold M into the interior of another *PL* manifold N is *locally knotted* at $x\in M$ if there is a triangulation J, L of N, $f(M)$ having $f(x)$ as a vertex and such that the ball or sphere $f(lk(x), L)$ is knotted in the sphere $lk(f(x), J)$. The pair $(lk(f(x), J), f(lk(x), L))$ is called the *local knot type* of f at x, and denoted by $\Sigma_f(x)$. If M and N are both orientable, we fix orientations for both, and in this case local knot type is to be understood to mean *oriented* local knot type. It is apparent that an imbedding of a compact 2–manifold into the interior of a 4–manifold can be locally knotted at only finitely many points, all of which are interior points. We say that imbeddings $f_1, f_2: M^2 \to N^4$ are *locally equivalent* if we may list the local knotting points $x_1, x_2, \cdots, x_n$ of f_1 and $y_1, \cdots, y_n$ of f_2 in such a way that the local knot type of f_1 at x_i is the same as that of f_2 at y_i, $i=1, 2, \cdots, n$. Concealed in Gugenheim's second 1953 paper is the following:

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Theorem 1. *If D_1 and D_2 are locally equivalently imbedded 2–cells in S^4, there is a homeomorphism $h: S^4 \to S^4$ such that $h(D_2)=D_1$.*

Corollary. *If $f_1, f_2: D^2 \to int\, N^4$ are locally equivalent imbeddings of a 2–cell into a 4–manifold, there is a homeomorphism $h: N^4 \to N^4$ such that $h(f_2(D))=f_1(D)$. Moreover h is isotopic to the identity rel ∂N (that is, the isotopy restricts to 1 on ∂N).*

Proof. Let Q_i be a regular neighborhood of $f_i(D)$ in int $N(i=1, 2)$. Then Q_1 and Q_2 are 4–balls in the interior of N, so by Newman's Theorem, we may ambient isotop (rel ∂N) Q_2 onto Q_1; thus we may (and do) assume $Q_1=Q_2=Q$. Now we have $f_1(D)$ and $f_2(D)$ both lying inside a 4–ball Q, and by Theorem 1, there will be a homeomorphism $h': Q \to Q$ such that $h'|\partial Q=1$ and $h'f_2(D)=f_1(D)$. Extend h' to $\bar{h}: N \to N$ by $\bar{h}|N-Q=1$. Now h' is isotopic to the identity rel ∂Q and we extend this isotopy to all of N by the identity outside of Q, showing that $\bar{h}$ is isotopic (rel ∂N) to the identity.

Our main result is the following:

Theorem 2. *If M_1 and M_2 are homeomorphic 2–manifolds with nonempty boundary, locally equivalently imbedded in the interior of a simply connected 4–manifold N, then there is an ambient isotopy of N rel ∂N carrying M_2 onto M_1.*

Proof. We give the proof in the case where M is connected (M_1 and M_2 homeomorphic to M); in the other cases of course, one must assume that each component of M has nonempty boundary, but the proof is essentially the same. We also note that if one assumes M_1 and M_2 to be homotopic in N, one need make no connectivity assumptions on N (N may, in fact, be non-orientable); however, the extra technical detail involved does not seem worth the gain.

By the classification of two manifolds, a bounded 2–manifold can be written as a 2–cell D with (possibly twisted) handles $H_1, H_2, \cdots, H_m$ attached; let α_i be the indexing arc of H_i, let $\partial\alpha_i=\alpha_i \cap D=\alpha_i \cap \partial D=\{a_{i1}, a_{i2}\}$, and let $H_i \cap D= H_i \cap \partial D=\beta_{i1} \cup \beta_{i2}$, where β_{ij} are disjoint arcs with $a_{ij} \in \text{int}\, \beta_{ij}$ (see fig. 1). Now we may choose homeomorphisms $f_i: M \to M_i$ such that all the local knotting occurs at points of the interior of D. The proof of the theorem is by induction on the number n of handles; for $n=0$, we appeal to the corollary to Theorem 1; at this stage, by redefining the imbeddings (but not altering the images), we may assume $f_1|D=f_2|D$. Thus let us assume that all but a single handle H with indexing arc α have been "unknotted" (i.e., $f_1|M-H=f_2|M-H$). Now let $\bar{N}$ be a regular neighborhood of $f_1(Cl(M-H))$ mod its boundary, and let $N'=Cl(N-\bar{N})$. Thus we have the two different imbeddings of H dangling inside N', attached to its boundary in the same two arcs $f_1(\beta_1)$ and $f_1(\beta_2)$ and $f_1|\beta_i=f_2|\beta_i$. If we can ambient isotop $f_1|H$ to $f_2|H$ in N' rel $\partial N'$, we could extend this isotopy to all of N by setting it equal to the identity on $\bar{N}$ (hence leaving fixed what we had already unknotted), and the proof would be complete. First unknot the indexing arcs; one can do this since $f_1|\alpha$ and $f_2|\alpha$ are homotopic rel the end points (since N is simply connected) and α lies in the trivial range (i.e., $2(1)+2\leq 4$).

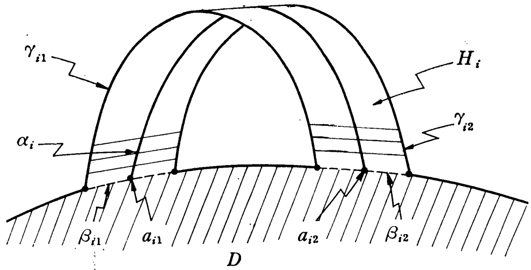

Fig. 1 A twisted handle

We denote by γ_1, γ_2 the complementary arcs to β_1, β_2 in ∂H; i.e., $Cl(\partial H-\beta_1-\beta_2)=\gamma_1\cup\gamma_2$ (see fig. 1). Let us triangulate N' by a complex J having subcomplexes covering $f_1(H)$ and $f_2(H)$, and take the following relative second derived neighborhoods: $Q_i=N(f_i(H)-f_i(\gamma_1\cup\gamma_2), J'')$ $(i=1, 2)$. Now Q_1 and Q_2 are also regular neighborhoods of $f_1(\alpha)$ which intersect $\partial N'$ in the same set (namely a regular neighborhood of $f_1(\beta_1\cup\beta_2)$ mod $f_1(\partial\beta_1\cup\partial\beta_2)$ in $\partial N'$) and hence we may carry Q_2 onto Q_1 by an ambient isotopy rel $\partial N'$; so we may as well assume that $Q_2=Q_1=Q$. Also $(Q, f_1(H))$ and $(Q, f_2(H))$ are locally unknotted proper ball pairs with the big ball collapsing to the smaller one, and hence are unknotted pairs (see [4]). We need to define a homeomorphism of Q onto itself carrying f_2 onto f_1, which is the identity on $Q\cap\partial N$ and which extends into the rest of N' so as to be isotopic rel $\partial N'$ to the identity. This could all be done if we had an isotopy h_t of $P=Cl(\partial Q-\partial N')$ onto itself which started at a homeomorphism h_0 carrying $f_2|(\gamma_1\cup\gamma_2)$ to $f_1|(\gamma_1\cup\gamma_2)$, ended at the identity $(h_1=1)$, and stayed the identity on $\partial P=P\cap\partial N'$ at all times. To see why this would do the trick consider the following: extend h_0 to all of ∂Q by setting it the identity on $Q\cap\partial N'$; then extend this to a homeomorphism of Q onto itself carrying f_2 to f_1 by using lemma 18 of [8]. Now P is collared in $Cl(N'-Q)$, and we use the isotopy h_t to extend the homeomorphism of Q to a homeomorphism of the ball $Q'=Q\cup$collar on P (see fig. 2) which is the identity on $\partial Q'$ and extend outside Q' by the identity. The homeomorphism of Q' is isotopic rel $\partial Q'$ to the identity and the isotopy may be extended to an isotopy of N' rel $\partial N'$ by setting it equal to the identity on $N'-Q'$. Thus the only thing missing is the construction of the isotopy h_t of P rel ∂P.

A moments reflection will show that we have the following situation: two different unknotted 1-spheres $(f_i(\partial H))$ in the three sphere (∂Q) which agree on a pair of subarcs β_1 and β_2; two 3-balls B_1, $B_2(Q\cap\partial N=B_1\cup B_2)$ containing

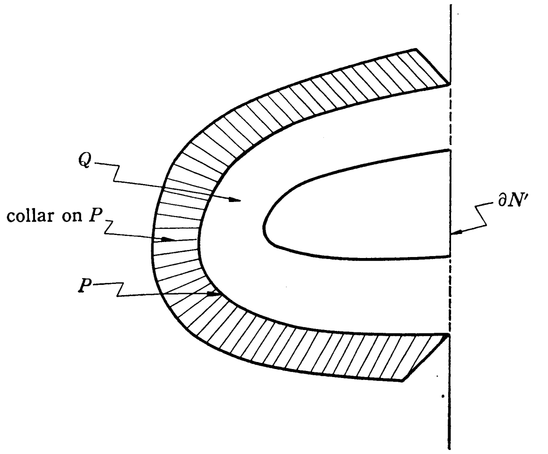

Fig. 2

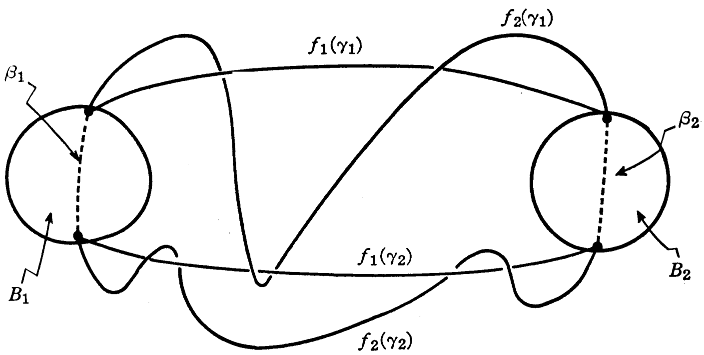

Fig. 3

β_1, β_2 (respectively) such that B_i intersects both 1–spheres in β_i only (see fig. 3). P thus is an annulus, $P=Cl(S^3-B_1-B_2)$. The classical method of showing that any arc is unknotted keeping its end points fixed shows that we may isotop $f_2|\gamma_1$ to $f_1|\gamma_1$ by an isotopy of S^3 ($=\partial Q$) which is the identity at all tines on $B_1\cup B_2$ ($=Q\cap\partial N'$), so this isotopy restricts to an isotopy of P rel ∂P. Thus we may assume that $f_2|\gamma_1=f_1|\gamma_1$. To unknot $f_2|\gamma_2$, proceed as follows: let R be a regular neighborhood of $f_1(\gamma_1)$ missing both $f_1(\gamma_2)$ and $f_2(\gamma_2)$. Then if we let $W=Cl(P-R)$, we see that $(W, f_1(\gamma_2))$ and $(W, f_2(\gamma_2))$ are both 3, 1 ball pair subsets of an unknotted 3, 1 sphere pair, and hence are unknotted. Moreover, they have the same boundary pairs, so we can isotop $f_2|\gamma_2$ to $f_1|\gamma_1$ in W by an ambient isotopy of W rel ∂W, and if we extend this isotopy to all of P by setting it equal to the identity on $R=Cl(P-W)$, we will have ambient isotoped $f_2|\gamma_2$ to $f_1|\gamma_1$ in P rel ∂P without moving $f_1(\gamma_1)$. Hence we have ambient isotoped $f_2|\gamma_1 \cup \gamma_2$ to $f_2|\gamma_1 \cup \gamma_2$ rel ∂P, completing the proof of Theorem 2.

REMARK. The techniques used in this paper have been used by the author to prove the following: homotopic imbeddings of a manifold M^m in the interior of another manifold N^n are ambient isotopic (rel ∂N) if M has a spine of dimension $p<n-m$ and $n-m\geq 3$ [7]. Also, there is no new difficulty to extending the results of the present paper to 1-flat imbeddings of balls with (index 1) handles in codimension 2.

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