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ELLIPTIC COMPLEXES ON CERTAIN HOMOGENEOUS SPACES

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Introduction

The main purpose of this paper is to construct elliptic complexes on symmetric spaces of inner type, which are very analogous to Dolbeault complexes. When a symmetric space of the above type has a homogeneous vector bundle satisfying a certain condition, we can canonically associate to it an elliptic complex with length of half a dimension of the space, whose first term coincides with the given vector bundle (Theorem 3.1). In particular, if the symmetric space has an invariant complex structure, one can see that the elliptic complex associated in such a way is no other than the Dolbeault complex for the given holomorphic vector bundle.

In more detail, let $X=G/K$ be a symmetric space of inner type, i.e., (G, K) is a symmetric pair and rank $G=$rank K. To an irreducible K-module V, there is the homogeneous vector bundle $\mathcal{V}$ over X associated. When we denote by $\mathcal{P}$ the complex cotangent bundle over X, we define an invariant first order differential operator

$$D^0\colon C^\infty(\mathcal{V}) \to C^\infty(\mathcal{V}\otimes\mathcal{P})$$

as the covariant differentiation induced from the invariant connection determined by the Cartan decomposition of the Lie algebra of G with respect to (G, K). Here $C^\infty(\cdot)$ denotes the space of infinitely differentiable sections of a vector bundle. It is known that this operator extends uniquely to a differential operator

$$D^q\colon C^\infty(\mathcal{V}\otimes\wedge^q\mathcal{P}) \to C^\infty(\mathcal{V}\otimes\wedge^{q+1}\mathcal{P}),$$

such that $D^q(s\varphi)=D^0s\wedge\varphi+sd\varphi$ for $s\in C^\infty(\mathcal{V})$, $\varphi\in C^\infty(\wedge^q\mathcal{P})$. For a lexicographical order of the root system of the complexification of the Lie algebra of G, we choose a homogeneous vector bundle $\mathcal{U}^q\subset\mathcal{V}\otimes\wedge^q\mathcal{P}$ for every $q\geq 1$, satisfying the following property. First it holds $D^q(C^\infty(\mathcal{U}^q))\subset C^\infty(\mathcal{U}^{q+1})$;

putting $\mathcal{V}^q=\mathcal{V}\otimes\wedge^q\mathcal{P}/\mathcal{U}^q$, $\mathcal{V}^0=\mathcal{V}$, we then obtain invariant first order differential operators

$$\mathcal{D}^q\colon C^\infty(\mathcal{V}^q)\to C^\infty(\mathcal{V}^{q+1})$$

by the quotient of D^q for every $q\geqslant 0$.
Then the resulting sequence

$$0\to C^\infty(\mathcal{V})\xrightarrow{\mathcal{D}^0}C^\infty(\mathcal{V}^1)\xrightarrow{\mathcal{D}^1}\cdots\xrightarrow{\mathcal{D}^{n-1}}C^\infty(\mathcal{V}^n)\to 0$$

is an elliptic complex over X under a certain condition on the K-module V (Definition 1, §2), when a lexicographical order of the root system is chosen appropriately ("admissible" in Definition 3, §3). Here the length of the complex is $n=\dfrac{1}{2}\dim_{R}X$.

The arguments in §2 are devoted to the preparations for determining $\mathcal{U}^q$ where the generalized Borel-Weil theorem proved by Bott plays a fundamental role, and in §3 we construct the above elliptic complexes (Theorem 3.1). Even if X is not symmetric or a lexicographical order of the root system is not admissible, we obtain an elliptic complex consisting of two homogeneous vector bundles determined by an irreducible K-module, which is similar to the elliptic complex over a symmetric space described as above. Here we consider such a homogeneous space $X=G/K$ that G is a connected real semi-simple Lie group, K a connected compact subgroup and rank $G=$rank K. In fact, the indices of these elliptic complexes are calculated in quite the same way (§§4, 5). Therefore *we do not assume that the homogeneous space X is symmetric till* §5 *except in* §3.

In §4, the homogeneous indices of the elliptic complexes constructed in §3 are calculated in case G is compact by means of Bott's theorem in [5] (Theorem 4.2). This result is very analogous to the Borel-Weil-Bott theorem for a compact kählerian homogeneous space. In fact, the homogeneous index of our elliptic complex associated to an irreducible K-module is equal to the element determined by *one* irreducible G-module in the character ring of G. In §5, we shall first generalize Hirzebruch's proportionality principle to the case of general elliptic complexes on a compact locally homogeneous space (Theorem 5.2) and then calculate the indices of the elliptic complexes constructed in §3 over such a space (Theorem 5.3). We use the index theorem of Atiyah-Singer [1] for this purpose. If the theory of automorphic functions on a real symmetric space of inner type would be meaningful, the results in §5 might give some indications to it.

Now let $X=G/K$ be a symmetric space of inner type again, V_λ the irreducible K-module with lowest weight $\lambda+2\rho_{\mathfrak{k}}$ where $\rho_{\mathfrak{k}}$ is half a sum of positive compact roots under some order of the root system. W. Schmid considered in [16], [17] an invariant first order differential operator

$$\mathcal{D}: C^\infty(\mathcal{V}_\lambda) \to C^\infty(\mathcal{V}_\lambda^1),$$

where $\mathcal{V}_\lambda$ is the associated homogeneous vector bundle to V_λ, $\mathcal{V}_\lambda^1$ determined by $\mathcal{V}_\lambda$ through the above order of the root system. He proved the ellipticity of this operator $\mathcal{D}$ under certain regularity conditions, which we shall show in §§2, 3 under the weaker condition than his. The differential operator $\mathcal{D}$ is equal to the operator $\mathcal{D}^0$ of the first term in our elliptic complex when the order is admissible. As a matter of fact, this work is stimulated by his in this sense. The ellipticity of $\mathcal{D}$ under the weaker condition improves most of his results. We shall illustrate it in §6. Especially, the following result seems to be somewhat striking. Introducing an invariant volume element on X, one can consider square-integrable sections of $\mathcal{V}_\lambda$. We denote by $\mathfrak{H}_\lambda$ the linear space consisting of the square-integrable sections of $\mathcal{V}_\lambda$ annihilated by $\mathcal{D}$. When $\mathcal{D}$ is elliptic, $\mathfrak{H}_\lambda$ has a structure of Hilbert space, which gives a unitary representation of G. When the group G is assumed to be a type of the generalized Lorentz groups with a compact Cartan subgroup, it follows from Schmid's argument in [16] and our estimate for the ellipticity of $\mathcal{D}$ that all the unitary representations of the discrete series of G are exhibited by these $\mathfrak{H}_\lambda$ where λ runs over an appropriate domain (Theorem 6.3 and its Corollary).

In concluding the introduction, it is a pleasant duty for the author to express his gratitude to Professors S. Murakami and M. Takeuchi for helpful discussions with them. Especially, an arrangement of §2 and the proof of Lemma 3.2 are indebted to Prof. M. Takeuchi. The author also thanks to Prof. M. Ise who communicated the reference related to §5, and to Prof. W. Schmid who kindly pointed out an oversight in §3 in the first manuscript.

1. Preliminaries and notation

Let $\mathfrak{g}_0$ be a real semi-simple Lie algebra, $\mathfrak{k}_0$ a compactly imbedded subalgebra of $\mathfrak{g}_0$, i.e., $\mathfrak{k}_0$ generates a compact subgroup in the adjoint group of $\mathfrak{g}_0$. We assume, throughout this paper, that rank $\mathfrak{g}_0$=rank $\mathfrak{k}_0$ and keep fixed a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. Denote by B the killing form of $\mathfrak{g}_0$. Then the restriction of B to the subalgebra $\mathfrak{k}_0$ is a negative-definite invariant bilinear form on $\mathfrak{k}_0$ (Helgason [9]). Therefore we have a direct sum decomposition

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$
$$[\mathfrak{k}_0, \mathfrak{p}_0] \subset \mathfrak{p}_0,$$

where $\mathfrak{p}_0$ is the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to B. We notice that the restriction of B to $\mathfrak{p}_0$ is non-degenerate and $\mathfrak{k}_0$-invariant, hence we shall often identify $\mathfrak{p}_0$ with its dual space $\mathfrak{p}_0^*$ by B.

Denote the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0$ and $\mathfrak{p}_0$ by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$ and $\mathfrak{p}$ respectively. Through the adjoint action of $\mathfrak{h}$, we have a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^{\alpha},$$

where Δ denotes the root system of $\mathfrak{g}$ for $\mathfrak{h}$, $\mathfrak{g}^{\alpha}$ the one-dimensional eigenspace for a root $\alpha \in \Delta$. Since the Lie algebra $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{k}$ and the subspace $\mathfrak{p}$ is stable under the action of $\mathfrak{h}$, the root vector space $\mathfrak{g}^{\alpha}$ is contained either in $\mathfrak{k}$ or in $\mathfrak{p}$. Hence we have

$$\mathfrak{k} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_{\mathfrak{k}}} \mathfrak{g}^{\alpha}$$
$$\mathfrak{p} = \sum_{\beta \in \Delta_{\mathfrak{p}}} \mathfrak{g}^{\beta}$$

and a disjoint union

$$\Delta = \Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}},$$

where we put

$$\Delta_{\mathfrak{k}} = \{\alpha \in \Delta \mid \mathfrak{g}^{\alpha} \subset \mathfrak{k}\}$$
$$\Delta_{\mathfrak{p}} = \{\beta \in \Delta \mid \mathfrak{g}^{\beta} \subset \mathfrak{p}\}.$$

We now consider a compact connected Lie group K whose Lie algebra is $\mathfrak{k}_0$. Let H be the Cartan subgroup of K corresponding to the Cartan subalgebra $\mathfrak{h}_0$, Λ the character group of the toral group H. Then Λ is identified with a lattice in $\mathfrak{h}_R^*$, the dual space of $\mathfrak{h}_R = \sqrt{-1}\,\mathfrak{h}_0$. One can regard the root system Δ as a subset in Λ. If we denote by σ the conjugation of $\mathfrak{g}$ with respect to a real form $\mathfrak{g}_0$, then for every root vector space $\mathfrak{g}^{\alpha}$ we have

$$\sigma\mathfrak{g}^{\alpha} = \mathfrak{g}^{-\alpha},$$

because α has imaginary values on the real part $\mathfrak{h}_0$. We fix, once and for all, a lexicographical order on the vector space $\mathfrak{h}_R^*$ and denote by Δ^+, $\Delta_{\mathfrak{k}}^+$, $\Delta_{\mathfrak{p}}^+$ the subsets consisting of the positive roots of Δ, $\Delta_{\mathfrak{k}}$, $\Delta_{\mathfrak{p}}$ with respect to this order. As usual, when we refer to the *highest* or the *lowest* weight of a $\mathfrak{g}$ (or $\mathfrak{k}$)-module, it means to be the highest or the lowest element with respect to the above order on $\mathfrak{h}_R^*$ among the set of the weights of the representation. The Killing form B induces a positive-definite inner product $(\ ,\ )$ on $\mathfrak{h}_R^*$ and we shall call $\lambda \in \Lambda$ *singular* if $(\lambda, \alpha) = 0$ for some $\alpha \in \Delta$ (resp. $\Delta_{\mathfrak{k}}$), *regular* if $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Delta$ (resp. $\Delta_{\mathfrak{k}}$), *dominant* if $(\lambda, \alpha) \geqslant 0$ for all $\alpha \in \Delta^+$ (resp. $\Delta_{\mathfrak{k}}^+$), with respect to $\mathfrak{g}$ (resp. $\mathfrak{k}$). It is well known that $\lambda \in \Lambda$ is the highest weight of some representation if and only if λ is dominant. We choose Borel subalgebras $\mathfrak{b}$, $\mathfrak{b}'$ of the complex Lie algebras $\mathfrak{g}$, $\mathfrak{k}$ such as

$$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}^{-\alpha}$$
$$\mathfrak{b}' = \mathfrak{h} \oplus \sum_{\alpha \in \Delta_{\mathfrak{k}}^+} \mathfrak{g}^{-\alpha} .$$

We shall now recall Bott's generalized Borel-Weil theorem for compact Lie groups. Let K^C be the complexification of the compact Lie group K, B' the Borel subgroup of K^C whose Lie algebra is $\mathfrak{b}'$. Then the homogeneous space $S=K/H$ is diffeomorphic to the complex manifold K^C/B'. We shall hereafter fix the choice of a complex structure on S with this identification. Further note that $\dim_C S$ is equal to the number of elements of the set $\Delta_{\mathfrak{k}}^+$, and put $s=\dim_C S$. For a character $\lambda \in \Lambda$ of H, λ can be extended uniquely to the holomorphic character of B', which is denoted by the same letter λ. If we denote by L_λ a homogeneous line boundle over $S=K^C/B'$ associated to λ, then L_λ is a holomorphic line bundle on which K^C acts as an automorphism group. Hence the cohomology group $H^*(S, L_\lambda)$, with coefficients in the sheaf of germs of holomorphic sections of L_λ, has a structure of a K^C-module. We denote by W_K the Weyl group of K, by $n(w)$ the index of $w \in W_K$, i.e., the number of elements of the set $\{\alpha \in \Delta_{\mathfrak{k}}^+ \mid w\alpha < 0\}$, and put

$$\rho_{\mathfrak{k}} = \frac{1}{2} \sum_{\alpha \in \Delta_{\mathfrak{k}}^+} \alpha .$$

We then know the following Borel-Weil theorem proved by Bott.

Theorem 1.1 (Bott [4]). *For a character $\lambda \in \Lambda$, let L_λ be the associate homogeneous line bundle over S. As for the K^C-module $H^*(S, L_\lambda)$, the following holds. If $\lambda+\rho_{\mathfrak{k}}$ is singular, then*

$$H^q(S, L_\lambda) = 0 \quad \textit{for every } q .$$

If $\lambda+\rho_{\mathfrak{k}}$ is regular, then there exists the unique element $w \in W_K$ such that $w(\lambda+\rho_{\mathfrak{k}})$ is dominant, and we have

$$H^q(S, L_\lambda) = 0 \quad \textit{for every } q \neq n(w) .$$

For $q=n(w)$, $H^{n(w)}(S, L_\lambda)$ is an irreducible K^C-module with highest weight $w(\lambda+\rho_{\mathfrak{k}})-\rho_{\mathfrak{k}}$. Here the words "singular, regular and dominant" mean so with respect to $\mathfrak{k}$.

REMARK. This theorem gives an explicit construction of all irreducible K^C-modules. For example, the irreducible K^C-module with highest weight λ is given by $H^0(S, L_\lambda)$, and the one with lowest weight μ is given by $H^s(S, L_{\mu-2\rho_{\mathfrak{k}}})$ where $s=\dim_C S$.

2. Exact sequences of homogeneous vector bundles and their cohomology groups

In this section, we shall treat certain sequences of homogeneous vector bundles over the flag manifold $S=K^C/B'$ and their cohomology groups. These arguments prepare a decomposition law and exact sequences of K^C (or K)-modules, which will be indispensable for the construction of elliptic complexes. We retain the situation and notation introduced in §1.

Let $\mathfrak{p}$ be the K^C-module defined in §1. We denote by P the homogeneous vector bundle over S associated to the B'-module $\mathfrak{p}$ induced by the restriction of K^C to B'. Then P is a holomorphically trivial bundle. In fact, the homogeneous vector bundle P is, by definition, the quotient space of $K^C\times\mathfrak{p}$ by the equivalence relation:

$$(kb, X)\sim(k, bX) \quad \text{for} \quad k\in K^C, \quad b\in B', \quad X\in\mathfrak{p},$$

where $X\mapsto bX$ is the action of $b\in B'$ on $X\in\mathfrak{p}$ with respect to the B'-module structure of $\mathfrak{p}$. We shall denote by $K^C\times_{B'}\mathfrak{p}$ this expression of P as a quotient space. Then the map $kB'\mapsto(k, k^{-1}X)$ of S to $K^C\times_{B'}\mathfrak{p}$ is well-defined for a fixed $X\in\mathfrak{p}$ and is a holomorphic section of P' which vanishes nowhere if $X\neq 0$. Denote by $s_X\in H^0(S, P)$ this holomorphic section determined by $X\in\mathfrak{p}$. Then the map $X\mapsto s_X$ gives a K^C-module isomorphism

$$\mathfrak{p}\cong H^0(S, P)$$

and a trivialization of P as a holomorphic vector bundle. When we put $\mathfrak{p}_-=\mathfrak{p}\cap\mathfrak{b}$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$ defined in §1, $\mathfrak{p}_-$ is a B'-submodule of $\mathfrak{p}$. Define $\mathfrak{p}_+=\mathfrak{p}/\mathfrak{p}_-$ as the quotient B'-module. Denoting by n the number of elements of the set $\Delta_{\mathfrak{p}}^+$, we then see

$$\dim_C\mathfrak{p}_- = \dim_C\mathfrak{p}_+ = n\,.$$

We denote by π^q a canonical projection $\wedge^q\mathfrak{p}\to\wedge^q\mathfrak{p}_+$ of the q-th exterior product of $\mathfrak{p}$ onto that of $\mathfrak{p}_+$, by $\mathfrak{n}^q$ the kernel of π^q ($\mathfrak{n}^0=\{0\}$, $\mathfrak{n}^1=\mathfrak{p}_-$). We then have an exact sequence of B'-modules,

$$0\to\mathfrak{n}^q\to\wedge^q\mathfrak{p}\xrightarrow{\pi^q}\wedge^q\mathfrak{p}_+\to 0\,.$$

Hence, denoting by P_+, N^q the homogeneous vector bundles over S associated to $\mathfrak{p}_+$, $\mathfrak{n}^q$, we have an exact sequence as K^C-homogeneous vector bundles over S,

$$0\to N^q\to\wedge^q P\to\wedge^q P_+\to 0\,.$$

Next we define a vector bundle map

$$\alpha_\xi^q: \wedge^q P \to \wedge^{q+1} P \quad \text{for } \xi \in \mathfrak{p}$$

as follows. We express $\wedge^q P$, $\wedge^{q+1} P$ as the quotient spaces $K^C \times_{B'} \wedge^q \mathfrak{p}$, $K^C \times_{B'} \wedge^{q+1} \mathfrak{p}$ respectively. Consider the map

$$\alpha_\xi^q: K^C \times \wedge^q \mathfrak{p} \to K^C \times \wedge^{q+1} \mathfrak{p} \quad \text{for } \xi \in \mathfrak{p}$$

defined by $\alpha_\xi^q((k, X)) = (k, \varepsilon(k^{-1}\xi)X)$, where $\varepsilon(k^{-1}\xi)$ is the exterior multiplication on $\wedge \mathfrak{p}$ by $k^{-1}\xi \in \mathfrak{p}$. Then α_ξ^q clearly determines a well-defined map α_ξ^q: $K^C \times_{B'} \wedge^q \mathfrak{p} \to K^C \times_{B'} \wedge^{q+1} \mathfrak{p}$, and consequently a vector bundle map $\alpha_\xi^q: \wedge^q P \to \wedge^{q+1} P$. Then we have

Lemma 2.1. *If $\xi \neq 0$ in $\mathfrak{p}$, then the sequence*

$$0 \to \mathbf{1} \xrightarrow{\alpha_\xi^0} P \xrightarrow{\alpha_\xi^1} \wedge^2 P \to \cdots \xrightarrow{\alpha_\xi^{2n-1}} \wedge^{2n} P \to 0$$

is exact, where $\mathbf{1}$ *denotes the trivial line bundle with fibre* $\boldsymbol{C}$.

The lemma is clear from the property of exterior multiplications and the fact that $\wedge^q P$ is holomorphically trivial.

We note that $\alpha_\xi^q(N^q) \subset N^{q+1}$ for every $\xi \in \mathfrak{p}$, where N^q, N^{q+1} are regarded as the subbundles in $\wedge^q P$, $\wedge^{q+1} P$. In fact, the canonical projection $\pi^{q+1}: \wedge^{q+1} P \to \wedge^{q+1} P_+$ is given by $\pi^{q+1}((k, X)) = (k, \pi^{q+1} X) \in K^C \times_{B'} \wedge^{q+1} \mathfrak{p}_+$ for $(k, X) \in K^C \times_{B'} \wedge^{q+1} \mathfrak{p}$, making use of the projection $\pi^{q+1}: \wedge^{q+1} \mathfrak{p} \to \wedge^{q+1} \mathfrak{p}_+$ defined before. Hence, if $(k, X) \in K^C \times_{B'} \mathfrak{n}^q$, then

$$\pi^{q+1} \circ \alpha_\xi^q((k, X)) = (k, 0)\,,$$

since $\pi^{q+1} \circ \alpha_\xi^q((k, X)) = (k, \varepsilon(\pi^1(k^{-1}\xi))\pi^q X)$ and $\pi^q X = 0$ for $X \in \mathfrak{n}^q$. Thus we have

$$\alpha_\xi^q(N^q) \subset N^{q+1}\,.$$

Here we put $\beta_\xi^q = \alpha_\xi^q \mid N^q$, the restriction of α_ξ^q to the subbundle N^q. We clearly have

$$\beta_\xi^{q+1} \circ \beta_\xi^q = 0\,.$$

Moreover, one can therefore define the vector bundle map

$$\gamma_\xi^q: \wedge^q P_+ \to \wedge^{q+1} P_+$$

as the quotient map of α_ξ^q by β_ξ^q, and we clearly have

$$\gamma_\xi^{q+1} \circ \gamma_\xi^q = 0\,.$$

Thus we obtain the following commutative diagram of the homogeneous vector bundles over S for $\xi\in\mathfrak{p}$.

$$\begin{tikzcd}
 & 0 & 0 & & 0 & & & & \\
0 & \mathbf{1} \arrow[l] \arrow[r] \arrow[u] & P_+ \arrow[r, "\gamma^1_\xi"] \arrow[u] & \cdots \arrow[r, "\gamma^{n-1}_\xi"] & \wedge^n P_+ \arrow[r] \arrow[u] & 0 & & 0 & \\
0 \arrow[r] & \mathbf{1} \arrow[r] \arrow[u] & P \arrow[r, "\alpha^1_\xi"] \arrow[u] & \cdots \arrow[r, "\alpha^{n-1}_\xi"] & \wedge^n P \arrow[r] \arrow[u] & \wedge^{n+1} P \arrow[r] \arrow[u] & \cdots \arrow[r] & \wedge^{2n} P \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[r] \arrow[u] & N^1 \arrow[r, "\beta^1_\xi"] \arrow[u] & \cdots \arrow[r, "\beta^{n-1}_\xi"] & N^n \arrow[r] \arrow[u] & N^{n+1} \arrow[r] \arrow[u] & \cdots \arrow[r] & N^{2n} \arrow[r] \arrow[u] & 0 \\
 & & 0 \arrow[u] & & 0 \arrow[u] & 0 \arrow[u] & & 0 \arrow[u] &
\end{tikzcd} \qquad (\mathrm{I})$$

We already know the exactness of the column short sequences and the row α^*_ξ-sequence if $\xi\neq 0$ in $\mathfrak{p}$ (Lemma 2.1). Further we have

Proposition 2.1. *Identify $\mathfrak{p}_0$ canonically with the real subspace of its complexification $\mathfrak{p}$. If $\xi\neq 0$ in $\mathfrak{p}_0$, then in the above diagram* (I) *the β^*_ξ-sequence and the γ^*_ξ-sequence are both exact.*

Proof. First observe the following fact.

Lemma 2.2. *Assume that the α^*_ξ-sequence is exact. Then the β^*_ξ-sequence is exact if and only if the γ^*_ξ-sequence is exact.*

Proof. For a vector bundle E over S, we denote by E_s the fibre of E over $s\in S$. Fix $s\in S$ and let $\mathfrak{N}_s$, $\mathfrak{P}_s$, $\mathfrak{P}^+_s$ be the complexes consisting of the sequences of vector spaces

$$\begin{aligned}
&0\to N^1_s\to\cdots\to N^{2n}_s\to 0\,,\\
&0\to \boldsymbol{C}\to P_s\to\cdots\to(\wedge^{2n}P)_s\to 0,\\
&0\to \boldsymbol{C}\to (P_+)_s\to\cdots\to(\wedge^{2n}P_+)_s\to 0\,,
\end{aligned}$$

respectively. We then have a short exact sequence

$$0\to\mathfrak{N}_s\to\mathfrak{P}_s\to\mathfrak{P}^+_s\to 0$$

of the three complexes in view of the diagram (I). By the assumption, we see that the cohomology group of the complex $\mathfrak{P}_s$ vanishes. Considering the cohomology exact sequence associated to the above short exact sequence, we see that $H^i(\mathfrak{N}_s)=0$ if and only if $H^{i-1}(\mathfrak{P}^+_s)=0$. This implies Lemma 2.2.

We shall return to the proof of Proposition 2.1. Take $x\in N^{q+1}$ such that $\beta^{q+1}_\xi(x)=0$ for a fixed $\xi\neq 0$ in $\mathfrak{p}_0$. By the exactness of the α^*_ξ-sequence, there exist $X\in\wedge^q\mathfrak{p}$, $k\in K^C$ such that $x=(k,\varepsilon(k^{-1}\xi)X)\in K^C\times_{B'}\mathfrak{n}^{q+1}=N^{q+1}$ where $\varepsilon(k^{-1}\xi)X\in\mathfrak{n}^{q+1}$. It suffices to show that there exists $X_0\in\mathfrak{n}^q$ such that

$$\varepsilon(k^{-1}\xi)X_0=\varepsilon(k^{-1}\xi)X\,.$$

In fact, if we take $(k, X_0)\in K^C\times_{B'}\mathfrak{n}^q=N^q$, then $\beta^q_\xi((k, X_0))=x$, which will imply the exactness of the β^*_ξ-sequence.

In the first place, suppose $k\in K$, the compact form of K^C. We consider the next direct decomposition as an H-module

$$\wedge^q\mathfrak{p} = \mathfrak{n}^q\oplus\mathfrak{q}^q ,$$

where $\mathfrak{q}^q$ is the subspace $\sum_{\beta_1,\cdots,\beta_q\in\Delta^+_{\mathfrak{p}}}\mathfrak{g}^{\beta_1}\wedge\cdots\wedge\mathfrak{g}^{\beta_q}$ of $\wedge^q\mathfrak{p}$ spanned by the q-th exterior products of the root vectors for the roots belonging to $\Delta^+_{\mathfrak{p}}$. We note that $\mathfrak{q}^q$ is isomorphic to $\wedge^q\mathfrak{p}_+$ as H-modules. In particular for $q=1$, we have

$$\mathfrak{p} = \mathfrak{p}_-\oplus\mathfrak{q}^1 ,$$

where $\mathfrak{q}^1=\sum_{\beta\in\Delta^+_{\mathfrak{p}}}\mathfrak{g}^\beta$. If we denote by σ the conjugation of $\mathfrak{p}$ with respect to the real form $\mathfrak{p}_0$, then we have

$$\sigma\mathfrak{p}_- = \mathfrak{q}^1 ,$$

since $\sigma\mathfrak{g}^\beta=\mathfrak{g}^{-\beta}$ as noticed in §1. We decompose $\xi\in\mathfrak{p}$ according to the above decomposition and put $\xi=\xi_-+\xi_+$ where $\xi_-\in\mathfrak{p}_-,\xi_+\in\mathfrak{q}^1$. If $\xi\in\mathfrak{p}_0$, then $\sigma\xi=\xi$; hence we have $\sigma\xi_-=\xi_+$. Thus $\xi\neq0$ for $\xi\in\mathfrak{p}_0$ if and only if $\xi_\pm\neq0$. Since $\mathfrak{p}_0$ is a K-module, we have

$$(k^{-1}\xi)_\pm\neq0 \quad \text{for } k\in K$$

when $\xi\neq0$ in $\mathfrak{p}_0$. The decomposition of $X\in\wedge^q\mathfrak{p}$ is also denoted by

$$X = X_-+X_+$$

where $X_-\in\mathfrak{n}^q$, $X_+\in\mathfrak{q}^q$. Then we have

$$\varepsilon(k^{-1}\xi)X = \varepsilon(k^{-1}\xi)X_-+\varepsilon((k^{-1}\xi)_-)X_++\varepsilon((k^{-1}\xi)_+)X_+ .$$

Suppose $\varepsilon(k^{-1}\xi)X\in\mathfrak{n}^{q+1}$. Then it holds that $\varepsilon((k^{-1}\xi)_+)X_+\in\mathfrak{n}^{q+1}$ since

$$\varepsilon(k^{-1}\xi)X_-+\varepsilon((k^{-1}\xi)_-)X_+\in\mathfrak{n}^{q+1} .$$

On the other hand, we have $\varepsilon((k^{-1}\xi)_+)X_+\in\mathfrak{q}^{q+1}$, which implies $\varepsilon((k^{-1}\xi)_+)X_+=0$. Therefore there exists $Y\in\wedge^{q-1}\mathfrak{p}$ such that $X_+=\varepsilon((k^{-1}\xi)_+)Y$ if $\xi\neq0$ in $\mathfrak{p}_0$. Put $X_0=X_--\varepsilon((k^{-1}\xi)_-)Y\in\mathfrak{n}^q$. We then have $\varepsilon(k^{-1}\xi)X_0=\varepsilon(k^{-1}\xi)X$. Thus the assertion has been shown for $k\in K$.

For $k\in K^C$, we put $k=k'b$ where $k'\in K$ and $b\in B'$. If $\varepsilon(k^{-1}\xi)X\in\mathfrak{n}^{q+1}$, then $\varepsilon(k'^{-1}\xi)bX\in\mathfrak{n}^{q+1}$ since $\mathfrak{n}^{q+1}$ is a B'-module. Hence there exists $Y_0\in\mathfrak{n}^q$ such that $\varepsilon(k'^{-1}\xi)Y_0=\varepsilon(k'^{-1}\xi)bX$. If we put $X_0=b^{-1}Y_0$, then $X_0\in\mathfrak{n}^q$ and $\varepsilon(k^{-1}\xi)X_0=\varepsilon(k^{-1}\xi)X$. This shows the assertion in general. Thus one can see the exactness of the β_ξ^*-sequence for $\xi \neq 0$ in $\mathfrak{p}_0$, which implies Proposition 2.1 in view of Lemma 2.2.

Proposition 2.2. *Let*

$$0 \to E_0 \to \cdots \to E_m \to 0$$

be an exact sequence of holomorphic vector bundles E_q, $q=0, 1, \cdots, m$ over a complex manifold X of dimension s. Then the following holds.

i) *If $H^i(X, E_q)=0$ for any $i>0$ and every q, then the induced sequence*

$$0 \to H^0(X, E_0) \to \cdots \to H^0(X, E_m) \to 0$$

is also exact.

ii) *If $H^i(X, E_q)=0$ for any $i<s$ and every q, then the induced sequence*

$$0 \to H^s(X, E_0) \to \cdots \to H^s(X, E_m) \to 0$$

is also exact.

Proof. The fact i) is rather natural in the view-point of homological algebra. Here we shall show ii). Note that the exactness of the sequence of vector bundles implies by the induction on q that the map $\varphi^q: E_q \to E_{q+1}$ in the given sequence is of constant rank for every q. Therefore one can consider the kernel of φ^q (=the image of φ^{q-1}) as a holomorphic vector subbundle of E_q, which is denoted by $\operatorname{Ker}\varphi^q$. We intend to show that the sequence

$$H^s(X, E_{q-1}) \xrightarrow{\varphi_*^{q-1}} H^s(X, E_q) \xrightarrow{\varphi_*^q} H^s(X, E_{q+1})$$

is exact when $H^{s-1}(X, \operatorname{Ker}\varphi^{q+2})=0$. Consider the following three exact sequences;

$$0 \to \operatorname{Ker}\varphi^{q-1} \xrightarrow{i^{q-1}} E_{q-1} \xrightarrow{j^{q-1}} \operatorname{Ker}\varphi^q \to 0 \qquad (1)$$

$$0 \to \operatorname{Ker}\varphi^q \xrightarrow{i^q} E_q \xrightarrow{j^q} \operatorname{Ker}\varphi^{q+1} \to 0 \qquad (2)$$

$$0 \to \operatorname{Ker}\varphi^{q+1} \xrightarrow{i^{q+1}} E_{q+1} \xrightarrow{j^{q+1}} \operatorname{Ker}\varphi^{q+2} \to 0 \qquad (3).$$

From (1) and (2), we have a commutative diagram

$$\begin{tikzcd} H^s(X, E_{q-1}) \arrow[r, "j_*^{q-1}"] \arrow[dr, "\varphi_*^{q-1}"'] & H^s(X, \operatorname{Ker}\varphi^q) \arrow[d, "i_*^q"] \\ & H^s(X, E_q) \end{tikzcd},$$

where j_*^{q-1} is surjective, which implies that

the image of φ_*^{q-1}=the image of i_*^q.

From (2) and (3), we have a commutative diagram

$$\begin{tikzcd} H^s(X, E_q) \arrow[r, "j_*^q"] \arrow[dr, "\varphi_*^q"'] & H^s(X, \operatorname{Ker}\varphi^{q+1}) \arrow[d, "i_*^{q+1}"] \\ & H^s(X, E_{q+1}) . \end{tikzcd}$$

On the other hand, in view of the cohomology exact sequence induced from (3), i_*^{q+1} is injective if $H^{s-1}(X, \operatorname{Ker}\varphi^{q+2})=0$. This implies

the kernel of φ_*^q=the kernel of j_*^q,

which shows the assertion since

the kernel of j_*^q=the image of i_*^q

by (2). In order to prove the proposition, it remains to deduce the vanishing of the cohomology groups of Ker φ^* from that of E_*. For q sufficiently large, for example $q>m$, we have Ker $\varphi^q=0$. Suppose $H^i(X, \operatorname{Ker}\varphi^{q+1})=0$ for $i<s-1$. Then $H^i(X, \operatorname{Ker}\varphi^q)=0$ for $i<s$ from the cohomology exact sequence related to (3) and the assumption. Therefore, by the induction on q, we have

$$H^i(X, \operatorname{Ker}\varphi^q) = 0 \quad \text{for } i<s \text{ and every } q .$$

This implies Proposition 2.2.

Let $\mathfrak{a}$ be a K^C-module and consider $\mathfrak{a}$ as a B'-module by the restriction. Then we know that the associated homogeneous vector bundle $A=K^C\times_{B'}\mathfrak{a}$ over S is holomorphically trivial and $H^0(S, A)\cong\mathfrak{a}$ as K^C-modules. Suppose that there is given another homogeneous vector bundle E over S. In general, the cup product with respect to coefficients in sheaves

$$H^i(S, E)\otimes H^0(S, A) \to H^i(S, E\otimes A)$$

gives a K^C-module homomorphism. In our case this homomorphism becomes to be a K^C-module isomorphism because of the triviality of A. Thus we have a K^C-module isomorphism

$$H^i(S, E\otimes A) \cong H^i(S, E)\otimes\mathfrak{a} .$$

This isomorphism is unique up to identifications of $H^0(S, A)$ with $\mathfrak{a}$.

REMARK. Fix an identification $\mathfrak{p}\cong H^0(S, P)$ as defined in the beginning of this section. Then the identification $\wedge^q\mathfrak{p}\cong H^0(S, \wedge^q P)$ is canonically induced. Defining α_ξ^q for $\xi\in\mathfrak{p}$ as before, we see easily that for a homogeneous vector bundle E, the diagram

$$\begin{array}{ccc} H^i(S, E\otimes\wedge^q P) & \xrightarrow{(1\otimes\alpha_\xi^q)_*} & H^i(S, E\otimes\wedge^{q+1}P) \\ \wr\Vert & & \wr\Vert \\ H^i(S, E)\otimes\wedge^q\mathfrak{p} & \xrightarrow{1\otimes\varepsilon(\xi)} & H^i(S, E)\otimes\wedge^{q+1}\mathfrak{p} \end{array}$$

is commutative, where the column isomorphisms are as above.

Now we take a character $\lambda\in\Lambda$ of the maximal torus H of K, and consider the homogeneous line bundle L_λ over S associated to $\lambda\in\Lambda$ as in §1. By tensoring of L_λ to the diagram (I), we obtain the following commutative diagram;

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & L_\lambda & \to & L_\lambda & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_\lambda\otimes N^1 & \to & L_\lambda\otimes P & \to & L_\lambda\otimes P_+ & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_\lambda\otimes N^n & \to & L_\lambda\otimes\wedge^n P & \to & L_\lambda\otimes\wedge^n P_+ & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & L_\lambda\otimes N^{n+1} & \to & L_\lambda\otimes\wedge^{n+1} P & \to & 0 & & \\ & & \downarrow & & \downarrow & & & & \\ & & \vdots & & \vdots & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array} \qquad (\mathrm{I}_\lambda)\,.$$

As for the cohomology groups of the vector bundles in (I_λ), we have

Proposition 2.3. *Suppose $\xi\neq0$ in $\mathfrak{p}_0$. Then the following holds.*

i) *If $H^i(S, L_\lambda)=0$ and $H^i(S, L_\lambda\otimes N^q)=0$ for any $i>0$ and every q, then in the commutative diagram*

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & 0 & \to & H^0(S, L_\lambda) & \to & H^0(S, L_\lambda) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^0(S, L_\lambda\otimes N^1) & \to & H^0(S, L_\lambda\otimes P) & \to & H^0(S, L_\lambda\otimes P_+) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^0(S, L_\lambda\otimes N^n) & \to & H^0(S, L_\lambda\otimes\wedge^n P) & \to & H^0(S, L_\lambda\otimes\wedge^n P_+) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & 0 & & \end{array} \qquad (\mathrm{II}_\lambda^0)\,,$$

all the column and row sequences are exact.

ii) *If $H^i(S, L_\lambda\otimes\wedge^q P_+)=0$ for any $i<s$ and every q, then in the commutative diagram*

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
0 & \to & H^s(S, L_\lambda) & \to & H^s(S, L_\lambda) & \to 0 & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to H^s(S, L_\lambda \otimes N^1) & \to & H^s(S, L_\lambda \otimes P) & \to & H^s(S, L_\lambda \otimes P_+) & \to 0 & \\
\downarrow & & \downarrow & & \downarrow & & (\mathrm{II}_\lambda^s)\,, \\
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
0 \to H^s(S, L_\lambda \otimes N^n) & \to & H^s(S, L_\lambda \otimes \wedge^n P) & \to & H^s(S, L_\lambda \otimes \wedge^n P_+) & \to 0 & \\
\downarrow & & \downarrow & & \downarrow & & \\
\vdots & & \vdots & & \vdots & &
\end{array}$$

all the column and row sequences are exact, where $s = \dim_C S$.

Proof. As for i), the assumption implies $H^i(S, L_\lambda \otimes \wedge^q P) \cong H^i(S, L_\lambda \otimes \wedge^q P_+)$ for any $i > 0$ in view of the cohomology exact sequence related to the row short exact sequence of (I_λ). On the other hand, $H^i(S, L_\lambda \otimes \wedge^q P) \cong H^i(S, L_\lambda) \otimes \wedge^q \mathfrak{p}$ since P is trivial. Thus $H^i(S, L_\lambda \otimes \wedge^q P_+) = H^i(S, L_\lambda \otimes \wedge^q P) = 0$ for any $i > 0$ and every q. This implies i) by Proposition 2.2, i). The proof of ii) is quite similar to that of i), using Proposition 2.2, ii).

REMARK. In the diagram (II_λ^0) and (II_λ^s), the row three-term sequences consist of K^C-module homomorphisms, but the column sequences are not so.

Finally, we shall discuss the vanishing conditions in Proposition 2.3. For this purpose, we make use of Schmid's method in [16]. Let us introduce the Grothendieck group $R(K)$ of finite-dimensional K-modules, which is regarded as the free abelian group generated by the set of equivalence classes of irreducible K-modules. Suppose that a homogeneous vector bundle E over S is given. We shall correspond to E the Euler characteristic

$$\chi(E) = \sum_{i=0}^{s} (-1)^i [H^i(S, E)] \in R(K)\,,$$

where $[V]$ denotes the element of $R(K)$ corresponding to a K-module V. If $0 \to E' \to E \to E'' \to 0$ is an exact sequence of homogeneous vector bundles over S as holomorphic K^C-vector bundles, then we have $\chi(E) = \chi(E') + \chi(E'')$ in $R(K)$. Here we shall show the following proposition partly due to W. Schmid.

Proposition 2.4. *Let* $\lambda \in \Lambda$.

i) *Suppose that*

$$(\lambda \pm \beta_1 \pm \cdots \pm \beta_r, \alpha) \geqslant 0 \quad \textit{for all } \alpha \in \Delta_{\mathfrak{k}}^+$$

whenever $\beta_1, \cdots, \beta_r \in \Delta_{\mathfrak{p}}^+$ *are mutually distinct for every* $r \leqslant q$. *Then* $H^i(S, L_\lambda \otimes N^r) = 0$ *and* $H^i(S, L_\lambda \otimes \wedge^r P_+) = 0$ *for any* $i > 0$ *and every* $r \leqslant q$. *Moreover, the irreducible components of the* K*-module* $H^0(S, L_\lambda \otimes \wedge^q P_+)$ *consist of the*

K-modules with highest weights $\lambda+\beta_1+\cdots+\beta_q$ *where* $\beta_1,\cdots,\beta_q\in\Delta_{\mathfrak{p}}^+$ *are mutually distinct.*

ii) (Schmid [16], Lemma 5.5). *Suppose that*

$$(\lambda+\rho_{\mathfrak{k}}+\beta_1+\cdots+\beta_q,\alpha)\leqslant 0 \quad \textit{for all } \alpha\in\Delta_{\mathfrak{k}}^+$$

whenever $\beta_1,\cdots,\beta_q\in\Delta_{\mathfrak{p}}^+$ *are mutually distinct. Then* $H^i(S, L_\lambda\otimes{}^qP_+)=0$ *for any* $i<s$, *and the irreducible components of K-module* $H^s(S, L_\lambda\otimes\wedge^qP_+)$ *consist of the K-modules with lowest weights* $\lambda+2\rho_{\mathfrak{k}}+\beta_1+\cdots+\beta_q$ *where* $\beta_1,\cdots,\beta_q\in\Delta_{\mathfrak{p}}^+$ *are mutually distinct.*

Proof. There exists a sequence of holomorphic homogeneous vector subbundles

$$0=E_0\subset E_1\subset\cdots\subset E_l=L_\lambda\otimes N^q$$

such that $L_{\lambda_i}=E_i/E_{i-1}$ is the homogeneous line bundle associated to a holomorphic character $\lambda_i\in\Lambda$ of B' because of the solvability of B'. We clearly have

$$\lambda_i=\lambda\pm\beta_1\pm\cdots\pm\beta_r \quad (0\leqslant r\leqslant q)$$

where $\beta_1,\cdots,\beta_r\in\Delta_{\mathfrak{p}}^+$ are mutually distinct and if $r=q$, then all the signs in the above formula are not positive. Consider the exact sequence

$$0\to E_{i-1}\to E_i\to L_{\lambda_i}\to 0\,.$$

Then we have $H^p(S, E_i)\simeq H^p(S, L_{\lambda_i})$ for $p>0$ when $H^p(S, E_{i-1})=0$ for $p>0$. Using Theorem 1.1 of Bott, we therefore have

$$H^p(S, L_\lambda\otimes N^q)=0 \quad \text{for } p>0$$

inductively by the assumption. Moreover this clearly implies the assertion also in case that $r\leqslant q$. The vanishing of $H^i(S, L_\lambda\otimes\wedge^rP_+)$ is deduced from that of $H^i(S, L_\lambda\otimes N^r)$ in view of the cohomology exact sequence.

Next, $L_\lambda\otimes\wedge^qP_+$ has also an sequence of the subbundles

$$0=F_0\subset F_1\subset\cdots\subset F_m=L_\lambda\otimes\wedge^qP_+$$

such that $L_{\lambda_j}=F_j/F_{j-1}$ is the homogeneous line bundle associated to the character

$$\lambda_j=\lambda+\beta_1+\cdots+\beta_q$$

where $\beta_1,\cdots,\beta_q\in\Delta_{\mathfrak{p}}^+$ are mutually distinct. We then have

$$\chi(L_\lambda\otimes\wedge^qP_+)=\sum_{j=1}^m\chi(L_{\lambda_j}) \quad \text{in } R(K)\,.$$

By the vanishing condition, we have

$$\chi(L_\lambda \otimes \wedge^q P_+) = [H^0(S, L_\lambda \otimes \wedge^q P_+)]$$

and

$$\chi(L_{\lambda_j}) = [H^0(S, L_{\lambda_j})] .$$

Thus we have

$$[H^0(S, L_\lambda \otimes \wedge^q P_+)] = \sum_{j=1}^{m} [H^0(S, L_{\lambda_j})] ,$$

which shows our assertion i). As for ii), the vanishing is due to Schmid. The proof of the rest part is similar to that of i).

Related to the conditions in Proposition 2.4, i), we easily see that there exists a positive number $c^\flat > 0$ such that, for $\lambda \in \Lambda$, if $(\lambda, \alpha) > c^\flat$ for all $\alpha \in \Delta_{\mathfrak{k}}^+$, then λ satisfies the vanishing conditions in i) for all q. Similarly, there exists a positive number $c^\sharp > 0$ such that, for $\lambda \in \Lambda$, if $(\lambda, \alpha) < -c^\sharp$ for all $\alpha \in \Delta_{\mathfrak{k}}^+$, then λ satisfies the vanishing conditions in ii) for all q.

DEFINITION 1. We shall say that $\lambda \in \Lambda$ *satisfies the condition* $(\flat)$ (*resp.* $(\sharp)$), if λ satisfies the vanishing conditions in Proposition 2.4, i) (resp. ii)) for all q. Choose the numbers $c^\flat$, $c^\sharp$ as above. Then we may say that if $(\lambda, \alpha) > c^\flat$ (*resp.* if $(\lambda, \alpha) < -c^\sharp$) for all $\alpha \in \Delta_{\mathfrak{k}}^+$, then λ satisfies the condition $(\flat)$ (*resp.* $(\sharp)$).

3. Construction of elliptic complexes

This section will be devoted to the construction of elliptic complexes. As for the notion of differential operators on vector bundles and their symbols, we refer the reader to Palais [15], Chapt. IV.

Let G be a connected real semi-simple Lie group with a compact Cartan subgroup. We denote by $\mathfrak{g}_0$ the Lie algebra corresponding to G. We shall consider a compactly imbedded subalgebra $\mathfrak{k}_0$ in $\mathfrak{g}_0$ whose rank is maximal as in §1. Then the connected Lie group K generated by $\mathfrak{k}_0$ is compact and a maximal torus contained in K is seen to be a Cartan subgroup of G (see [16], §2). We fix a compact Cartan subgroup H of G contained in K and keep the other situation and notation as in the previous sections. We notice that, in the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha$$

in §1, $B(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) \neq 0$ if and only if $\alpha = -\beta$. Hence we choose a root vector $E_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Delta$ such that

$$B(E_\alpha, E_\beta) = \delta_{\alpha, -\beta} .$$

To a complex K-module V, one can associate the differentiable homogeneous vector bundle $\mathcal{V}$ over $X=G/K$. We denote by $C^\infty(\mathcal{V})$ the space of the infinitely differentiable sections of $\mathcal{V}$ and identify $C^\infty(\mathcal{V})$ with the following subspace of the infinitely differentiable V-valued functions on G;

$$C^\infty(\mathcal{V}) = \{s: G \xrightarrow{C^\infty} V \mid s(gk) = k^{-1}s(g) \quad \text{for } g\in G,\ k\in K\}\,.$$

For a vector valued function s on G and $X\in\mathfrak{g}$, we put

$$\nu(X)s = Xs\,,$$

where $s\mapsto Xs$ denotes the operation of X as a left-invariant complex vector field on G. Define $\mathfrak{p}$ as in §1 and let $\mathcal{V}\otimes\wedge^q\mathcal{P}$ be the homogeneous vector bundle associated to a K-module $V\otimes\wedge^q\mathfrak{p}$ for $q=0, 1, \cdots, 2n$. Choosing a base $\{E_\beta\}_{\beta\in\Delta_\mathfrak{p}}$ of $\mathfrak{p}$ as before, we shall define a G-invariant differential operator

$$D^q: C^\infty(\mathcal{V}\otimes\wedge^q\mathcal{P}) \to C^\infty(\mathcal{V}\otimes\wedge^{q+1}\mathcal{P})$$

by $D^q=\sum_{\beta\in\Delta_\mathfrak{p}} \nu(E_\beta)\otimes\varepsilon(E_{-\beta})$, where $\varepsilon(E_{-\beta})$ is the exterior multiplication by $E_{-\beta}\in\mathfrak{p}$. Particularly for $q=0$, $\varepsilon(E_{-\beta})$ should be a mere tensoring operator by $E_{-\beta}$. In effect, let s be a $V\otimes\wedge^q\mathfrak{p}$-valued function on G such that $s(gk)=k^{-1}s(g)$ for $g\in G$, $k\in K$. Then it is easily checked that D^qs has the same property since

$$\sum_{\beta\in\Delta_\mathfrak{p}} E_\beta\otimes E_{-\beta} = \sum_{\beta\in\Delta_\mathfrak{p}} kE_\beta\otimes kE_{-\beta} \quad \text{for } k\in K\,.$$

Thus D^q is a well-defined differential operator from $\mathcal{V}\otimes\wedge^q\mathcal{P}$ to $\mathcal{V}\otimes\wedge^{q+1}\mathcal{P}$. We note that this definition of D^q is independent of the choice of the base $\{E_\beta\}_{\beta\in\Delta_\mathfrak{p}}$ as far as $B(E_\alpha, E_\beta)=\delta_{\alpha,-\beta}$ for $\alpha, \beta\in\Delta_\mathfrak{p}$. As for the symbol of D^q, we regard $\mathfrak{p}_0$ as the tangent space at the origin $eK\in X$ and identify $\mathfrak{p}_0$ with its dual space through the restriction $B|\mathfrak{p}_0$ of the Killing form. Thus we may also regard $\mathfrak{p}_0$ as the cotangent space at the origin. Then for a cotangent vector $\xi\in\mathfrak{p}_0$, the symbol map

$$\sigma_\xi(D^q): V\otimes\wedge^q\mathfrak{p} \to V\otimes\wedge^{q+1}\mathfrak{p}$$

of D^q at the origin appears to be

$$\sigma_\xi(D^q) = 1\otimes\varepsilon(\xi)\,,$$

where $\xi\in\mathfrak{p}_0$ is considered as an element of $\mathfrak{p}$. In fact, if we take a function f on X such that $f(eK)=0$ and $\xi=(df)_{eK}$, then we have

$$D^q(fs) = fD^qs + \sum_{\beta\in\Delta_\mathfrak{p}} (E_\beta f)(1\otimes\varepsilon(E_{-\beta}))s$$

for $s\in C^\infty(\mathcal{V}\otimes\wedge^q\mathcal{P})$. On the other hand, we may consider that

$$df = \sum_{\beta \in \Delta_{\mathfrak{p}}} (E_\beta f) E_{-\beta}$$

under the identification of $\mathcal{P}$ with the complex cotangent bundle of X by the Killing form, since $Xf=0$ for $X \in \mathfrak{k}$ and $B(E_\alpha, E_\beta)=\delta_{\alpha,-\beta}$. Therefore we obtain

$$D^q(fs)_{eK} = (1 \otimes \mathcal{E}(\xi))(s_{eK}),$$

which implies our assertion.

Now, suppose that there is given an irreducible K-module V_λ whose highest weight is λ (here the order on $\mathfrak{h}_R^*$ is fixed as in §1). By Theorem 1.1, we then have a K-module isomorphism

$$V_\lambda \cong H^0(S, L_\lambda).$$

Then it induces the K-module isomorphism

$$\varphi_q \colon V_\lambda \otimes \wedge^q \mathfrak{p} \xrightarrow{\ \sim\ } H^0(S, L_\lambda \otimes \wedge^q P)$$

for every q. We put

$$U_\lambda^q = \varphi_q^{-1}(H^0(S, L_\lambda \otimes N^q)) \quad \text{for } q>0$$

as a K-submodule of $V_\lambda \otimes \wedge^q \mathfrak{p}$. Here we notice that the determination of the K-submodule U_λ^q is independent of the choice of the isomorphism $V_\lambda \cong H^0(S, L_\lambda)$ because of the irreducibility of V_λ and Schur's lemma. Next, we define a K-module V_λ^q by

$$V_\lambda^q = V_\lambda \otimes \wedge^q \mathfrak{p} / U_\lambda^q \quad \text{for } q>0,$$

and $V_\lambda^0 = V_\lambda$ for $q=0$. Assume that the highest weight $\lambda \in \Lambda$ satisfies the condition ($\flat$) in Definition 1 in §2. Then, by Propositions 2.3, i) and 2.4, i), we have a K-module isomorphism

$$V_\lambda^q \cong H^0(S, L_\lambda \otimes \wedge^q P_+)$$

and, by the diagram (II_λ^0), a commutative diagram of K-modules for $\xi \in \mathfrak{p}_0$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & 0 & \\
 & & \uparrow & & \uparrow & & & \uparrow & \\
0 & \to & V_\lambda & \to & V_\lambda^1 & \to \cdots \to & & V_\lambda^n & \to 0 \\
 & & \uparrow & & \uparrow & & & \uparrow & \\
0 & \to & V_\lambda & \to & V_\lambda \otimes \mathfrak{p} & \to \cdots \to & & V_\lambda \otimes \wedge^n \mathfrak{p} & \to \cdots \\
 & & \uparrow & & \uparrow & & & \uparrow & \\
 & & 0 & \to & U_\lambda^1 & \to \cdots \to & & U_\lambda^n & \to \cdots \\
 & & & & \uparrow & & & \uparrow & \\
 & & & & 0 & & & 0 &
\end{array} \qquad (\mathrm{III}_\lambda^\flat),$$

where both the columns and rows are all exact if $\xi \neq 0$ in $\mathfrak{p}_0$. As noticed in Remark in §2, a diagram

$$\begin{array}{ccc} H^0(S, L_\lambda \otimes \wedge^q P) & \xrightarrow{(1\otimes\alpha_\xi^q)_*} & H^0(S, L_\lambda \otimes \wedge^{q+1} P) \\ \wr\Vert & & \wr\Vert \\ H^0(S, L_\lambda)\otimes \wedge^q \mathfrak{p} & \xrightarrow{1\otimes\varepsilon(\xi)} & H^0(S, L_\lambda)\otimes \wedge^{q+1}\mathfrak{p} \end{array}$$

is commutative, and therefore the map

$$V_\lambda \otimes \wedge^q \mathfrak{p} \to V_\lambda \otimes \wedge^{q+1}\mathfrak{p}$$

in the diagram ($\mathrm{III}_\lambda^\flat$) is nothing but the exterior multiplication $1\otimes\varepsilon(\xi)$ by $\xi \in \mathfrak{p}_0$. We denote by σ_ξ^q the map $V_\lambda^q \to V_\lambda^{q+1}$ in ($\mathrm{III}_\lambda^\flat$). Thus if λ satisfies the condition ($\flat$), then we have the exact sequence

$$0 \to V_\lambda \xrightarrow{\sigma_\xi^0} V_\lambda^1 \xrightarrow{\sigma_\xi^1} \cdots \to V_\lambda^n \to 0$$

for $\xi \neq 0$ in $\mathfrak{p}_0$, where $n = \frac{1}{2}\dim_{\boldsymbol{R}} X$.

Alternatively, if we denote by V_λ an irreducible K-module whose lowest weight is $\lambda + 2\rho_{\mathfrak{k}}$ ($\rho_{\mathfrak{k}}$ is as in §1), then we have a K-module isomorphism

$$V_\lambda \cong H^s(S, L_\lambda)\,.$$

When $\lambda \in \Lambda$ satisfies the condition ($\sharp$) in Definition 1, in the same way we have a commutative diagram of exact sequences for $\xi \neq 0$ in $\mathfrak{p}_0$

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & 0 & \\ & & \uparrow & & \uparrow & & & \uparrow & \\ 0 & \to & V_\lambda & \to & V_\lambda^1 & \to & \cdots \to & V_\lambda^n & \to 0 \\ & & \uparrow & & \uparrow & & & \uparrow & \\ 0 & \to & V_\lambda & \to & V_\lambda \otimes \mathfrak{p} & \to & \cdots \to & V_\lambda \otimes \wedge^n \mathfrak{p} & \to \cdots \\ & & \uparrow & & \uparrow & & & \uparrow & \\ & & 0 & \to & U_\lambda^1 & \to & \cdots \to & U_\lambda^n & \to \cdots \\ & & & & \uparrow & & & \uparrow & \\ & & & & 0 & & & 0 & \end{array} \qquad (\mathrm{III}_\lambda^\sharp)\,,$$

where

$$U_\lambda^q \cong H^s(S, L_\lambda \otimes N^q)$$
$$V_\lambda^q \cong H^s(S, L_\lambda \otimes \wedge^q P_+)$$

as K-modules and the map $V_\lambda \otimes \wedge^q \mathfrak{p} \to V_\lambda \otimes \wedge^{q+1}\mathfrak{p}$ is $1\otimes\varepsilon(\xi)$ for $\xi \in \mathfrak{p}_0$. This follows similarly from Propositions 2.3, ii) and 2.4, ii).

DEFINITION 2. Let V_λ be an irreducible K-module whose highest (resp. lowest) weight is λ (resp. $\lambda + 2\rho_{\mathfrak{k}}$). Assume that $\lambda \in \Lambda$ satisfies the condition ($\flat$) (resp. ($\sharp$)) in Definition 1 in §2. We then obtain a commutative diagram

($\mathrm{III}_\lambda^\flat$) (resp. ($\mathrm{III}_\lambda^\sharp$)) consisting of exact sequences for $\xi \neq 0$ in $\mathfrak{p}_0$. We shall call the exact sequence in ($\mathrm{III}_\lambda^\flat$) (resp. ($\mathrm{III}_\lambda^\sharp$))

$$0 \to V_\lambda \xrightarrow{\sigma_\xi^0} V_\lambda^1 \xrightarrow{\sigma_\xi^1} \cdots \to V_\lambda^n \to 0$$

the canonical ($\flat$) (*resp.* ($\sharp$))*-sequence associated to* V_λ.

Most of our future arguments are valid for both ($\flat$) and ($\sharp$) cases. Consequently, when we do not refer to ($\flat$) or ($\sharp$) for simplification, it will be valid for either case.

The situations being kept as above, we denote by $\mathcal{U}_\lambda^q$, $\mathcal{V}^q$ the homogeneous vector bundles over X associated to the K-module U_λ^q, V_λ^q. Regarding $\mathcal{U}_\lambda^q$ as a subbundle of $\mathcal{V}_\lambda \otimes \wedge^q \mathcal{P}$, hence $C^\infty(\mathcal{U}_\lambda^q)$ as a subspace of $C^\infty(\mathcal{V}_\lambda \otimes \wedge^q \mathcal{P})$, we see that

$$D^q(C^\infty(\mathcal{U}_\lambda^q)) \subset C^\infty(\mathcal{U}_\lambda^{q+1})$$

for every q. In fact, $\nu(E_\beta) \otimes 1$ transforms the space of U_λ^q-valued functions on G into themselves. On the other hand, the exterior multiplication by an element of $\mathfrak{p}$ transforms U_λ^q into U_λ^{q+1} as noticed in §2. Therefore the above inclusion relation holds. Choose a splitting injection $i^q \colon V_\lambda^q \to V_\lambda \otimes \wedge^q \mathfrak{p}$ of the exact sequence of K-modules

$$0 \to U_\lambda^q \to V_\lambda \otimes \wedge^q \mathfrak{p} \to V_\lambda^q \to 0\,.$$

Denoting by i^q and p^q the injection $C^\infty(\mathcal{V}_\lambda^q) \to C^\infty(\mathcal{V}_\lambda \otimes \wedge^q \mathcal{P})$ induced from the above splitting and the projection $C^\infty(\mathcal{V}_\lambda \otimes \wedge^q \mathcal{P}) \to C^\infty(\mathcal{V}_\lambda^q)$, we define a G-invariant differential operator

$$\mathcal{D}^q \colon C^\infty(\mathcal{V}_\lambda^q) \to C^\infty(\mathcal{V}_\lambda^{q+1})$$

by $\mathcal{D}^q = p^{q+1} \circ D^q \circ i^q$. Then $\mathcal{D}^q$ is clearly independent of the choice of a splitting i^q.

Lemma 3.1. *The symbol map of* $\mathcal{D}^q$ *at the origin* $eK \in X$ *for* $\xi \in \mathfrak{p}_0$ *is the map*

$$\sigma_\xi^q \colon V_\lambda^q \to V_\lambda^{q+1}$$

in the canonical sequence associated to V_λ.

Proof. The operator i^q and p^{q+1} are of order zero as differential operators. Since $\sigma_\xi(D^q) = 1 \otimes \varepsilon(\xi)$ and the diagram

$$\begin{array}{ccc} V_\lambda^q & \xrightarrow{\sigma_\xi^q} & V_\lambda^{q+1} \\ {\scriptstyle i^q}\big\downarrow & & \big\uparrow \\ V_\lambda \otimes \wedge^q \mathfrak{p} & \xrightarrow{1 \otimes \varepsilon(\xi)} & V_\lambda \otimes \wedge^{q+1} \mathfrak{p} \end{array}$$

is commutative, our assertion is clear.

REMARK 1. When we look back upon the construction of the differential operators $\mathcal{D}^q$, these operators depend on the choice of a lexicographical order of the root system. It does not hold that $\mathcal{D}^{q+1}\circ\mathcal{D}^q=0$ in general, but we shall see that in case of symmetric spaces it holds if we choose a special order of the root system.

We recall that the homogeneous space $X=G/K$ is called *symmetric* (*of inner type*) if

$$[\mathfrak{p}_0, \mathfrak{p}_0]\subset\mathfrak{k}_0$$

in the decomposition of $\mathfrak{g}_0$ in §1. We then have $[\mathfrak{p}, \mathfrak{p}]\subset\mathfrak{k}$. For a symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$, we shall consider the following special order of the root system Δ of $\mathfrak{g}$.

DEFINITION 3. We shall call a lexicographical order of Δ *admissible* with respect to a symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$ of inner type, when

$$\beta_1+\cdots+\beta_q\neq\gamma_1+\cdots+\gamma_r$$

for $\beta_1,\cdots, \beta_q, \gamma_1,\cdots, \gamma_r$ in $\Delta_{\mathfrak{p}}^+$ and $q\neq r$.

The existence of such an order is guaranteed by the following lemma.

Lemma 3.2 (Takeuchi). *There exists an admissible lexicographical order with respect to a symmetric pair of inner type.*

Proof. We may assume that $\mathfrak{g}_0$ is simple without loss of generality. Let ι be the Cartan involution of a symmetric pair $(\mathfrak{g}_0, \mathfrak{k}_0)$ of inner type. Then one can choose a system of simple roots $\Pi=\{\alpha_1,\cdots, \alpha_l\}$ such as the following holds (Borel-Siebenthal [3], Murakami [14], Théorème 1). Let $m_1,\cdots, m_l$ be the positive integers such that $\sum_{i=1}^{l} m_i\alpha_i$ is the highest root in Δ with respect to Π. If we choose such an appropriate number k as $m_k=1$ or 2, then we have

$$\iota = \exp 2\pi\sqrt{-1}\,\mathrm{ad}\,H_\iota,$$

where $H_\iota\in\mathfrak{h}_R$ is defied as $\alpha_i(H_\iota)=\frac{1}{2}\delta_{ik}$ for $i=1,\cdots, l$.

We shall now show that the lexicographical order determined by the above system of simple roots is admissible. Let E_β be a root vector for a root $\beta=\sum_{i=1}^{l} n_i\alpha_i$ which is positive with respect to the above order. We then have

$$\iota E_\beta = e^{\pi\sqrt{-1}n_k}E_\beta,$$

extending ι to an involution on $\mathfrak{g}$ complex linearly. Hence it follows that

$$n_k \equiv 0 \mod 2 \quad \text{if } \beta \in \Delta_{\mathfrak{k}}^+ ,$$
$$n_k \equiv 1 \mod 2 \quad \text{if } \beta \in \Delta_{\mathfrak{p}}^+ ,$$

because $\iota = 1$ on $\mathfrak{k}$, and $\iota = -1$ on $\mathfrak{p}$. Since $0 \leqq n_k \leqq 2$, we see that a positive root $\beta = \sum_{i=1}^{l} n_i \alpha_i$ is in $\Delta_{\mathfrak{p}}^+$ if and only if $n_k = 1$. Therefore, with respect to this order, if $\beta_1, \cdots, \beta_q$ are in $\Delta_{\mathfrak{p}}^+$, then the coefficient of α_k in $\beta_1 + \cdots + \beta_q$ is q. This shows Lemma 3.2.

Theorem 3.1. *Suppose that $X = G/K$ is a symmetric space (of inner type) and choose an admissible lexicographical order of the root system with respect to the symmetric pair. Let V_λ be an irreducible K-module whose highest (resp. lowest) weight is λ (resp. $\lambda + 2\rho_{\mathfrak{k}}$). If λ satisfies the condition ($\flat$) (resp. ($\sharp$)), then the sequence constructed above*

$$0 \to C^\infty(\mathcal{V}_\lambda) \xrightarrow{\mathcal{D}^0} C^\infty(\mathcal{V}_\lambda^1) \xrightarrow{\mathcal{D}^1} \cdots \xrightarrow{\mathcal{D}^{n-1}} C^\infty(\mathcal{V}_\lambda^n) \to 0$$

is an elliptic complex over X whose symbol sequence for $\xi \in \mathfrak{p}_0$ at the origin is the canonical ($\flat$) (resp. ($\sharp$))-sequence associated to V_λ. Here $n = \frac{1}{2} \dim_{\mathbf{R}} X$.

Proof. We have already shown in Lemma 3.1 that the symbol sequence for $\xi \in \mathfrak{p}_0$ at the origin is the canonical sequence associated to V_λ and it is exact for $\xi \neq 0$. Since each $\mathcal{D}^q$ is an invariant differential operator on homogeneous vector bundles, the sequence in the theorem will be an elliptic complex if we show that it is a differential complex. Thus it only remains to show that $\mathcal{D}^{q+1} \circ \mathcal{D}^q = 0$ under the conditions of the theorem. For $s \in C^\infty(\mathcal{V}_\lambda^q)$, we have

$$D^q s = p^{q+1} \circ D^q s + r^{q+1} \circ D^q s$$

where r^{q+1} is the projection from $C^\infty(\mathcal{V}_\lambda \otimes \wedge^{q+1} \mathcal{P})$ onto $C^\infty(\mathcal{U}_\lambda^{q+1})$ by fixing some splitting $i^{q+1}\colon V_\lambda^{q+1} \to V_\lambda \otimes \wedge^{q+1} \mathfrak{p}$. Since $D^{q+1}(C^\infty(\mathcal{U}_\lambda^{q+1})) \subset C^\infty(\mathcal{U}_\lambda^{q+2})$, we then have

$$p^{q+2} \circ D^{q+1} \circ D^q \circ i^q s = p^{q+2} \circ D^{q+1} \circ i^{q+1} \circ p^{q+1} \circ D^q \circ i^q s$$

for $s \in C^\infty(\mathcal{V}_\lambda^q)$, which implies

$$\mathcal{D}^{q+1} \circ \mathcal{D}^q = p^{q+2} \circ D^{q+1} \circ D^q \circ i^q .$$

On the other hand, it holds that

$$D^{q+1} \circ D^q = \frac{1}{2} \sum_{\beta, \gamma \in \Delta_{\mathfrak{p}}} \nu([E_\beta, E_\gamma]) \otimes \varepsilon(E_{-\beta}) \varepsilon(E_{-\gamma}) .$$

Here we see that $[E_\beta, E_\gamma] \in \mathfrak{k}$ since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ by the assumption. Therefore the differential operator $D^{q+1} \circ D^q$ becomes of order zero. Since p^{q+2} and i^q are of order zero, $\mathcal{D}^{q+1}\circ\mathcal{D}^q$ is also of order zero, that is, induced by some vector bundle map $\rho^q\colon \mathcal{V}^q_\lambda\to\mathcal{V}^{q+2}_\lambda$. Moreover $\mathcal{D}^{q+1}\circ\mathcal{D}^q$ is G-invariant, hence $\rho^q\colon \mathcal{V}^q_\lambda\to\mathcal{V}^{q+2}_\lambda$ is induced by a K-module homomorphism

$$\hat{\rho}^q\colon V^q_\lambda \to V^{q+2}_\lambda .$$

Thus in order to show that $\mathcal{D}^{q+1}\circ\mathcal{D}^q=0$, it suffices to show that $\hat{\rho}^q=0$. By Proposition 2.4, the highest (resp. lowest) weight of an irreducible component of V^q_λ is $\lambda+\beta_1+\cdots+\beta_q$ (resp. $\lambda+2\rho_{\mathfrak{k}}+\beta_1+\cdots+\beta_q$) where $\beta_1,\cdots,\beta_q\in\Delta^+_{\mathfrak{p}}$ are mutually distinct in case ($\flat$) (resp. ($\sharp$)). Because the order of Δ is admissible, no irreducible component of V^q_λ is equivalent to any irreducible component of V^{q+2}_λ. Hence there exists no non-zero K-module homomorphism from V^q_λ into V^{q+2}_λ, which implies $\hat{\rho}^q=0$. Thus we complete the proof of Theorem 3.1.

REMARK 2. When X is hermitian symmetric, one can choose an admissible lexicographical order of the root system such as $m_k=1$ in the proof of Lemma 3.2. Under this order, the B'-modules $\mathfrak{p}_-$, $\mathfrak{p}_+$ in §2 are moreover K-modules, and we have a K-module decomposition

$$\mathfrak{p} = \mathfrak{p}_+\oplus\mathfrak{p}_-$$

Considering $\mathfrak{p}_+$ as the holomorphic tangent space of X at the origin, one can give a G-invariant complex structure on X, and our vector bundle $\mathcal{V}_\lambda$ is endowed with the structure of a holomorphic vector bundle over X. Then our differential complex in Theorem 3.1 becomes the so-called Dolbeault complex ($\bar{\partial}$-complex) associated to the holomorphic vector bundle $\mathcal{V}_\lambda$ in either case ($\flat$) or ($\sharp$). In fact, the B'-module $\mathfrak{p}_+$ is, then, a K^C-module and therefore we have an isomorphism as K^C-modules

$$H^i(S, L_\lambda\otimes\wedge^q P_+) \cong H^i(S, L_\lambda)\otimes\wedge^q\mathfrak{p}_+ ,$$

which gives an isomorphism as K-modules:

$$V^q_\lambda \cong V_\lambda\otimes\wedge^q\mathfrak{p}_+ .$$

Therefore $\mathcal{V}^q_\lambda\cong\mathcal{V}_\lambda\otimes\wedge^q\mathcal{P}_+$ is regarded as the vector bundle of anti-holomorphic cotangent vectors of X of type $(0, q)$. Under this circumstance, it is seen that

$$\mathcal{D}^q = \sum_{\beta\in\Delta^+_{\mathfrak{p}}} \nu(E_{-\beta})\otimes\varepsilon(E_\beta)$$

and this is exactly the Cauchy-Riemann operator $\bar{\partial}$.

In case that $X=G/K$ is not symmetric or a lexicographical order is not admissible, we shall consider the following differential operator between two hermitian vector bundles. We give V_λ and $\mathfrak{p}$ K-invariant hermitian inner products, which make $\mathcal{V}_\lambda \otimes \wedge^q \mathcal{P}$ a homogeneous vector bundle with an invariant hermitian metric. This induces an invariant hermitian metric on $\mathcal{V}_\lambda^q$; in this way consider $\mathcal{V}_\lambda^q$ as a hermitian vector bundle. We put

$$\mathcal{W}_\lambda^e = \bigoplus_{q\equiv 0 \bmod 2} \mathcal{V}_\lambda^q \qquad (\mathcal{V}_\lambda^0 = \mathcal{V}_\lambda)$$
$$\mathcal{W}_\lambda^o = \bigoplus_{q\equiv 1 \bmod 2} \mathcal{V}_\lambda^q .$$

These are hermitian vector bundles over X with the hermitian metrics induced by those of $\mathcal{V}_\lambda^q$, $q=0, 1, \cdots, n$. Endowing X with an invariant volume element, one can define the formal adjoint operator of $\mathcal{D}^q$:

$$*\mathcal{D}^q : C^\infty(\mathcal{V}_\lambda^{q+1}) \to C^\infty(\mathcal{V}_\lambda^q) .$$

The symbol map of $*\mathcal{D}^q$ for $\xi \in \mathfrak{p}_0$ at the origin is

$$-*\sigma_\xi^q : V_\lambda^{q+1} \to V_\lambda^q ,$$

where $*\sigma_\xi^q$ denotes the adjoint of $\sigma_\xi^q : V_\lambda^q \to V_\lambda^{q+1}$ (see Palais [15]). We define a G-invariant differential operator

$$\mathcal{L} : C^\infty(\mathcal{W}_\lambda^e) \to C^\infty(\mathcal{W}_\lambda^o)$$

by the formula

$$\mathcal{L}(s_0, s_2, \cdots, s_k) = (\mathcal{D}^0 s_0 + *\mathcal{D}^1 s_2, \cdots, \mathcal{D}^{k-2} s_{k-2} + *\mathcal{D}^{k-1} s_k) ,$$

where $s_q \in C^\infty(\mathcal{V}_\lambda^q)$ and $k=2\left[\frac{n}{2}\right]$. Then the symbol map of $\mathcal{L}$ for $\xi \in \mathfrak{p}_0$ at the origin $\sigma_\xi(\mathcal{L}) : \bigoplus_{q\equiv 0 \bmod 2} V_\lambda^q \to \bigoplus_{q\equiv 1 \bmod 2} V_\lambda^q$ ($V_\lambda^0 = V_\lambda$) is clearly given by

$$\sigma_\xi(\mathcal{L})(v_0, v_1, \cdots, v_k) = (\sigma_\xi^0 v_0 - *\sigma_\xi^1 v_2, \cdots, \sigma_\xi^{k-2} v_{k-2} - *\sigma_\xi^{k-1} v_k) .$$

It is easy from the exactness of the canonical sequence associated to V_λ to see that, for $\xi \neq 0$ in $\mathfrak{p}_0$, $\sigma_\xi(\mathcal{L})$ is bijective, which shows that $\mathcal{L}$ is an invariant elliptic operator from $\mathcal{W}_\lambda^e$ to $\mathcal{W}_\lambda^o$.

DEFINITION 4. Let V_λ be an irreducible K-module whose highest (resp. lowest) weight is λ (resp. $\lambda + 2\rho_{\mathfrak{k}}$). Assume that $\lambda \in \Lambda$ satisfies the condition ($\flat$) (resp. ($\sharp$)). We then obtain the elliptic complex

$$0 \to C^\infty(\mathcal{V}_\lambda) \xrightarrow{\mathcal{D}^0} C^\infty(\mathcal{V}_\lambda^1) \xrightarrow{\mathcal{D}^1} \cdots \to C^\infty(\mathcal{V}_\lambda^n) \to 0$$

when X is symmetric and the lexicographical order of the root system is admisssible, or

$$0 \to C^\infty(\mathcal{W}_\lambda^e) \xrightarrow{\mathcal{L}} C^\infty(\mathcal{W}_\lambda^o) \to 0$$

otherwise. We shall call the above elliptic complex *the* ($\flat$) (resp. ($\sharp$))-*complex associated to the homogeneous vector bundle* $\mathcal{V}_\lambda$.

REMARK 3. The invariant operator $D^0\colon C^\infty(\mathcal{V}_\lambda)\to C^\infty(\mathcal{V}_\lambda\otimes\mathcal{P})$ is considered as an invariant connection on the homogeneous vector bundle $\mathcal{V}_\lambda$, under the identification $\mathcal{P}$ with the complex contagent bundle of X. One can see that an connection D^0 extends uniquely to an differential operator

$$\bar{D}^q\colon C^\infty(\mathcal{V}_\lambda\otimes\wedge^q\mathcal{P})\to C^\infty(\mathcal{V}_\lambda\otimes\wedge^{q+1}\mathcal{P})$$

such that

$$\bar{D}^q(s\varphi)=D^0s\wedge\varphi+sd\varphi\quad\text{for } s\in C^\infty(\mathcal{V}_\lambda),\ \varphi\in C^\infty(\wedge^q\mathcal{P})$$

and $\bar{D}^{q+1}\circ\bar{D}^q$ is an operator of order zero (Bott-Chern [6]). When X is symmetric, we can see that $\bar{D}^q=D^q$. If we adopt $\bar{D}^q$ instead of D^q, it may be possible to construct an differential complex even in case that X is non-symmetric.

4. The index of elliptic complexes over compact homogeneous spaces

In this section, we shall chalculate the indices of the elliptic complexes constructed in §3, making use of Bott's theorem in [5]. We shall review this theorem for invariant elliptic operators.

Let M be a compact connected Lie group. We denote by $R(M)$ the Grothendieck ring of M, i.e., the free abelian group generated by the equivalence classes of finite-dimensional irreducible M-modules with the multiplicative structure induced by tensor products. For an M-module V, we denote by $[V]\in R(M)$ the element of $R(M)$ determined by V. One can define a symmetric bilinear form $\langle\ ,\ \rangle_M$ on $R(M)$ by

$$\langle[V],[W]\rangle_M=\dim_C \mathrm{Hom}_M(V,W),$$

for M-modules V, W. If K is another compact connected Lie group and $\varphi\colon K\to M$ is a homomorphism, then by the restriction it induces the canonical homomorphism

$$\varphi^*\colon R(M)\to R(K).$$

When we denote by $\mathcal{J}(M)$ the equivalence classes of irreducible M-modules, we have $R(M)=\sum_{x\in\mathcal{J}(M)}\boldsymbol{Z}x$, where $\boldsymbol{Z}$ is the ring of integers. We define the formal ring $\hat{R}(M)=\prod_{x\in\mathcal{J}(M)}\boldsymbol{Z}x$ and regard $R(M)$ as a subring of $\hat{R}(M)$. Then for a homomorphism $\varphi\colon K\to M$, we define the formal induced representation

$$\varphi_*: R(K) \to \hat{R}(M)$$

by $\varphi_*(y) = \sum_{x \in \mathcal{I}(M)} \langle \varphi^* x, y \rangle_K x$ for $y \in R(K)$, where an element in $\hat{R}(M)$ is denoted by an infinite sum.

Now, let K be a connected closed subgroup of M, V^q a finite-dimensional K-module for $q=0, 1, \cdots, n$. We denote by $\mathcal{V}^q$ the homogeneous vector bundle over the compact homogeneous space $X=M/K$ associated to the K-module V^q. For an M-invariant elliptic complex over X

$$0 \to C^\infty(\mathcal{V}^0) \xrightarrow{\mathcal{D}^0} C^\infty(\mathcal{V}^1) \xrightarrow{\mathcal{D}^1} \cdots \xrightarrow{\mathcal{D}^{n-1}} C^\infty(\mathcal{V}^n) \to 0,$$

a vector space

$$H^q(\mathcal{V}^*) = \operatorname{Ker} \mathcal{D}^q / \operatorname{Im} \mathcal{D}^{q-1}$$

is a finite-dimensional M-module for each q, since X is compact. We then define *the homogeneous index* $\chi(\mathcal{V}^*) \in R(M)$ of this elliptic complex by

$$\chi(\mathcal{V}^*) = \sum_{q=0}^n (-1)^q [H^q(\mathcal{V}^*)].$$

The following theorem for the homogeneous index is due to Bott.

Theorem 4.1 (Bott [5]). *Let*

$$0 \to C^\infty(\mathcal{V}^0) \to \cdots \to C^\infty(\mathcal{V}^n) \to 0$$

be an invariant elliptic complex over $X=M/K$, where $\mathcal{V}^0, \cdots, \mathcal{V}^n$ are the homogeneous vector bundles associated to K-modules $V^0, \cdots, V^n$. Define the homogeneous symbol $\sigma(\mathcal{V}^*) \in R(K)$ by $\sigma(\mathcal{V}^*) = \sum_{q=0}^n (-1)^q [V^q]$. If $i_*: R(K) \to \hat{R}(M)$ is the formal induced homomorphism determined by the inclusion $i: K \to M$, then the following holds:

a) $i_* \sigma(\mathcal{V}^*) \in R(M)$

b) $\chi(\mathcal{V}^*) = i_* \sigma(\mathcal{V}^*)$.

REMARK. In [5], the theorem is stated in case that $n=1$. The above generalization is easy in the usual manner of reducing of the problem of index on an elliptic complex to that on an elliptic operator between two vector bundles.

Now, we assume that a connected compact Lie group M is semi-simple and a connected closed subgroup K of M is of maximal rank. Fix a Cartan subgroup H of M in K, and keep the other situation and notation as in §§1, 2, 3, substituting M for G. Given an irreducible K-module V_λ whose highest (resp. lowest) weight λ (resp. $\lambda+2\rho_{\mathfrak{k}}$) satisfies the condition ($\flat$) (resp. ($\sharp$)), we have an invariant elliptic complex called the ($\flat$) (resp. ($\sharp$))-complex associated to the homogeneous vector bundle $\mathcal{V}_\lambda$ in Definition 4 in §3. More precisely, if $X=M/K$ is symmetric and the order is admissible, then we have an invariant elliptic complex

$$0 \to C^\infty(\mathcal{V}_\lambda) \xrightarrow{\mathcal{D}^0} C^\infty(\mathcal{V}_\lambda^1) \xrightarrow{\mathcal{D}^1} \cdots \to C^\infty(\mathcal{V}_\lambda^n) \to 0$$

where $n=\dfrac{1}{2}\dim_{\boldsymbol{R}} X$, and otherwise, an invariant elliptic operator

$$C^\infty(\mathcal{W}_\lambda^e) \xrightarrow{\mathcal{L}} C^\infty(\mathcal{W}_\lambda^o),$$

under a K-invariant hermitian inner products on V_λ and $\mathfrak{p}$. For the sake of notational simplification we denote by $\chi(\mathcal{V}_\lambda^*)\in R(M)$ and $\sigma(\mathcal{V}_\lambda^*)\in R(K)$ the homogeneous index and the homogeneous symbol of the above ($\flat$) (resp. ($\sharp$))-complex associated to $\mathcal{V}_\lambda$, whether X is symmetric or not.

Lemma 4.1. *The homogeneous symbol of the* ($\flat$)*-complex is*

$$\sigma(\mathcal{V}_\lambda^*) = \sum_{q=0}^{n}(-1)^q[H^0(S, L_\lambda\otimes\wedge^q P_+)]\in R(K),$$

and that of the ($\sharp$)*-complex is*

$$\sigma(\mathcal{V}_\lambda^*) = \sum_{q=0}^{n}(-1)^q[H^s(S, L_\lambda\otimes\wedge^q P_+)]\in R(K).$$

Proof. Recall that $V_\lambda^q\simeq H^i(S, L_\lambda\otimes\wedge^q P_+)$ where $i=0$ in ($\flat$)-case or $i=s$ in ($\sharp$)-case. Lemma 4.1 is then clear from the definition of homogeneous symbols in Theorem 4.1.

We consider the inclusion maps

$$H \xrightarrow[j]{} K \xrightarrow[i]{} M,$$

and endow $S=K/H$ and $Y=M/H$ with the complex structures determined by the fixed order on the root system Δ of M. For a finite-dimensional complex H-module $\mathfrak{e}$, one can associate the holomorphic homogeneous vector bundles E^S, E^Y on S, Y. Put

$$\mathfrak{n}_+ = \sum_{\alpha\in\Delta_{\mathfrak{k}}^+} \mathfrak{g}^\alpha$$

which is H-module, and let N_+ be the *differentiable* homogeneous vector bundle over S associated to the H-module $\mathfrak{n}_+$, which is, here, regarded as the anti-holomorphic cotangent bundle of S. Then the Dolbeault complex associated to the holomorphic vector bundle E^S is

$$0 \to C^\infty(E^S) \xrightarrow{\bar\partial} C^\infty(E^S \otimes N_+) \xrightarrow{\bar\partial} \cdots \to C^\infty(E^S \otimes \wedge^s N_+) \to 0$$

and its homogeneous symbol is

$$\sigma_S(\mathfrak{e}) = \sum_{p=0}^{s} (-1)^p [\mathfrak{e} \otimes \wedge^p \mathfrak{n}_+] \in R(H) .$$

Similarly, the homogeneous symbol of the Dolbeault complex associated to E^Y is

$$\sigma_Y(\mathfrak{e}) = \sum_{q=0}^{n+s} (-1)^q [\mathfrak{e} \otimes \wedge^q (\mathfrak{n}_+ \oplus \mathfrak{p}_+)] \in R(H) .$$

We then have

$$\sigma_Y(\mathfrak{e}) = \sum_{q=0}^{n} (-1)^q \sigma_S(\mathfrak{e} \otimes \wedge^q \mathfrak{p}_+) \quad \text{in } R(H) .$$

By Theorem 4.1, we have

$$j_* \sigma_S(\mathfrak{e}) = \sum_{p=0}^{s} (-1)^p [H^p(S, E^S)] \in R(K) .$$

Therefore by Lemma 4.1 and Proposition 2.4, we have

Lemma 4.2. *Let $\mathfrak{l}_\lambda$ be an irreducible H-module whose character $\lambda \in \Lambda$ satisfies the condition ($\flat$) (resp. ($\sharp$)). Then we have*

$$j_* \sigma_Y(\mathfrak{l}_\lambda) = \sigma(\mathcal{V}_\lambda^*) \quad (\textit{resp. } (-1)^s \sigma(\mathcal{V}_\lambda^*)) ,$$

where $\mathcal{V}_\lambda^*$ is the ($\flat$) (resp. ($\sharp$))-complex associated to $\mathcal{V}_\lambda$.

With these preparations, we have the following theorem for the homogeneous index of ($\flat$)- or ($\sharp$)-complex.

Theorem 4.2. *Let $\mathcal{V}_\lambda^*$ be a ($\flat$) (resp. ($\sharp$))-complex associated to $\mathcal{V}_\lambda$, and $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$. Then the homogeneous index $\chi(\mathcal{V}_\lambda^*) \in R(M)$ is given by the following formulae. If $\lambda + \rho$ is singular with respect to M, then in either case*

$$\chi(\mathcal{V}_\lambda^*) = 0 .$$

If $\lambda + \rho$ is regular with respect to M, then there exists the unique element w of the Weyl group of M such that $w(\lambda + \rho) - \rho$ is dominant with respect to M. We then have

$$\chi(\mathcal{V}_\lambda^*) = (-1)^{n(w)} [w(\lambda + \rho) - \rho] \ (\textit{resp. } (-1)^{s + n(w)} [w(\lambda + \rho) - \rho]) ,$$

where $n(w)$ denotes the number of elements of the set $\{\alpha \in \Delta^+ \mid w\alpha < 0\}$, $[\mu] \in R(M)$

for $\mu\in\Lambda$ *the equivalence class of the irreducible M-module whose highest weight is* $\mu\in\Lambda$.

Proof. By Theorem 4.1,

$$\chi(\mathcal{C}V_\lambda^*) = i_*\sigma(\mathcal{C}V_\lambda^*)\,.$$

Therefore by Lemma 4.2, we have

$$\chi(\mathcal{C}V_\lambda^*) = (i\circ j)_*\sigma_Y(\mathrm{I}_\lambda) \qquad \text{in } (\flat)\text{-case}$$

or

$$\chi(\mathcal{C}V_\lambda^*) = (-1)^s(i\circ j)_*\sigma_Y(\mathrm{I}_\lambda) \quad \text{in } (\sharp)\text{-case},$$

since $(i\circ j)_*x=i_*\circ j_*x$ for $x\in R(H)$ when both sides have the meaning. Applying Theorem 4.1 to the inclusion $i\circ j\colon H\rightarrow M$, it holds

$$(i\circ j)_*\sigma_Y(\mathrm{I}_\lambda) = \sum_{q=0}^{n+s}(-1)^q[H^q(Y, L_\lambda^Y)]\in R(M)\,,$$

where L_λ^Y is the holomorphic homogeneous line bundle over $Y=M/H$ associated to $\lambda\in\Lambda$. Thus by Theorem 1.1*), Theorem 4.2 is now clear.

5. The index of elliptic complexes over compact locally homogeneous spaces; proportionality principle

In this section, we shall generalize Hirzebruch's proportionality principle (see Hirzebruch [10], Ise [12], Griffiths [7]) for elliptic complexes and evaluate the indices of the ($\flat$)- or ($\sharp$)-complexes constructed in §3 over compact locally homogeneous spaces. For this purpose, we make use of a variant of the index theorem due to Atiyah-Singer [1].

Let K be a compact Lie group, V a fixed real oriented K-module. We shall then say that an oriented manifold X has a K-structure when there exists a principal K-bundle P over X such that

$$P\times_K V \cong TX$$

as oriented vector bundles. Here $P\times_K V$ denotes the vector bundle over X associated to P by the K-module V, TX the tangent bundle of X. Then we have a canonical homomorphism

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*) The referee pointed out the following matter. Instead of utilizing Theorem 1.1, one can complete the proof of the theorem straight from the Weyl's character formula. In fact, by means of Weyl's integral formula (see [5], Theorem B), $(i\circ j)_*\sigma_Y(\mathrm{I}_\lambda)$ can be calculated directly and in view of the character formula, it is seen to coincide with the element of $R(M)$ stated in Theorem 4.2.

$$\alpha_P : K_K(V) \to K_K(P \times V) = K(TX),$$

where $K_K(\cdot)$ denotes the usual equivariant K-functor for the compact group K, $K(TX)$ the K-group of TX. For an elliptic complex on X, we shall say that its symbol class is *associated to the K-structure* when the symbol class in $K(TX)$ belongs to the image of α_P. Here we have considered symbol classes as elements of $K(TX)$ through an identification of TX with the cotangent bundle of X by means of an adequate Riemann metric (see [1]). Now suppose $\dim_{\boldsymbol{R}} X = 2n$, n integer, and fix a K-invariant inner product on the oriented K-module V. We then have a representation

$$\rho : K \to SO(2n).$$

Denoting by $H_K^*(\boldsymbol{Q})$, $H_{SO(2n)}^*(\boldsymbol{Q})$ the rings of characteristic classes of K, $SO(2n)$ with coefficient field $\boldsymbol{Q}$ of the rational numbers, we have the induced homomorphism

$$\rho^* : H_{SO(2n)}^*(\boldsymbol{Q}) \to H_K^*(\boldsymbol{Q}).$$

For $\rho^*(e) \in H_K^*(\boldsymbol{Q})$ where $e \in H_{SO(2n)}^*(\boldsymbol{Q})$ denotes the Euler class, one can see that

$$\rho^*(e) \neq 0$$

if a maximal torus of K fixes no non-zero vector in V. Let $E^0, \cdots, E^l$ be complex K-modules, $\mathcal{E}^0, \cdots, \mathcal{E}^l$ the associated vector bundles over X and suppose that

$$0 \to C^\infty(\mathcal{E}^0) \to \cdots \to C^\infty(\mathcal{E}^l) \to 0$$

is an elliptic complex whose symbol class is associated to the K-structure. Denoting by $\operatorname{ch} E^i \in H_K^*(\boldsymbol{Q})$ the Chern character of the K-module E^i, one knows that $\sum_{i=0}^{l} (-1)^i \operatorname{ch} E^i \in H_K^*(\boldsymbol{Q})$ is then divisible by $\rho^*(e) \in H_K^*(\boldsymbol{Q})$ when $\rho^*(e) \neq 0$, and we have a characteristic class

$$\frac{\sum_{i=0}^{l} (-1)^i \operatorname{ch} E^i}{\rho^*(e)} \in H_K^*(\boldsymbol{Q}).$$

According to [1], we shall call the Todd class of $TX \otimes \boldsymbol{C}$ the Index class of X and denote it by $\mathcal{J}(X)$.

We then know

Theorem 5.1 (Atiyah-Singer [1], Proposition 2.17). *Let X be an oriented compact manifold of dimension $2n$ with a K-structure, i.e., there exist a principal K-bundle P over X and a real oriented K-module V such that TX is associated to P by V. Assume that maximal torus of K has no fixed non-zero vector in V. Let*

$E^0, \cdots, E^l$ be complex K-modules, $\mathcal{E}^0, \cdots, \mathcal{E}^l$ the associated vector bundles and suppose

$$0 \to C^\infty(\mathcal{E}^0) \to \cdots \to C^\infty(\mathcal{E}^l) \to 0$$

is an elliptic complex whose symbol class is associated to the K-structure. Then the index of this complex is given by

$$(-1)^n \left\{ \frac{\sum_{i=0}^{l} \mathrm{ch}\, E^i}{\rho^*(e)} (P) \mathcal{J}(X) \right\} [X] ,$$

where $\{\cdot\}[X]$ denotes the evaluation of a cohomology class on the fundamental cycle of X.

Here we shall recall some fundamental results on the expression of characteristic classes by means of differential forms (see, for example, Kobayashi-Nomizu [13], Chapt. XII). For a compact Lie group K, let $\mathfrak{k}_0$ be its Lie algebra, $S(\mathfrak{k}_0^*)^K$ the ring of polynomials on $\mathfrak{k}_0$ invariant under the adjoint action of K. We then have a canonical identification

$$S(\mathfrak{k}_0^*)^K \cong H_K^*(\boldsymbol{R}) ,$$

of $S(\mathfrak{k}_0^*)^K$ with the ring of characteristic classes of K with coefficient field $\boldsymbol{R}$ of the real numbers. Let P be a differentiable principal K-bundle over a differentiable manifold X. If we give a connection θ on P, then one can define the curvature form $\kappa(\theta)$ of θ, which is an exterior differential 2-form on P with values in the Lie algebra $\mathfrak{k}_0$. For $f \in S(\mathfrak{k}_0^*)^K$, $f(\kappa(\theta))$ is an exterior differential form on P, which is moreover closed and projectable to X. Thus we have a de Rham cohomology class

$$[f(\kappa(\theta))] \in H^*(X, \boldsymbol{R}) ,$$

which is known to be independent of the choice of the connection θ. In effect, for $c \in H_K^*(\boldsymbol{R})$ we denote by $f_c \in S(\mathfrak{k}_0^*)^K$ the invariant polynomial by the above identification. Then one knows that

$$c(P) = [f_c(\kappa(\theta))]$$

in the cohomology group $H^*(X, \boldsymbol{R})$. This map $f \mapsto [f(\kappa(\theta))]$ is an algebra homomorphism from $S(\mathfrak{k}_0^*)^K$ into $H^*(X, \boldsymbol{R})$ called the Weil homomorphism. We shall use this expression of characteristic classes.

Let G be a non-compact connected semi-simple Lie group with a faithful representation, K its compact connected subgroup. Assume, as in the previous sections, that G has a Cartan subgroup contained in K. Let Γ be a discrete subgroup of G and suppose that Γ acts freely and properly discontinuously on the homogeneous space $X = G/K$, and the quotient $\Gamma\backslash G$ is compact (the existence of such a group is guaranteed by Borel [2]). Thus we have a compact locally homogeneous manifold ${}_\Gamma X=\Gamma\backslash X$, and a principal K-bundle ${}_\Gamma P=\Gamma\backslash G$ over ${}_\Gamma X$. Let $V^0,\cdots, V^l$ be complex K-modules, $\mathcal{V}^0,\cdots,\mathcal{V}^l$ the associated homogeneous vector bundles over X. For a G-invariant differential operator

$$\mathcal{D}^i: C^\infty(\mathcal{V}^i) \to C^\infty(\mathcal{V}^{i+1}),$$

we have a differential operator

$${}_\Gamma\mathcal{D}^i: C^\infty({}_\Gamma\mathcal{V}^i) \to C^\infty({}_\Gamma\mathcal{V}^{i+1})$$

over ${}_\Gamma X$, where ${}_\Gamma\mathcal{V}^i$ is a vector bundle over ${}_\Gamma X$, obtained by the quotient of $\mathcal{V}^i$ by Γ. We notice that ${}_\Gamma\mathcal{V}^i$ is also regarded as the associated vector bundle to a principal K-bundle ${}_\Gamma P$ by the K-module V^i. When a differential complex

$$0 \to C^\infty({}_\Gamma\mathcal{V}^0) \to\cdots\to C^\infty({}_\Gamma\mathcal{V}^l) \to 0$$

over ${}_\Gamma X$ is given by the above procedure out of G-invariant operators from $C^\infty(\mathcal{V}^i)$ into $C^\infty(\mathcal{V}^{i+1})$ for each i, we shall call this complex an *invariant* differential complex over ${}_\Gamma X$.

We now denote by M the compact Lie group which is dual to the non-compact semi-simple group G (see Helgason [9]). Then one can consider K also as a subgroup of M. Thus we get the compact homogeneous space $\hat{X}=M/K$, so to say "a dual compact form" corresponding to X. For K-modules $V^i,\cdots, V^l$, we denote by $\hat{\mathcal{V}}^0,\cdots,\hat{\mathcal{V}}^l$ the associated homogeneous vector bundles over the compact form $\hat{X}$. A differential complex

$$0 \to C^\infty(\hat{\mathcal{V}}^0) \to\cdots\to C^\infty(\hat{\mathcal{V}}^l) \to 0$$

is then called *invariant* when each differential operator $C^\infty(\hat{\mathcal{V}}^i)\to C^\infty(\hat{\mathcal{V}}^{i+1})$ is M-invariant.

Under these situations, we obtain the following analogy to Hirzebruch's proportional principle.

Theorem 5.2. *Let $V^0,\cdots, V^l$ be K-modules and suppose that there associate, as above, two invariant elliptic complexes*

$$0 \to C^\infty({}_\Gamma\mathcal{V}^0) \to\cdots\to C^\infty({}_\Gamma\mathcal{V}^l)\to 0 \quad \textit{over } {}_\Gamma X$$

and

$$0 \to C^\infty(\hat{\mathcal{V}}^0) \to\cdots\to C^\infty(\hat{\mathcal{V}}^l) \to 0 \quad \textit{over } \hat{X},$$

${}_\Gamma X$, $\hat{X}$ being as before. Then, as for the indices $\chi({}_\Gamma\mathcal{V}^*)$, $\chi(\hat{\mathcal{V}}^*)$ of these elliptic complexes, we have

$$\chi({}_\Gamma\mathcal{V}^*) = \frac{E({}_\Gamma X)}{E(\hat{X})}\chi(\hat{\mathcal{V}}^*),$$

where $E({}_{\Gamma}X)$, $E(\hat{X})$ denote the Euler numbers of ${}_{\Gamma}X$, $\hat{X}$ respectively.

Proof. Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ be the Lie algebras of G, K,

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

the decomposition in §1. Choose a maximal compact subgroup $\tilde{K} \supset K$ of G and denote its Lie algebra by $\tilde{\mathfrak{k}}_0$. Then we have a Cartan decomposition

$$\mathfrak{g}_0 = \tilde{\mathfrak{k}}_0 \oplus \tilde{\mathfrak{p}}_0 ,$$

and we can regard that

$$\mathfrak{m}_0 = \tilde{\mathfrak{k}}_0 \oplus \sqrt{-1}\, \tilde{\mathfrak{p}}_0 ,$$

where $\mathfrak{m}_0$ denotes the Lie algebra of M. We easily see that

$$\mathfrak{p}_0 = \tilde{\mathfrak{k}}_0 \cap \mathfrak{p}_0 \oplus \tilde{\mathfrak{p}}_0$$

and when we put

$$\hat{\mathfrak{p}}_0 = \tilde{\mathfrak{k}}_0 \cap \mathfrak{p}_0 \oplus \sqrt{-1}\, \tilde{\mathfrak{p}}_0,$$

we have

$$\mathfrak{m}_0 = \mathfrak{k}_0 \oplus \hat{\mathfrak{p}}_0 .$$

Denoting by ι the Cartan involution of the above Cartan decomposition $\mathfrak{g}_0 = \tilde{\mathfrak{k}}_0 \oplus \tilde{\mathfrak{p}}_0$, we define a K-invariant inner product on $\mathfrak{p}_0$ by

$$(X, Y) = -B(X, \iota Y) \quad \text{for } X, Y \in \mathfrak{p}_0 ,$$

where B denotes the Killing form of $\mathfrak{g}_0$. We also define a K-invariant inner product on $\hat{\mathfrak{p}}_0$ by

$$(X, Y) = -B(X, Y) \quad \text{for } X, Y \in \hat{\mathfrak{p}}_0 ,$$

where B also denotes the Killing form of $\mathfrak{m}_0$. If we decompose $X \in \mathfrak{p}_0$ as

$$X = X_1 \oplus X_2 , \quad X_1 \in \tilde{\mathfrak{k}}_0 \cap \mathfrak{p}_0 , \quad X_2 \in \tilde{\mathfrak{p}}_0 ,$$

then the map $X_1 \oplus X_2 \mapsto X_1 \oplus \sqrt{-1}\, X_2$ is an isometry of $\mathfrak{p}_0$ onto $\hat{\mathfrak{p}}_0$ as K-modules. We fix the orientations of $\mathfrak{p}_0$ and $\hat{\mathfrak{p}}_0$ such that this map is an isomorphism as oriented K-modules, which determine the orientations of ${}_{\Gamma}X$ and $\hat{X}$. Then, for the two orthogonal representations

$$\rho : K \to SO(2n) \quad \text{on } \mathfrak{p}_0$$

and

$$\hat{\rho} : K \to SO(2n) \quad \text{on } \hat{\mathfrak{p}}_0 ,$$

we have $\rho^*(e) = \hat{\rho}^*(e)$ in $H_K^*(\boldsymbol{Q})$, where ρ^*, $\hat{\rho}^*$ are induced by ρ, $\hat{\rho}$ and e is the

Euler class in $H^*_{SO(2n)}(\boldsymbol{Q})$.

We shall introduce invariant connections on the principal K-bundles ${}_\Gamma P=\Gamma\backslash G$ over ${}_\Gamma X$ and $\hat{P}=M$ over $\hat{X}$ as follows. Let

$$\theta\colon \mathfrak{g}_0 \to \mathfrak{k}_0$$

and

$$\hat{\theta}\colon \mathfrak{m}_0 \to \mathfrak{k}_0$$

be the projection such that $\operatorname{Ker}\theta=\mathfrak{p}_0$ and $\operatorname{Ker}\hat{\theta}=\hat{\mathfrak{p}}_0$. Since θ, $\hat{\theta}$ are K-module homomorphisms, the projections θ, $\hat{\theta}$ give the invariant connections on ${}_\Gamma P$, $\hat{P}$ respectively. In order to give explicit formulae of the curvature forms, we extend the above connections to those with complex coefficients, i.e., both θ and $\hat{\theta}$ extends to the projection $\mathfrak{g}\to\mathfrak{k}$ whose kernel is $\mathfrak{p}$, where $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ are the complexifications of $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{p}_0$ and $\mathfrak{g}$ is identified with the complexification of $\mathfrak{m}_0$. We denote by the same letters θ, $\hat{\theta}$ the connections with complex coefficients on ${}_\Gamma P$, $\hat{P}$ respectively defined by the above projection. Choose a base $\{H_i, E_\alpha\}$ of $\mathfrak{g}$ such that $\{H_i\}$ forms a base of the fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$, E_α a root vector for a root $\alpha\in\Delta$, and put

$$[E_\alpha, E_\beta] = N_{\alpha,\beta}E_\gamma \quad \text{if } \gamma=\alpha+\beta\in\Delta\,,$$
$$[E_\alpha, E_{-\alpha}] = H_\alpha\in\mathfrak{h}\,.$$

The dual base of $\{H_i, E_\alpha\}$ then gives complex valued invariant 1–forms on the Lie groups G, M. Here we notice that an invariant 1–form on G determines the 1–form on ${}_\Gamma P=\Gamma\backslash G$. Denote by $\{\omega_\alpha\}_{\alpha\in\Delta}$, $\{\hat{\omega}_\alpha\}_{\alpha\in\Delta}$ the systems of complex valued 1–forms on ${}_\Gamma P=\Gamma\backslash G$, $\hat{P}=M$ such hat ω_α, $\hat{\omega}_\alpha$ are determined by the dual of E_α. Then we see easily from the Cartan structural equation that the curvature forms $\kappa(\theta)$, $\kappa(\hat{\theta})$ on ${}_\Gamma P$, $\hat{P}$ are, as $\mathfrak{k}$-valued 2–forms,

$$\kappa(\theta) = \sum_{\alpha\in\Delta_{\mathfrak{p}}^+} H_\alpha\cdot\omega_\alpha\wedge\omega_{-\alpha} + \sum_{\substack{\alpha+\beta\in\Delta_{\mathfrak{k}}\\ \alpha,\beta\in\Delta_{\mathfrak{p}},\,\alpha<\beta}} N_{\alpha,\beta}E_{\alpha+\beta}\cdot\omega_\alpha\wedge\omega_\beta$$

$$\kappa(\hat{\theta}) = \sum_{\alpha\in\Delta_{\mathfrak{p}}^+} H_\alpha\cdot\hat{\omega}_\alpha\wedge\hat{\omega}_{-\alpha} + \sum_{\substack{\alpha+\beta\in\Delta_{\mathfrak{k}}\\ \alpha,\beta\in\Delta_{\mathfrak{p}},\,\alpha<\beta}} N_{\alpha,\beta}E_{\alpha+\beta}\cdot\hat{\omega}_\alpha\wedge\hat{\omega}_\beta$$

where $\Delta_{\mathfrak{p}}$, $\Delta_{\mathfrak{k}}$ denote the subsets of Δ defined in §1.

Thus for a K-invariant polynomial $f\in S(\mathfrak{k}^*)^K$, if we have

$$f(\kappa(\theta)) = \sum_{\substack{\alpha_1,\cdots,\alpha_k\in\Delta_{\mathfrak{p}}\\ \alpha_1<\cdots<\alpha_k}} c_{\alpha_1\cdots\alpha_k}\omega_{\alpha_1}\wedge\cdots\wedge\omega_{\alpha_k}$$

where $c_{\alpha_1\cdots\alpha_k}\in\boldsymbol{C}$, then

$$f(\kappa(\hat{\theta})) = \sum_{\substack{\alpha_1,\cdots,\alpha_k\in\Delta_{\mathfrak{p}}\\ \alpha_1<\cdots<\alpha_k}} c_{\alpha_1\cdots\alpha_k}\hat{\omega}_{\alpha_1}\wedge\cdots\wedge\hat{\omega}_{\alpha_k}\,.$$

That is, $f(\kappa(\theta))$ is transferred to $f(\kappa(\hat{\theta}))$ through the substitution ω_α by $\hat{\omega}_\alpha$.

With these preparations, we shall prove the theorem, making use of Theorem 5.1. We notice first that the symbol class of an invariant elliptic complex is associated to the K-structure. In fact, through the identification $\hat{\mathfrak{p}}_0$ with its dual $\hat{\mathfrak{p}}_0^*$, the symbol map for $\xi\in\hat{\mathfrak{p}}_0$ at the origin

$$\sigma_\xi^i : V^i \to V^{i+1}$$

satisfies

$$\sigma_{k\xi}^i = k\sigma_\xi^i k^{-1} \quad \text{for } k\in K$$

by the invariance of the differential operator. On the other hand, the symbol sequence

$$0 \to V^0 \xrightarrow{\sigma_\xi^0} V^1 \xrightarrow{\sigma_\xi^1} \cdots \to V^l \to 0$$

is exact for $\xi\neq 0$ in $\hat{\mathfrak{p}}_0$ from the ellipticity, this sequence represents an element $\sigma(V^*)\in K_K(\hat{\mathfrak{p}}_0)$. We see easily that the symbol class of the invariant elliptic complex

$$0 \to C^\infty(\hat{\mathcal{V}}^0) \to \cdots \to C^\infty(\hat{\mathcal{V}}^l) \to 0$$

is the image of $\sigma(V^*)$ by

$$\alpha_{\hat{P}} : K_K(\hat{\mathfrak{p}}_0) \to K_K(\hat{P}\times\hat{\mathfrak{p}}_0) \simeq K(T\hat{X})$$

defined at the beginning of this section. In case of ${}_\Gamma X$, it is quite similar to see so. Next, the maximal torus H of K has no non-zero fixed vector in $\mathfrak{p}_0$ and $\hat{\mathfrak{p}}_0$. In fact, if $X\in\mathfrak{p}_0$ or $\hat{\mathfrak{p}}_0$, and $hX=X$ for every $h\in H$, then X is contained in the centralizer of the Cartan subalgebra $\mathfrak{h}$. Hence $X\in\mathfrak{h}$, which implies $X=0$. Therefore the conditions of Theorem 5.1 are satisfied in our case.

Now the assumption being as in the theorem, the characteristic classes

$$\frac{\sum_{i=0}^{l}(-1)^i \operatorname{ch} V^i}{\rho^*(e)}, \quad \frac{\sum_{i=0}^{l}(-1)^i \operatorname{ch} V^i}{\hat{\rho}^*(e)} \quad \text{in } H_K^*(\boldsymbol{Q})$$

coincide for the oriented manifolds ${}_\Gamma X$, $\hat{X}$ since $\rho^*(e)=\hat{\rho}^*(e)$ as noticed before. Denote this characteristic class by $v\in H_K^*(\boldsymbol{Q})$ and through the extension of the coefficient from $\boldsymbol{Q}$ to $\boldsymbol{C}$, consider v as an element of $S(\mathfrak{k}^*)^K\simeq H_K^*(\boldsymbol{C})$. Then we have

$$v({}_\Gamma P) = [v(\kappa(\theta))]\in H^*({}_\Gamma X, \boldsymbol{C})$$
$$v(P) = [v(\kappa(\hat{\theta}))]\in H^*(\hat{X}, \boldsymbol{C}),$$

where in the left-hand sides v is considered as in $H_K^*(\boldsymbol{C})$, in the right-hand sides in $S(\mathfrak{k}^*)^K$. On the other hand, though the Index class $\mathcal{J}$ is, by definition, in $H^*_{O(2n)}(\boldsymbol{Q})$, by the reduction of $O(2n)$ to K we get

$$\mathcal{J}_K = \rho^*\mathcal{J} \in H^*_K(\boldsymbol{Q}),$$

where ρ is the composition $K \to SO(2n) \to O(2n)$. Therefore one can use the similar expression of the Index class by the de Rham cohomology class as follows. When we denote by $i \in S(\mathfrak{k}^*)^K$ the corresponding element to $\mathcal{J}_K \in H^*_K(\boldsymbol{C})$, we then have

$$\mathcal{J}({}_\Gamma X) = [i(\kappa(\theta))] \in H^*({}_\Gamma X, \boldsymbol{C})$$
$$\mathcal{J}(\hat{X}) = [i(\kappa(\hat{\theta}))] \in H^*(\hat{X}, \boldsymbol{C}).$$

Compare the differential forms $v(\kappa(\theta))\cdot i(\kappa(\theta))$ and $v(\kappa(\hat{\theta}))\cdot i(\kappa(\hat{\theta}))$. Then, from the fact noticed before, if

$$v(\kappa(\theta))\cdot i(\kappa(\theta)) = \sum_{\substack{\alpha_1,\cdots,\alpha_k \in \Delta_{\mathfrak{p}} \\ \alpha_1 > \cdots > \alpha_k}} c_{\alpha_1\cdots\alpha_k}\omega_{\alpha_1}\wedge\cdots\wedge\omega_{\alpha_k}$$

where $c_{\alpha_1\cdots\alpha_k} \in \mathbf{C}$, then

$$v(\kappa(\hat{\theta}))\cdot i(\kappa(\hat{\theta})) = \sum_{\substack{\alpha_1,\cdots,\alpha_k \in \Delta_{\mathfrak{p}} \\ \alpha_1 > \cdots > \alpha_k}} c_{\alpha_1\cdots\alpha_k \in}\hat{\omega}_{\alpha_1}\wedge\cdots\wedge\hat{\omega}_{\alpha_k}.$$

Now we denote by $c \in \boldsymbol{C}$ the common coefficients of the highest terms of the above differential forms and put

$$\omega = \omega_{\alpha_1}\wedge\cdots\wedge\omega_{\alpha_n}\wedge\omega_{-\alpha_n}\wedge\cdots\wedge\omega_{-\alpha_1}$$
$$\hat{\omega} = \hat{\omega}_{\alpha_1}\wedge\cdots\wedge\hat{\omega}_{\alpha_n}\wedge\hat{\omega}_{-\alpha_n}\wedge\cdots\wedge\hat{\omega}_{-\alpha_1}$$

where $\{\alpha_1\cdots,\alpha_n\} = \Delta_{\mathfrak{p}}^+$ and $\alpha_1 > \cdots > \alpha_n$. Then by Theorem 5.1 we have

$$\chi({}_\Gamma\mathcal{CV}^*) = (-1)^n\int_{{}_\Gamma X} v(\kappa(\theta))\cdot i(\kappa(\theta)) = (-1)^n c\int_{{}_\Gamma X}\omega$$
$$\chi(\hat{\mathcal{CV}}^*) = (-1)^n\int_{\hat{X}} v(\kappa(\hat{\theta}))\cdot i(\kappa(\hat{\theta})) = (-1)^n c\int_{\hat{X}}\hat{\omega}.$$

Therefore we have

$$\chi({}_\Gamma\mathcal{CV}^*) = \left(\int_{{}_\Gamma X}\omega\Big/\int_{\hat{X}}\hat{\omega}\right)\chi(\hat{\mathcal{CV}}^*),$$

where $\int_{{}_\Gamma X}\omega\Big/\int_{\hat{X}}\hat{\omega}$ is independent of the elliptic complexes. If we choose the de Rham complexes for $\hat{\mathcal{CV}}^*$, ${}_\Gamma\mathcal{CV}^*$, then we have

$$\chi(\hat{\mathcal{CV}}^*) = E(\hat{X})$$
$$\chi({}_\Gamma\mathcal{CV}^*) = E({}_\Gamma X).$$

We know by Hopf-Samelson [11] that

$$E(\hat{X}) = \#|W_M/W_K|>0$$

where W_M, W_K denote the Weyl groups of M, K. Hence we have

$$\int_{\Gamma X}\omega\Big/\int_{\hat{X}}\hat{X} = E(_\Gamma X)/E(\hat{X}),$$

which implies Theorem 5.2.

The following two corollaries*) are also analogous to Hirzebruch [10].

Corollary 1. *Let m be the number of the positive non-compact roots**) of G. Then the sign of the Euler number $E(_\Gamma X)$ of the compact locally homogeneous spaces $_\Gamma X$ coincides with $(-1)^m$. In particular, notice that if X is symmetric, then $m=\frac{1}{2}\dim_R X$.*

Corollary 2. *Let Γ_1, Γ_2 be two discrete subgroups of G such that $_{\Gamma_1}X=\Gamma_1\backslash X$, $_{\Gamma_2}X=\Gamma_2\backslash X$ are compact manifolds, $_{\Gamma_1}\mathcal{V}^*$, $_{\Gamma_2}\mathcal{V}^*$ the invariant elliptic complexes over $_{\Gamma_1}X$, $_{\Gamma_2}X$ associated to the same family of K-modules $V^1,\cdots,V^l$. Denoting by v_1, v_2 the volumes of $_{\Gamma_1}X$, $_{\Gamma_2}X$ with respect to the same invariant volume element of X, we have*

$$\chi(_{\Gamma_1}\mathcal{V}^*)=\frac{v_1}{v_2}\chi(_{\Gamma_2}\mathcal{V}^*).$$

Proof of Corollaries 1 and 2. The notation being as in the proof of Theorem 5.2, let σ be the conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$. One can then choose the root vectors of $\mathfrak{g}$ such that $\sigma E_\alpha=-E_{-\alpha}$ if α is a compact root, $\sigma E_\alpha=E_{-\alpha}$ if α is a non-compact root (Helgason [9]). Therefore we may choose a base of $\mathfrak{p}_0$ consisting of the vectors such as

$$X_\alpha=\sqrt{-1}\,(E_\alpha+E_{-\alpha}),\quad Y_\alpha=E_\alpha-E_{-\alpha}$$

where α runs over compact roots,

$$X_\alpha=E_\alpha+E_{-\alpha},\quad Y_\alpha=\sqrt{-1}\,(E_\alpha-E_{-\alpha})$$

where α runs over non-compact roots. The base of $\hat{\mathfrak{p}}_0$ then consists of

*) As for the proportionality principle for the Euler number, see also S. Bochner, *Euler-Poincaré characteristic for locally homogeneous and complex spaces*, Ann. of Math. **51** (1950), 241–261. This reference was communicated by Prof. M. Ise.

**) Let (G, K) be a symmetric pair of inner type which is of non-compact type, and $\Delta\mathfrak{k}$, $\Delta\mathfrak{p}$ the sets of roots determined by the *symmetric* pair $(\mathfrak{g}_0, \mathfrak{k}_0)$. An element of $\Delta\mathfrak{k}$ (resp. $\Delta\mathfrak{p}$) is then said to be a *compact* (resp. *noncompact*) root of G.

X_α, Y_α where α runs over compact roots, $\sqrt{-1}\,X_\alpha$, $\sqrt{-1}\,Y_\alpha$ where α runs over non-compact roots. Let φ_α, ψ_α be the invariant 1-forms on G dual to X_α, Y_α for $\alpha\in\Delta_{\mathfrak{p}}^+$, and denote by the same letters the ones on ${}_\Gamma P=\Gamma\backslash G$ determined by φ_α, ψ_α. We then have

$$\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\varphi_\alpha\wedge\psi_\alpha=(-2\sqrt{-1})^n\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\omega_\alpha\wedge\omega_{-\alpha}\,,$$

where ω_α for $\alpha\in\Delta_{\mathfrak{p}}$ is as in the proof of Theorem 5.2. We choose an orientation of ${}_\Gamma X$ such that

$$\int_{{}_\Gamma X}\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\varphi_\alpha\wedge\psi_\alpha>0\,.$$

Denoting by $\hat{\varphi}_\alpha$, $\hat{\psi}_\alpha$ the invariant 1-forms on $\hat{P}=M$ dual to X_α, Y_α or $\sqrt{-1}\,X_\alpha$, $\sqrt{-1}\,Y_\alpha$ accordingly whether α is compact or non-compact, the orientation of $\hat{X}$ should then be chosen such as

$$\int_{\hat{X}}\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\hat{\varphi}_\alpha\wedge\hat{\psi}_\alpha>0\,.$$

Now we have

$$\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\hat{\varphi}_\alpha\wedge\hat{\psi}_\alpha=(-1)^m(-2\sqrt{-1})^n\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\hat{\omega}_\alpha\wedge\hat{\omega}_{-\alpha}\,,$$

where $\hat{\omega}_\alpha$ is as before, m the number of the positive non-compact roots of G. Seeing that

$$\omega=\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\omega_\alpha\wedge\omega_{-\alpha}$$
$$\hat{\omega}=\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\hat{\omega}_\alpha\wedge\hat{\omega}_{-\alpha}\,,$$

we have

$$(-1)^m\Big(\int_{{}_\Gamma X}\omega\Big/\int_{\hat{X}}\hat{\omega}\Big)>0\,,$$

which implies Corollary 1 from the last formula in the proof of Theorem 5.2.

The invariant volume element of X is unique up to scalar multiplications, and therefore we may assume that the volume element of X is $\prod_{\alpha\in\Delta_{\mathfrak{p}}^+}\varphi_\alpha\wedge\psi_\alpha$. Hence Corollary 2 is straightforward from the proof of Theorem 5.2.

We shall now apply Theorem 5.2 to the invariant elliptic complex ($\flat$) or ($\sharp$) constructed in §3. Given an irreducible K-module V_λ whose highest (resp. lowest) weight λ (resp. $\lambda+2\rho_{\mathfrak{k}}$) satisfies the condition ($\flat$) (resp. ($\sharp$)), we have the invariant elliptic ($\flat$) (resp. ($\sharp$))-complex on ${}_\Gamma X$

$$0 \to C^\infty({}_\Gamma\mathcal{V}_\lambda) \to \cdots \to C^\infty({}_\Gamma\mathcal{V}_\lambda^n) \to 0$$

if X is symmetric and the order is admissible,

$$0 \to C^\infty({}_\Gamma\mathcal{W}_\lambda^e) \to C^\infty({}_\Gamma\mathcal{W}_\lambda^o) \to 0$$

otherwise.

As for the index of this elliptic complex, we have

Theorem 5.3. *Let $\chi({}_\Gamma\mathcal{V}_\lambda^*)$ be the index of the* ($\flat$) *(resp.* ($\sharp$)*)-complex on* ${}_\Gamma X$. *If* $\lambda+\rho\in\Lambda$ *is singular with respect to* $\mathfrak{g}$, *then*

$$\chi({}_\Gamma\mathcal{V}_\lambda^*) = 0\,.$$

If $\lambda+\rho\in\Lambda$ *is regular with respect to* $\mathfrak{g}$, *then*

$$\chi({}_\Gamma\mathcal{V}_\lambda^*) = c_\Gamma \frac{\prod_{\alpha\in\Delta^+}(\lambda+\rho, \alpha)}{\prod_{\alpha\in\Delta^+}(\rho, \alpha)}$$

$$\left(\textit{resp. } (-1)^s c_\Gamma \frac{\prod_{\alpha\in\Delta^+}(\lambda+\rho, \alpha)}{\prod_{\alpha\in\Delta^+}(\rho, \alpha)}\right)$$

where

$$c_\Gamma = \frac{E({}_\Gamma X)}{E(\hat{X})} = \frac{E({}_\Gamma X)}{{}^\sharp|W_M/W_K|}$$

and s is the number of elements of the set $\Delta_\mathfrak{k}^+$.

Proof. For $w\in W_M$, we see easily that

$$\prod_{\alpha\in\Delta^+}(\lambda+\rho, w^{-1}\alpha) = (-1)^{n(w)} \prod_{\alpha\in\Delta^+}(\lambda+\rho, \alpha)\,.$$

From this fact together with H. Weyl's dimension formula, Theorem 5.3 is clear in view of Theorems 4.2 and 5.2.

6. Some remarks on Schmid's results

In Theorem 3.1 we have constructed an elliptic complex associated to a homogeneous vector bundle over a symmetric space of inner type. This elliptic complex is determined by an admissible lexicographical order of the root system (Definition 3, §3). But the invariant first order differential operator $\mathcal{D}^0$ is defined for an *arbitrary* lexicographical order in §3 and it is elliptic under the condition in Definition 1 in §2. The works of W. Schmid [16], [17] are greatly indebted to the ellipticity of this operator $\mathcal{D}^0$ in ($\sharp$)-case, and he proved the ellipticity of $\mathcal{D}^0$ under a condition stronger than our condition ($\sharp$). Therefore, most of his results are improved and we shall here illustrate it.

We shall first recall some of his results in [16], [17]. Let G be a non-compact connected semi-simple Lie group with a compact Cartan subgroup H, which will be fixed once and for all. We fix a *maximal* compact subgroup K containing H. We denote by $\mathfrak{g}_0$, $\mathfrak{k}_0$, $\mathfrak{h}_0$ the Lie algebras of G, K, H, and by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{h}$ their complexifications. Consider the root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and fix an *arbitrary* order on Δ from now on. One can then endow the manifold $D=G/H$ with a G-invariant complex structure such that the holomorphic tangent space at the origin eH corresponds to $\sum_{\alpha\in\Delta^+}\mathfrak{g}^{\alpha}$, where Δ^+ denotes the set of positive roots, $\mathfrak{g}^{\alpha}$ the root vector space for $\alpha\in\Delta^+$ as before. Denote by $\Delta_{\mathfrak{k}}^+$ the set of positive compact roots, by s the number of elements of $\Delta_{\mathfrak{k}}^+$. Then the complex manifold D is $(s+1)$-complete in the sense of Andreotti-Grauert ([17], Theorem 1).

As in § 1, let Λ be the character group of H identified with a lattice in the dual space of $\mathfrak{h}_R=\sqrt{-1}\,\mathfrak{h}_0$. For $\lambda\in\Lambda$, we denote by L_λ the associated homogeneous line bundle on D, which has a structure of a holomorphic line bundle. If we denote by $H^q(D, L_\lambda)$ the q-th cohomology group with coefficients in the sheaf of germs of holomorphic sections of L_λ, then $H^q(D, L_\lambda)$ is a G-module with respect to the left translations. Under these circumstances, he obtained

Theorem 6.1 (Schmid [16], Theorem 6.1). *There exists a positive number $b>0$ such that the following holds: if $(\lambda, \alpha)<-b$ for every $\alpha\in\Delta_{\mathfrak{k}}^+$, then*

i) *$H^q(D, L_\lambda)=0$ for every $q\neq s$, and $H^s(D, L_\lambda)$ is an infinite dimensional Fréchet space on which G acts continuously. Moreover every vector in $H^s(D, L_\lambda)$ is a differentiable vector for the action of G.*

ii) *There exist closed K-invariant subspaces $\mathcal{H}_l$ $(l=0, 1, 2, \cdots)$ of $H^s(D, L_\lambda)$ such that*

$$\cdots\subset\mathcal{H}_l\subset\cdots\subset\mathcal{H}_1\subset\mathcal{H}_0=H^s(D, L_\lambda)\,,$$

each $\mathcal{H}_l$ is of finite codimension in $H^s(D, L_\lambda)$, and $X.\mathcal{H}_{l+1}\subset\mathcal{H}_l$ for every $X\in\mathfrak{g}_0$, where $X.\mathcal{H}_{l+1}$ denotes the image the infinitesimal action of X to the vectors in $\mathcal{H}_{l+1}$. Moreover, for $\mu\in\Lambda$ dominant with respect to K, the irreducible K-modules with highest weight μ occurs in the finite dimensional K-module $\mathcal{H}_l/\mathcal{H}_{l+1}$ with multiplicity

$$(-1)^s\sum_{w\in W_K}(-1)^{n(w)}Q_l(\lambda+\rho_{\mathfrak{k}}-w(\mu+\rho_{\mathfrak{k}}))\,.$$

Here for $\nu\in\Lambda$, $Q_l(\nu)$ denotes the number of distinct ways in which ν can be expressed precisely as a sum of positive non-compact roots, and W_K the Weyl group of K,

$$\rho_{\mathfrak{k}}=\frac{1}{2}\sum_{\alpha\in\Delta_{\mathfrak{k}}^+}\alpha\,.$$

REMARK 1. According to his proof in [16], the number $b>0$ in Theorem

6.1 must be chosen larger than the number $c^{\#}$ in the condition ($\#$) of Definition 1, §2.

We now consider the symmetric space (of inner type) $X=G/K$ and the irreducible K-module V_λ with lowest weight $\lambda+2\rho_{\mathfrak{k}}$. If λ satisfies the condition ($\#$), in §3 we obtain the invariant differential operator

$$\mathcal{D}^0: C^\infty(\mathcal{V}_\lambda) \to C^\infty(\mathcal{V}_\lambda^1),$$

which is elliptic by Lemma 3.1 whether the order is admissible or not. Here the ellipticity means that for every non-zero cotangent vector of X, the symbol map is injective. This differential operator $\mathcal{D}^0$ coincides with Schmid's operator $\mathcal{D}$ in [16], [17] and *we put* $\mathcal{D}=\mathcal{D}^0$ *hereafter*. He showed the ellipticity of $\mathcal{D}$ by direct computations of estimate under the stronger condition than our condition ($\#$) (see [16], Lemma 7.2). It reads that there exists a positive number $b'>0$ such that if $(\lambda, \alpha)<-b'$ for every $\alpha\in\Delta^+$, then $\mathcal{D}$ is elliptic. In comparison with this, we thus have the following key lemma.

Lemma 6.1. *Let $c^{\#}$ be the number as in Definition* 1, §2. *If $(\lambda, \alpha)<-c^{\#}$ for every $\alpha\in\Delta_{\mathfrak{k}}^+$, then Schmid's operator*

$$\mathcal{D}: C^\infty(\mathcal{V}_\lambda) \to C^\infty(\mathcal{V}_\lambda^1)$$

is elliptic.

Now we denote by $\mathcal{F}_\lambda$ the null space of $\mathcal{D}$, which is a Fréchet space as a closed subspace of C^∞ $(\mathcal{V}_\lambda)$ and is a G-module. The assumption being the same as in Theorem 6.1, $H^s(D, L_\lambda)$ is topologically isomorphic to $\mathcal{F}_\lambda$ as G-module ([16], Lemma 7.1). By this isomorphism, the subspace $\mathcal{H}_l$ in Theorem 6.1 corresponds to the subspace of $\mathcal{F}_\lambda$ consisting of the sections which vanish to order l at the origin $eK\in X$. Therefore, when $\mathcal{D}$ is elliptic, we see easily that

$$\bigcap_{l=0}^{\infty} \mathcal{H}_l = \{0\}$$

by the regularity theorem. By Lemma 6.1 and Remark 1 to Theorem 6.1, we see that $\mathcal{D}$ is elliptic under the same condition in Theorem 6.1. The fact that $\bigcap_{l=0}^{\infty} \mathcal{H}_l=\{0\}$ deduces the next theorem as Schmid worked in [16], §6.

Theorem 6.2. *Under the same condition as in Theorem* 6.1, *the following holds. As for the restriction to K of the G-module $H^s(D, L_\lambda)$, the irreducible K-module with highest weight $\mu\in\Lambda$ occurs with finite multiplicity*

$$(-1)^s \sum_{w\in W_K} \sum_{l=0}^{\infty} (-1)^{n(w)} Q_l(\lambda+\rho_{\mathfrak{t}}-w(\mu+\rho_{\mathfrak{t}})).$$

There exists a non-zero K-invariant, K-irreducible subspace of $H^s(D, L_\lambda)$ which is contained in every non-zero, closed, G-invariant subspace.

REMARK 2. Schmid's condition in this theorem is stronger than ours as stated before Lemma 6.1. Notice that our condition is related only to the compact positive roots but his to all the positive roots. Thus it is not too much to say that this is one of the improvements.

Next, we shall refer to the construction of unitary representations of G by means of the null space of $\mathcal{D}$. Introduce a G-invariant hermitian metric $(\, , \,)$ on $\mathcal{V}_\lambda$, induced by a K-invariant hermitian inner product on V_λ, and a G-invariant volume element dv on X. We denote by $\mathfrak{H}_\lambda$ the space of sections $s\in\mathcal{F}_\lambda$ such that $\int_X (s, s)dv<\infty$. Then for $s, s'\in\mathfrak{H}_\lambda$

$$\langle s, s'\rangle = \int_X (s, s')dv$$

defines an inner product on $\mathfrak{H}_\lambda$. If $\mathcal{D}$ is elliptic, then $\mathfrak{H}_\lambda$ is a Hilbert space by the regularity theorem. When $\mathfrak{H}_\lambda$ is non-trivial, the G-action on $\mathfrak{H}_\lambda$ gives a unitary representation of G.

From now on, we shall particularly consider the generalized Lorentz groups with compact Cartan subgroups. That is, let G be the identity component of $SO(2n, 1)$, or its two-fold universal covering group (we assume $n\geqslant 2$). The rank of G and half a dimension of the symmetric space $X=G/K$ (a hyperboloid of one sheet) are then n. For the root system of $\mathfrak{g}$, we know that one can choose the base $\{e_1, \cdots, e_n\}$ of $\mathfrak{h}_R^*$ such that

$$\begin{aligned}
(e_i, e_j) &= \frac{1}{2(2n-1)}\delta_{ij} \\
\Delta_{\mathfrak{p}} &= \{e_i, -e_i \,|\, 1\leqslant i\leqslant n\} \\
\Delta_{\mathfrak{k}} &= \{\pm(e_i\pm e_j) \,|\, 1\leqslant i<j\leqslant n\}\,.
\end{aligned}$$

With respect to this base the Weyl groups W of $\mathfrak{g}$ consists of the transformations $e_i\mapsto\varepsilon_i e_{\pi(i)}$ where π denotes a permutation in $\{1, \cdots, n\}$ and $\varepsilon_i=\pm 1$. The Weyl group W_K of $\mathfrak{k}$ consists of the elements of W such that $\prod_{i=1}^n \varepsilon_i=1$, which shows that the number of the cosets of W/W_K is 2. Moreover it is known that all the elements of W are generated by the automorphisms of G. By means of these explicitations, it is easy to see that any order of Δ is admissible in this case. Therefore any Schmid's operator $\mathcal{D}$ is the first term of the ($\sharp$)-complex in Theorem 3.1. The following lemma is included implicitely in Lemma 8.2 [16], but we shall give the proof for the sake of completeness.

Lemma 6.2. *Let* $\lambda\in\Lambda$ *and* $\rho=\frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. *If* $-(\lambda+\rho)$ *is regular and dominant with respect to* $\mathfrak{g}$, *i.e.*, $(\lambda+\rho, \alpha)<0$ *for every* $\alpha\in\Delta^+$, *then* λ *satisfies* *the condition (#) in Definition* 1, §2. *In particular, we then have the elliptic operator* $\mathcal{D}$.

Proof. In view of the above remark it suffices to show the statement for some fixed order of Δ. Hence we introduce an order on $\mathfrak{h}_R^*$ so that

$$\Delta^+ = \{e_i,\ e_j \pm e_k \,|\, 1 \leqslant i \leqslant n,\ 1 \leqslant k < j \leqslant n\},$$

and prove the lemma for this order.

Let $\lambda = \sum_{i=1}^{n} m_i e_i \in \Lambda$. Then it holds $2(\alpha, \lambda)/(\alpha, \alpha)^{-1} \in \boldsymbol{Z}$ for every $\alpha \in \Delta_{\mathfrak{k}}$, which implies that either all m_i must be integral or all m_i must be strictly semi-integral. Moreover, $-(\lambda+\rho)$ is regular and dominant if and only if $m_1 \leqslant 1$, $m_2 \leqslant m_1 - 2$, $m_3 \leqslant m_2 - 2, \cdots, m_n \leqslant m_{n-1} - 2$. Using the description of the elements of $\Delta_{\mathfrak{k}}^+$, $\Delta_{\mathfrak{p}}^+$ by e_i, one can then easily check that this satisfies the condition (#), i.e.,

$$(\lambda + \rho_{\mathfrak{k}} + \beta_1 + \cdots + \beta_q,\ \alpha) \leqslant 0 \qquad \text{or all} \quad \alpha \in \Delta_{\mathfrak{k}}^+$$

whenever $\beta_1, \cdots, \beta_q \in \Delta_{\mathfrak{p}}^+$ are distinct.

According to the proof of Theorem 8.1 in [16], Schmid showed the following facts. Let $\lambda \in \Lambda$ and assume that $-(\lambda+\rho)$ is regular and dominant with respect to $\mathfrak{g}$. *If the differential operator* $\mathcal{D}$ *is elliptic, then the action of* G *on the Hilbert space* $\mathfrak{H}_\lambda$ *determines an irreducible unitary representation which belongs to the discrete series, and its character is* $\Theta_{\lambda+\rho}$ *in the sense of Harish-Chandra.* Moreover, in the restriction of this representation to the maximal compact subgroup K, the irreducible K-module with the highest weght $\mu \in \Lambda$, occurs with multiplicity

$$\sum_{w \in W_K} \sum_{l=0}^{\infty} (-1)^{n(w)} Q_l(\lambda + \rho_{\mathfrak{k}} - w(\mu + \rho_{\mathfrak{k}})),$$

where the notation is as in Theorem 6.1. Combining with these results of Schmid and Lemma 6.2, we have

Theorem 6.3. *Let* G *be the identity component of the generalized Lorentz group* $SO(2n, 1)$, *or its two-fold universal covering group, an order on the root system* Δ *fixed. For any* $\lambda \in \Lambda$ *such that* $(\lambda+\rho, \alpha) < 0$ *for every* $\alpha \in \Delta^+$, *the Hilbert space* $\mathfrak{H}_\lambda$ *constructed above gives an irreducible unitary representation of* G *belonging to the discrete series and its character is* $\Theta_{\lambda+\rho}$. *Moreover as for the restriction of this representation to the maximal compact subgroup* K, *we have the multiplicity formula described above.*

Combining with Harish-Chandra's classification theory of the discrete series, this improvement of Schmid's theorem (Theorem 8.1 [16], Theorem 4 [17]) implies the following result which seems to be rather striking. For $\mu\in\Lambda$ which is regular, we choose an order on the root system Δ so that $(\mu, \alpha)<0$ for every $\alpha\in\Delta^{+}$. Put $\lambda=\mu-\rho$ and consider a unitary K-module V_{λ} whose lowest weight is $\lambda+2\rho_{\mathfrak{k}}$. We then obtain the Hilbert space $\mathfrak{H}_{\lambda}$ consisting of the square-integrable sections in the null space of the elliptic operator $\mathcal{D}$ on the hermitian vector bundle $\mathcal{V}_{\lambda}$. By Theorem 6.3, $\mathfrak{H}_{\lambda}$ then determines an irreducible unitary representation of G belonging to the discrete series, whose character is Θ_{μ}. In view of Theorem 16 in Harish-Chandra [8], we can state:

Corollary. *One can realize all the irreducible unitary representations belonging to the discrete series of the identity component of the generalized Lorentz group* $SO(2n, 1)$ *or its two-fold universal covering group by means of the Hilbert spaces constructed in Theorem* 6.3.

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Added in proof. Theorem 6.3 in §6, has been generalized for general semi-simple Lie groups with compact Cartan subgroups, under a certain regularity condition for $\lambda \in \Lambda$. This is stated as follows. Notation being as in §6, let $\mu \in \Lambda$ be a regular character with respect to $\mathfrak{g}$ and then choose an order on Δ such that $\{\alpha \in \Delta \mid (\mu, \alpha) < 0\}$ be a positive root system. Put $\lambda = \mu - \rho$ and consider the Schmid's operator $\mathcal{D}$ in Lemma 6.1 for λ and the above order. *There then exists a non-negative constant c such that the square-integrable null space $\mathfrak{H}_\lambda$ of $\mathcal{D}$ gives an irreducible unitary representation belonging to the discrete series, whose character is $(-1)^s \Theta_\mu$, when $|(\mu, \alpha)| > c$ for every $\alpha \in \Delta$.* (Notice that $(-1)^s = 1$ in case of the Lorentz group.)

This makes it possible to realize *"most"* discrete series representations for semi-simple Lie groups, in view of Theorem 16 in [8]. The above result was communicated without proof in the letter from Prof. Schmid, and the author has proved it independently. The author's proof is carried out through the method of alternating sum developed by M.S. Narasimhan and K. Okamoto (Ann. of Math. **91** (1970), 486–511). Moreover, this method allows us to realize the discrete series in another way, i.e. on certain eigenspaces of the Casimir operator over the symmetric space.