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PRIMARY DECOMPOSITION OF ELEMENTS IN COMPACTLY GENERATED INTEGRAL MULTIPLICATIVE LATTICES

Dedicated to Professor Keizo Asano on his 60th birthday

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1. Introduction

A complete lattice L is said to be compactly generated, when it has a subset Σ which satisfies that (1) if $x \leq \sup N$ for an element x of Σ and a subset N of Σ, there exists a finite number of elements $x_1, \cdots, x_n$ of N satisfying $x_1 \cup \cdots \cup x_n \geq x$, and (2) every element of L is expressible as a join (supremum) of a subset of Σ. Σ is called a compact generator system of L[1]. The purpose of the present paper is to investigate primary decompositions of elements in compactly generated integral multiplicative lattices[2].

Throughout this paper, we let L be a compactly generated integral multiplicative lattice with a compact generator system Σ. In Section 2 we define a μ-system as a suitable subset of Σ, which is somewhat different from the one introduced in [13]. By using the μ-systems, we define radicals of elements in L and consider meet decompositions of radicals by prime elements. In Section 3 right primary elements are defined by using radicals defined in Section 2. The result in this section is a uniqueness theorem of short decomposition for elements having right primary decompositions. Section 4 deals with right upper M-components of elements, where M is a μ-system. A right upper M-component is defined by using the concept of M-ν-systems, which are also somewhat different from the one introduced in [13]. It will be shown in this section that the right upper M-component of an element has two different representations (Theorem 3). Section 5 is mainly concerned with minimal primes of elements and decompositions of upper isolated p-components of elements. The results in this section are obtained under two conditions. The

1) In [12, §9] Σ is called an aj-system of L. It can be proved that a lattice is compactly generated in the sense of Dilworth and Crawley ([5], [6]), if and only if it has an aj-system, that is, it is compactly generated in our sense.

2) Cf. [3, CHAP. XIII].

one is the ascending chain condition for elements, and the other is the condition (N), which is concerned with weight and type of product-forms. Under some modified semi-modularity for L, it can be proved, in Section 6, that every element of L has right primary decomposition if and only if L has right weak Artin-Rees property. The proofs of the results obtained in this section are similar as in [7], [9] and [10]. But in order to make this paper self-contained we include proofs of the results. Section 7 lays two applications. The ideal theory in non-associative rings has been developed in [2] and [8]. The results obtained in the first half of this section are generalizations of the classical primary decompositions of ideals in commutative rings to ideals in (N)-rings (non-associative and non-commutative), and which are concerned with [2], [8], [10] and [15]. In [3] Birkhoff has pointed out that the lattice of normal subgroups of a group is a commutative integral residuated *cm*-lattice under the commutator-product and the set-inclusion. It is easy to see that the set of the normal subgroups with single generators is a compact generator system of the lattice. In the latter half of this section, primary decompositions of normal subgroups of (N)-groups are obtained as an application of the results in the preceding sections, where (N)-groups are regarded as a generalization of nilpotent groups. Recently primary decomposition theory has been studied in various algebraic systems ([1], [17], [18], etc.). In particular, the theory in groupoids is obtained, among others, in [1]. We shall note here that the results in Sections 2～6 are applicable to subsystems of some sorts of groupoids, but which is not collected in this paper.

Elements of L will be denoted, throughout this paper, by $a, b, c, \cdots$, and those of Σ, in particular, by $x, y, z, \cdots$ with or without suffices. The greatest element of L will be denoted by e, which is not necessarily multiplicative unit of L ([3, CHAP. XIII]). $ab \leq a$ and $ab \leq b$ are assumed for two elements a, b of L. An element a is said to be less than b if $a \leq b$. The symbols $\vee$ and $\wedge$ will denote the set-theoretic union and the intersection respectively. By $\{a \in A \mid a \text{ has property } P\}$ we mean the set of all elements a in A, each of which has property P.

2. Radicals of elements

Let a, b be any two elements of L. The set of the elements x of Σ such that $xb \leq a$ is not void. The join (supremum) of such elements x will be denoted by a/b, and called a (*right*) *quotient* of a by b. It is easily verified that a/b is not necessarily the join of the elements c of L such that $cb \leq a$. The quotient has the following properties: (1) $a \leq a/b$, (2) $(a/b)b \leq a$, (3) $b \leq a$ implies $a/b = e$, (4) $b \leq a$ implies $b/c \leq a/c$, (5) $c \leq b$ implies $a/b \leq a/c$, (6) $\inf_\lambda (a/b_\lambda) = a/(\sup_\lambda b_\lambda)$ and (7) $\inf_\lambda (a_\lambda/b) = (\inf_\lambda a_\lambda)/b$.

From now on, the symbols $P(a)$ and $\Sigma(a)$ will mean the sets $\{x\in\Sigma \mid a/x=a\}$ and $\{x\in\Sigma \mid x\leq a\}$, respectively. The complements of $P(a)$ and $\Sigma(a)$ in Σ will be denoted by $P'(a)$ and $\Sigma'(a)$ respectively. It is then easy to see that $P(a)$ is contained in $\Sigma'(a)$ for every element $a\neq e$.

DEFINITION 1. A subset M of Σ is called a *μ-system*, if there exists an z of M such that $z\leq xy$ for any two elements x, y of M. The void set is to be element considered as a μ-system.

An element p of L is said to be prime if whenever a product of two elements of L is less than p, then at least one of the factors is less than p.

Lemma 1. *The following conditions are equivalent to one another.*

1) *p is prime,*
2) *$xy\leq p$ $(x, y\in\Sigma)$ implies $x\leq p$ or $y\leq p$,*
3) *$\Sigma'(p)$ is a μ-system.*

Lemma 2. *An element p $(\neq e)$ is prime if and only if $P(p)=\Sigma'(p)$.*

Proofs. These two lemmas are immediate.

The following lemma is somewhat different from Lemma 1 in [14].

Lemma 3. *Let a be an element of L, and let M be a μ-system which does not meet $\Sigma(a)$. Then there exists an element p which is maximal in the set consisting of the elements b such that $b\geq a$ and $\Sigma(b)$ does not meet M. p is necessarily a prime element.*

Proof. Since L is compactly generated, we can show, by Zorn's lemma, the existence of p mentioned in the first part of the lemma. To prove the last part of the lemma, we suppose that $xy\leq p$, $x\not\leq p$ and $y\not\leq p$ for x, y in Σ. Then there exist x' and y' in M such that $x'\leq p\cup x$ and $y'\leq p\cup y$. Since there exists an element u of M such that $u\leq x'y'$, we obtain that $u\leq(p\cup x)(p\cup y)\leq p\cup xy=p$. This is a contradiction.

DEFINITION 2. Let a be an element of L. A ***radical*** of a, denoted by rad(a), is the join of all elements x of Σ having the property that every μ-system which contains x meets $\Sigma(a)$.

Theorem 1. *For every element a of L, rad(a) is the meet (infimum) of the primes p_λ such that $p_\lambda\geq a$.*

Proof. First we shall show that rad$(a)\leq p_\lambda$ for every prime p_λ such that $p_\lambda\geq a$. If we suppose that there exists p such that $p\geq a$ and $p\not\geq \mathrm{rad}(a)$, we can take an element x of Σ such that $x\not\leq p$ and $x\leq \mathrm{rad}(a)$. Then there exists a finite number of elements $x_1,\cdots, x_n$ such that $x\leq x_1\cup\cdots\cup x_n$ and each x_i has the property that every μ-system which contains x_i meets $\Sigma(a)$. Now, since there exists x_j such that $x_j \not\leq p$, $\Sigma'(p)$ meets $\Sigma(a)$, which is a contradiction. We have therefore $\mathrm{rad}(a) \leq \inf_{\lambda} p_\lambda$. Next, let x be any element of Σ such that it is not less than rad(a). Then there exists a μ-system M which does not meet $\Sigma(a)$ and contains x. Hence by using Lemma 3 we can take a prime element p such that $p \geq a$ and $\Sigma(p)$ does not meet M. Then evidently x is not less than p. Therefore x is not less than $\inf_{\lambda} p_\lambda$. This completes the proof.

DEFINITION 3. Let a be an element of L. A prime element p of L is said to be a *minimal prime belonging to* a, if (1) $p \geq a$ and (2) there exists no prime element p' such that $a \leq p' < p$.

Let p be a prime element such that $p \geq a$. Then it is proved that the set of the primes which are in the closed interval $[a, p]$ is inductive for downwards; that is, for every descending chain C consisting of primes in $[a, p]$, inf C is a prime in $[a, p]$. Hence Zorn's lemma assures the existence of a minimal prime belonging to a which is less than p. Therefore we obtain the following

Corollary. *For every element a of L, rad(a) is the meet of the minimal primes belonging to a.*

For radicals we can prove the following

Lemma 4. (1°) $a \leq \mathrm{rad}(a)$, (2°) $a \leq b$ *implies* $\mathrm{rad}(a) \leq \mathrm{rad}(b)$, (3°) $\mathrm{rad}(\mathrm{rad}(a)) = \mathrm{rad}(a)$, (4°) $\mathrm{rad}(a \cap b) = \mathrm{rad}(a) \cap \mathrm{rad}(b) = \mathrm{rad}(ab)$.

3. Elements with right primary decompositions

DEFINITION 4. An element q of L is said to be (*right*) *primary*, if whenever $xy \leq q$ and $y \not\leq \mathrm{rad}(q)$ for x, y in Σ, then $x \leq q$.

It is easy to see that q is primary if and only if $ab \leq q$ and $b \not\leq \mathrm{rad}(q)$ imply $a \leq q$ for a, b in L.

Lemma 5. *An element q of L is primary if and only if $\Sigma(rad(q))$ contains $P'(q)$.*

Proof. This is immediate.

Lemma 6. *If $q_1, \cdots, q_n$ is a finite number of primary elements with the same radicals, say $rad(q_i) = c (i=1, \cdots, n)$, then $q = q_1 \cap \cdots \cap q_n$ is primary and has the radical c.*

Proof. It is eivdent that $\mathrm{rad}(q) = c$ by the property (4°) in Lemma 4. In order to prove that q is primary, we suppose that $xy \leq q$ and $y \not\leq \mathrm{rad}(q) = c$. Then $xy \leq q_i$ and $y \not\leq \mathrm{rad}(q_i)$; hence $x \leq q_i$ for $i=1, \cdots, n$. We obtain therefore $x \leq q_1 \cap \cdots \cap q_n = q$, completing the proof.

Lemma 7. *Let $a = q_1 \cap \cdots \cap q_n$ be an irredundant decomposition of a into* *a finite number of primary elements q_i. If $rad(q_i)\neq rad(q_k)$ for some i and k, a is not primary.*

Proof. Put $t_j=q_1\cap\cdots\cap_{j-1}\cap q_{j+1}\cap\cdots\cap q_n$. Then $t_jq_j\leq a$. Since $t_j\not\leq a$, we have $q_j\leq\mathrm{rad}(a)=\cap_{i=1}^n\mathrm{rad}(q_i)$. Hence $\mathrm{rad}(q_j)\leq\cap_{i=1}^n\mathrm{rad}(q_i)$ for $j=1,\cdots,n$. We obtain therefore $\mathrm{rad}(q_1)=\cdots=\mathrm{rad}(q_n)$, a contradiction.

DEFINITION 5. An irredudant decomposition

$$a=q_1\cap\cdots\cap q_n \tag{$*$}$$

of a into primary elements q_i is called a *short decomposition* of a, if none of the meets of two (or more) of $q_1,\cdots,q_n$ are primary.

Theorem 2. *If an element a of L can be decomposed as a meet of a finite number of primary elements, a has a short decomposition. In any two short decompositions of a, the number of primary components as well as their radicals are necessarily the same.*

Proof. By Lemmas 6 and 7, a has a short decomposition. Now, let $(*)$ and $a=q_1^*\cap\cdots\cap q_m^*$ be any two short decompositions of a. Take a maximal element in the *po*-set $\{\mathrm{rad}(q_1),\cdots,\mathrm{rad}(q_n),\ \mathrm{rad}(q_1^*),\cdots,\mathrm{rad}(q_m^*)\}$. We may suppose, without loss of generality, that the maximal element is $\mathrm{rad}(q_1)$. We now show that $\mathrm{rad}(q_1)$ occurs among $\mathrm{rad}(q_k^*)$, $k=1,\cdots,m$. Assume that $\mathrm{rad}(q_1)\neq\mathrm{rad}(q_k^*)$ for all k. Then we have that $q_1\not\leq\mathrm{rad}(q_k^*)$ for all k. Because, if cotrary, we have a contradiction by using (2°), (3°) in Lemma 4 and the maximality of $\mathrm{rad}(q_1)$. On the other hand it is easily verified that $q_i/q_1=q_i$ for $i\neq 1$, and $q_k^*/q_1=q_k^*$ for $k=1,\cdots,m$. Hence we obtain that $a=q_1^*\cap\cdots\cap q_m^*=(q_1^*/q_1)\cap\cdots\cap(q_m^*/q_1)=(q_1/q_1)\cap(q_2/q_1)\cap\cdots\cap(q_n/q_1)=e\cap q_2\cap\cdots\cap q_n=q_2\cap\cdots\cap q_n$, which is a contradiction. We can now suppose, without loss of generality, that $\mathrm{rad}(q_1)=\mathrm{rad}(q_1^*)$, and make

$$(q_2/q_1)\cap\cdots\cap(q_n/q_1)=(q_1^*/q_1)\cap\cdots\cap(q_m^*/q_1) \tag{α}$$

Then since $q_1\not\leq\mathrm{rad}(q_i)$ for $i\neq 1$, and $q_1\not\leq\mathrm{rad}(q_k^*)$ for $k\neq 1$, we have $q_i/q_1=q_i$ $(i\neq 1)$, and $q_k^*/q_1=q_k^*$ $(k\neq 1)$. Hence by (α) we have $q_2\cap\cdots\cap q_n=(q_1^*/q_1)\cap q_2^*\cap\cdots\cap q_m^*$, and have

$$(q_2/q_1^*)\cap\cdots\cap(q_n/q_1^*)=((q_1^*/q_1)/q_1^*)\cap(q_2^*/q_1^*)\cap\cdots\cap(q_m^*/q_1^*) \tag{β}$$

Since it is easily verified that $q_1^*\not\leq\mathrm{rad}(q_i)$ for $i\neq 1$, and $q_1^*\not\leq\mathrm{rad}(q_k^*)$ for $k\neq 1$, and since $q_1^*/q_1\geq q_1^*$, we have $q_i/q_1^*=q_i$ for $i\neq 1$; $q_k^*/q_1^*=q_k^*$ for $k\neq 1$ and $(q_1^*/q_1)/q_1^*=e$. Hence by (β), we have

$$q_2\cap\cdots\cap q_n=q_2^*\cap\cdots\cap q_m^* \tag{γ}$$

Continuing an exactly similar argument for (γ), we atain after a finite number of steps that $m=n$, and $\mathrm{rad}(q_i)=\mathrm{rad}(q_i^*)$ for $i=1, \cdots, m=n$.

4. Isolated components of elements

DEFINITION 6. A subset N of Σ is called a (right) *M-ν-system,* if (1) N contains a μ-system M and (2) for every element u of N and every element x of M there exists an element z of N such that $z \leq ux$. If M is void, the only M-ν-system is, by definition, the void set itself.

Let a be an element of L and M a μ-system which does not meet $\Sigma(a)$. Then it is easily verified that the set-union N^* of all M-ν-systems, each of which does not meet $\Sigma(a)$, is the unique maximal M-ν-system which does not meet $\Sigma(a)$. N^* is uniquely determined by a and M.

Lemma 8. *Let a $(\neq e)$ be an element of L, M a μ-system, and N an M-ν-system. If $\Sigma(a)$ does not meet N, there exists an element q which is maximal in the set consisting of the elements c such that $c \geq a$ and $\Sigma(c)$ does not meet N, and $P(q)$ contains M.*

Proof. Since L is compactly generated, we can prove, by using Zorn's lemma, the existence of q mentioned in the first part of the lemma. In order to prove the last part of the lemma, it is sufficient to show that $q/x > q$ implies $x \notin M$. Take an element y of Σ such that $y \leq q/x$ and $y \not\leq q$. Then, since $q < q \cup y$, we can take an element v of N such that $v \leq q \cup y$. Hence we have that $vx \leq (q \cup y)x = qx \cup yx \leq q$. If we suppose that $x \in M$, we can choose an element z of N such that $z \leq vx$. Hence $z \leq q$, that is, $\Sigma(q)$ meets N, which is a contradiction.

Lemma 9. *Suppose that $M \neq \phi$, $a \neq e$. Then $\Sigma'(a)$ forms an M-ν-system if and only if $P(a)$ contains M.*

Proof. First we suppose that $\Sigma'(a)$ is an M-ν-system. If $P(a)$ does not contain M, we can take an element y such that $y \in M$ and $y \in P'(a)$. Since $a < a/y$, there exists an element x of Σ such that $x \leq a/y$ and $x \not\leq a$. Then we have that $z \leq a$ for every element z of Σ satisfying $z \leq xy$. On the other hand, since $\Sigma'(a)$ is an M-ν-system, there exists an element u of Σ such that $u \leq xy$ and $u \not\leq a$. This is a contradiction. Next, we suppose that $a \neq e$ and $P(a)$ contains M. Then, since $P(a)$ is contained in $\Sigma'(a)$, we have $u \not\leq a = a/x$ for any u of $\Sigma'(a)$ and x of M. Hence ux is not less than a. Therefore we can take an element z of Σ such that $z \leq ux$ and $z \not\leq a$. This shows that $\Sigma'(a)$ is an M-ν-system.

Lemma 10. *Let a $(\neq e)$ be an element of L, let M $(\neq \phi)$ be a μ-system such that it does not meet $\Sigma(a)$, let N^* be the unique maximal M-ν-system which does* *not meet $\Sigma(a)$, and let $S(a, M)$ be the set of the elements s of L having the properties that $s \geq a$ and $P(s)$ contains M. Then the join of the complement of N^* in Σ, say $sup(\Sigma \backslash N^*)$, is a minimal element in $S(a, M)$.*

Proof. By Lemma 8, there exists a maximal element q such that $q \geq a$ and $\Sigma(q)$ does not meet N^*, and $P(q)$ contains M. Since $\Sigma'(q)$ contains $P(q)$, it forms an M-ν-system by the "if part" of Lemma 9. Obviously $\Sigma'(q)$ contains N^*. Hence $\Sigma'(q)=N^*$ by the maximality of N^*. Hence $\Sigma(q)=\Sigma\backslash N^*$. Therefore we have that $q=\sup \Sigma(q)=\sup(\Sigma\backslash N^*)$. It remains to prove that $P(c)$ does not contain M for every c such that $q > c \geq a$. If we suppose that $P(c)$ contains M, then $\Sigma'(c)$ is an M-ν-system by Lemma 9, and meets $\Sigma(a)$. Hence we can find an element u of Σ such that u is less than a and not less than c, a contradiction.

Lemma 11. *Suppose that a, M, N^* and $S(a, M)$ are the same as in Lemma* 10. *If q is a minimal element in $S(a, M)$ and $q \neq e$ then $q=sup(\Sigma\backslash N^*)$.*

Proof. By Lemma 9, $\Sigma'(q)$ is an M-ν-system, and it is evident that $\Sigma'(q)$ does not meet $\Sigma(a)$. By using Lemma 10, we have that $a \leq \sup(\Sigma^*\backslash N) \equiv q'$, and q' is a minimal element such that $\Sigma(q')$ contains M. Then, since $\Sigma'(q)$ is contained in N^*, we have that $q=\sup \Sigma(q) \geq \sup(\Sigma\backslash N^*)=q'$. Therefore we obtain $q=q'$ by the minimality of q.

DEFINITION 7. Let a be an element of L, and let M be a μ-system which does not meet $\Sigma(a)$. A *(right) upper M-component* of a is the join of all elements x of Σ such that every M-ν-system which contains x meets $\Sigma(a)$. The upper M-component of a will be denoted by $u(a, M)$.

Theorem 3. *Let a be an element of L, M $(\neq \phi)$ a μ-system which does not meet $\Sigma(a)$, and N^* the unique maximal M-ν-system which does not meet $\Sigma(a)$. If $S(a, M)$ contains an element $\neq e$, then*

$$u(a, M) = inf(S(a, M)) = sup(\Sigma\backslash N^*) .$$

Proof. For simplisity, we put $q=\inf(S(a, M))$. First, we shall prove that $q=\sup(\Sigma\backslash N^*)$. Since $q/x=\inf_{s\in S(a,M)}\{s/x\}=\inf_{s\in S(a,M)}\{s\}=\inf(S(a, M))=q$ for every element x of M, $P(q)$ contains M. Hence, by Lemma 11 we obtain $q=\sup(\Sigma\backslash N^*)$. Next we prove that $u(a, M)=q$. Evidently every element of $\Sigma(q)$ is not contained in N^*. Since N^* is the unique maximal M-ν-system which does not meet $\Sigma(a)$, every M-ν-system which contains x of $\Sigma(q)$ meets $\Sigma(a)$, that is, x is less than $u(a, M)$. This implies that $q \leq u(a, M)$. Let x be any element of $\{x\in\Sigma \mid x\in N(M\text{-}\nu\text{-system}) \Rightarrow N \cap \Sigma(a) \text{ is not void}\}$. Then evidently x is not contained in N^*. Hence x is less than $\sup(\Sigma\backslash N^*)=q$. Therefore we have $u(a, M) \leq q$, completing the proof.

Corollary 1. *Let a, b be two elements of L such that $a \geq b$, and let M be a μ-system which does not meet $\Sigma(a)$. Then $u(a, M) \geq u(b, M)$.*

Proof. Since $S(b, M)$ contains $S(a, M)$, this is immediate by Theorem 3.

Corollary 2. *Let a be an element of L, and let M_1, M_2 be two μ-systems such that M_1 contains M_2, and M_1 does not meet $\Sigma(a)$. Then $u(a, M_1) \geq u(a, M_2)$.*

Proof. Let N_i^* be the maximal M_i-ν-systems ($i=1, 2$), each of which does not meet $\Sigma(a)$. Then it is easy to see that N_1^* is contained in N_2^*. Therefore we obtain that $u(a, M_1) = \sup(\Sigma \backslash N_1^*) \geq \sup(\Sigma \backslash N_2^*) = u(a, M_2)$.

DEFINITION 8. Let p be a prime element such that $p \geq a$, and let $M = \Sigma'(p)$. $u(a, M)$ is called a *(right) upper isolated p-component* of a, and denoted by $u(a, p)$.

Suppose that $(*)$ in §3 is a decomposition of a into primary elements q_i, and suppose that each $\Sigma'(\mathrm{rad}(q_i))$ contains the unique maximal μ-system M_i, $i=1, \cdots, n$. If p is a prime element such that $M_1 \supseteq \Sigma'(p), \cdots, M_s \supseteq \Sigma'(p)$, $M_{s+1} \not\supseteq \Sigma'(p), \cdots, M_n \not\supseteq \Sigma'(p)$, then $u(a, p) = q_1 \cap \cdots \cap q_s$. Because, for $i=1, \cdots, s$, we have $u(a, p) \leq u(a, M_i)$ by Corollary 2 to Theorem 3. Now by Lemma 5, we have $P(q_i) \supseteq \Sigma'(\mathrm{rad}\,(q_i))$. Hence $P(q_i)$ contains M_i. Hence we have, by Theorem 3, $u(a, M_i) \leq q_i$. Therefore $u(a, p) \leq q_1 \cap \cdots \cap q_s$. If $s=n$, we obtain $a \leq u(a, p) \leq q_1 \cap \cdots \cap q_n = a$, $u(a, p) = q_1 \cap \cdots \cap q_n$. If $s<n$, then, since $\Sigma'(p)$ is not contained in $\Sigma'(\mathrm{rad}(q_j))$, we have $\mathrm{rad}(q_j) \not\leq p$, and have $q_j \not\leq p$ for $j>s$ (by Theorem 1). Hence we can take elements x_j such that $x_j \leq q_j$ and $x_j \in \Sigma'(p)$, $j=s+1, \cdots, n$. Since $\Sigma'(p)$ is a μ-system, there exists a finite number of elements y_j in $\Sigma'(p)$ such that $y_{s+1} \leq x_{s+1} \cdot x_{s+2}$, $y_{s+2} \leq y_{s+1} \cdot x_{s+3}, \cdots, y_{n-1} \leq y_{n-2} x_n$. Then we have $y_{n-1} \leq (\cdots((x_{s+1} \cdot x_{s+2}) x_{s+3}) \cdots) x_n \leq q_{s+1} \cap \cdots \cap q_n$. Let z be an arbitrary element of Σ such that $z \leq q_1 \cap \cdots \cap q_s$. Then we obtain $z y_{n-1} \leq (q_1 \cap \cdots \cap q_s) \cap (q_{s+1} \cap \cdots \cap q_n) = a$. Now, take any $\Sigma'(p)$-ν-system N containing z. Then there exists an element v of N such that $v \leq z y_{n-1}$. Since $v \leq a$, N meets $\Sigma(a)$. Hence we have $z \leq u(a, p)$, $q_1 \cap \cdots \cap q_s \leq u(a, p)$. Therefore we obtain $q_1 \cap \cdots \cap q_s = u(a, p)$, completing the proof.

5. Ascending chain condition, Condition (N)

We shall assume, throughout this section, that the ascending chain condition (a. c. c.) holds for elements of L.

Lemma 12. *Every element of L has a finite number of minimal primes belonging to it.*

Proof. Let c be an element of L. If c is prime, the lemma is trivially evident. Suppose now that c is not prime. If there exists an infinite number of minimal primes p_λ belonging to c, then, since $a_1 \not\leq c$, $b_1 \not\leq c$ and $a_1 b_1 \leq c$ for suitable elements a_1, b_1 of L, a_1 or b_1 is less than p_λ for an infinite number of p_λ. Suppose that it is a_1, and put $c_1 = c \cup a_1$. Then evidently $c < c_1$ and $c_1 \leq p_\lambda$. c_1 is not prime. Hence c_1 has the same property as that of c. Continuing in this way, we obtain an ascending chain $c < c_1 < c_2 < \cdots$, which is a contradiction.

Lemma 13. *Let $p_1, \cdots, p_n$ be the minimal primes belonging to an element c of L. Then there exists a product $\mathfrak{P}(p_{i_1}, \cdots, p_{i_m})$ which is less than c, where $\mathfrak{P}$ denote a product-form of some type of weight m, and $i_1, \cdots, i_m$ is some finite permutation of $1, \cdots, n$ with repetitions allowed.*

Proof. The lemma is evident if c is prime. Suppose that c is not prime. Then there exist two elements x and y of Σ such that $x \not\leq c$, $y \not\leq c$ and $xy \leq c$. Put $a_1 = c \cup x$ and $b_1 = c \cup y$. Then $c < a_1$ and $c < b_1$. Now, let $p_1', \cdots, p_r'$ and $p_1'', \cdots, p_s''$ be the minimal primes belonging to a_1 and b_1 respectively. If we suppose that both a_1 and b_1 have the same property that we wish to prove of c, so that $\mathfrak{P}'(p'_{j_1}, \cdots, p'_{j_\lambda}) \leq a_1$ and $\mathfrak{P}''(p''_{k_1}, \cdots, p''_{k_\mu}) \leq b_1$, then, since $a_1 b_1 = (c \cup x) \cdot (c \cup y) \leq c \cup xy = c$, we have $\mathfrak{P}'(p'_{j_1}, \cdots, p'_{j_\lambda}) \cdot \mathfrak{P}''(p''_{k_1}), \cdots, p''_{k_\mu}) \leq c$. The interval $[c, p'_{j_\rho}]$ contains a minimal prime belonging to c, $\rho = 1, \cdots, \lambda$, and similarly for $[c, p''_{k_\sigma}]$, $\sigma = 1, \cdots, \mu$. Hence we have $\mathfrak{P}(p_{i_1}, \cdots, p_{i_m}) \leq c$, where $p_{i_1}, \cdots, p_{i_m}$ are minimal primes belonging to c, and $\mathfrak{P} = \mathfrak{P}' \cdot \mathfrak{P}''$. Hence, if the lemma is false for c, it is false for a_1 or for b_1. Continuing in this way, we atain a contradiction of the a. c. c.

DEFINITION 9. A product-form $\mathfrak{Q}(X_1, \cdots, X_m) = (\cdots((X_1 X_2) X_3) \cdots) X_m$ is called that it has a (*right*) *nested type* of weight m, where X_i are indeterminates over L.

We now consider the following condition:

(N) *For every product-form $\mathfrak{P}$ of weight n, and for every elements $c_1, \cdots, c_n$ (repetitions allowed) of L, there exists a product-form $\mathfrak{Q}$ with nested type of weight m such that*

$$\mathfrak{Q}(c_{i_1}, \cdots, c_{i_m}) \leq \mathfrak{P}(c_1, \cdots, c_n),$$

where $i_1 \leq \cdots \leq i_m$.

If L is associative, the condition (N) is satisfied trivially. But there are important examples which are compactly generated non-associative multiplicative lattices satisfying the condition (N), which will be shown in the last section of this paper.

Lemma 14. *Suppose that the condition (N) holds for L. Then, for every element a $(\neq e)$ of L, there exists a minimal prime p of a such that $a/p > a$.*

Proof. If a is prime, the lemma is trivially evident. Suppose that a is not prime. Then by Lemma 13 and the condition (N), there exist minimal primes $p_1, \cdots, p_m$ (not necessarily distinct) belonging to a and a product form $\mathfrak{Q}$ of nested type such that $\mathfrak{Q}(p_1, \cdots, p_m) \leq a$. It is then easy to see that $m > 1$, and that there exists p_i such that $a/p_i > a$. This completes the proof.

Behrens showed in [2] that the radicals of primary ideals in non-associative rings are not necessarily prime. He gave two examples in that paper. Each of those examples is a commutative algebra with some finite base over a field. It is now easily verified that the ideals in each of the algebras is a compactly generated multiplicative lattice. Accordingly, those examples assure the existence of the lattices in which the radicals of primary elements are not prime. Now we have the following

Theorem 4. *Suppose that the condition (N) holds for L. Then the radical of every primary element is prime.*

Proof. Let q be a primary element of L. If $q = e$, the theorem is evident. We suppose that $q < e$. Then by Lemma 14, we can find a minimal prime p belonging to q such that $x_z \cdot z \leq q$ and $x_z \not\leq q$ for an arbitrary element z of $\Sigma(p)$ and a suitable element x_z of Σ. Hence $z \leq \mathrm{rad}(q)$, and hence $p \leq \mathrm{rad}(q)$. On the other hand, since $\mathrm{rad}(q) \leq p$ by Theorem 1, we obtain $\mathrm{rad}(q) = p$, as desired.

REMARK. Under the condition (N) for L, we can show that if $\mathrm{rad}(c) = p$ is prime for an element $c (\neq e)$, then $c/p > c$. Because, the assertion is trivially evident if $p = c$. Hence we can suppose that $p > c$. Then by Lemma 13 and the condition (N), there exists a nested product $\mathfrak{Q}$ of p such that $\mathfrak{Q} \equiv \mathfrak{Q}' p \leq c$. If we suppose that $c/p = c$, then $\mathfrak{Q}' \equiv \mathfrak{Q}'' p \leq c$. Continuing in this way, we obtain $p = c$, which is a contradiction.

Theorem 5. *Suppose that (*) (in §3) is a decomposition of a into primary elements q_i with prime radicals p_i. Then the minimal primes belonging to a coincide with the minimal elements in the po-set $\{p_1, \cdots, p_n\}$.*

Proof. By Lemma 13, we have that $\mathfrak{P}^{(i)}(p_i) \equiv \mathfrak{P}^{(i)}(p_i, \cdots, p_i) \leq q_i$ for suitable product-forms $\mathfrak{P}^{(i)}$, $i = 1, \cdots, n$. Hence, for any product-form of n-th weight, we obtain $\mathfrak{P}(\mathfrak{P}^{(1)}(p_1), \cdots, \mathfrak{P}^{(n)}(p_n)) \leq \mathfrak{P}(q_1, \cdots, q_n) \leq q_1 \cap \cdots \cap q_n = a$. This implies the existence of p_i such that $p_i \leq p$ for any prime p satisfying $p \geq a$. In particular, any minimal prime belonging to a coincides with some p_i, and there is no p_j such that $p_j < p_i$. Conversely, let p_i be any minimal element in the *po*-set $\{p_i, \cdots, p_n\}$. If p is a prime element contained in $[a, p_i]$, we can show, similarly as above, the existence of a prime element p_k such that $p_k \leq p$. We obtain therefore $p_k \leq p_i$, $p_k = p_i$, completing the proof.

The following theorem have been established by the last part of §4.

Theorem 6. *Suppose that (*) is a decomposition of a into primary elements*

q_i with prime radicals p_i. If $p(\neq e)$ is a prime element such that $p_1 \leq p, \cdots, p_s \leq p$, $p_{s+1} \not\leq p, \cdots, p_n \not\leq p$, then

$$u(a, p) = q_1 \cap \cdots \cap q_s .$$

Theorem 7. *Suppose that (*) is a short decomposition of a into primary elements q_i with prime radicals p_i. If p is any minimal prime element belonging to a then $u(a, p)=q_i$ for some i, and $u(a, p)$ is primary.*

Proof. By Theorem 5, we have $p=p_i$ for some i. Since there exists no j such that $p_j \leq p$ $(j \neq i)$, we obtain $u(a, p)=q_i$ by Theorem 6.

REMARK. If $p=e$ in Theorem 7, a is primary such as $\mathrm{rad}(a)=e$.

Corollary 1. *Suppose that (*) is a short decomposition of a, and let $p_1, \cdots, p_s$ be the minimal primes belonging to a $(i=1, \cdots, s)$. Then*

$$a = u(a, p_1) \cap \cdots \cap u(a, p_s) \cap q_{s+1} \cap \cdots \cap q_n .$$

Corollary 2. *Suppose that a has a decomposition into primary elements with prime radicals. If $p_1, \cdots, p_s$ are the minimal primes belonging to a, then $u(a, p_1), \cdots, u(a, p_s)$ are primary.*

Proof. This is immediate by Theorems 2, 5 and 7.

Now let V be a compactly generated lattice with compact generator system Σ. If Σ is a join-semi-lattice, Σ is said to be join-closed. Let Σ be any compact generator system of V. Then it can be proved that the join-semi-lattice Σ' generated by Σ satisfies the conditions (1) and (2) in §1. Hence Σ' is a join-closed compact generator system of V.

In the rest of this section we suppose that Σ is join-closed. Then it is easy to see by the a.c.c. that Σ coincides with L. But it is convenient to remain the symbol Σ.

Lemma 15. *Let $p_1, \cdots, p_n$ be a finite number of prime elements of a compactly generated multiplicative lattice with a join-closed compact generator system. If $\Sigma(a)$ is contained in the set-union $\vee_{i=1}^{n} \Sigma(p_i)$, there exists p_i such that $p_i \geq a$.*

Proof. If $n=1$, the lemma is trivially evident. If $n=2$, then $\Sigma(a)$ is contained in $\Sigma(p_1) \vee \Sigma(p_2)$. Suppose that $a \not\leq p_1$ and $a \not\leq p_2$. Then we can take z_i of Σ such that $z_i \leq a$, $z_i \leq p_i$ $(i=1, 2)$, $z_1 \not\leq p_2$ and $z_2 \not\leq p_1$. Since $z_1 \cup z_2$ is less than a, $\Sigma(z_1 \cup z_2)$ is contained in $\Sigma(p_1)$ or $\Sigma(p_2)$. This implies $z_2 \leq p_1$ or $z_1 \leq p_2$, which is a contradiction. If $n \geq 3$, we can assume, no loss of generality, that $\Sigma(a)$ is contained in $\vee_{i=1}^{m} \Sigma(p_i)$ $(m \leq n)$, and not contained in $\vee_{i=1}^{k-1} \Sigma(p_i) \vee \vee_{i=k+1}^{m} \Sigma(p_i)$ for every $k=2, \cdots, m-1$. Then we can take elements z_k of Σ such that $z_k \leq a$, $z_k \leq p_k$ and $z_k \not\leq p_i$ for $i \neq k$; $i, k=1, \cdots, m$. Since $z_2, \cdots, z_m$ are contained in a μ-system $\Sigma'(p_1)$, we can find a finite number of elements

$v_1, \cdots, v_{m-2}$ of $\Sigma'(p_1)$ such that $v_1 \leq z_2 z_3$, $v_2 \leq v_1 z_4, \cdots, v_{m-2} \leq v_{m-3} z_m$. Then we have $v^{(1)} \equiv v_{m-2} \leq ((\cdots((z_2 \cdot z_3) z_4) \cdots) z_{m-1}) z_m \leq p_j$ for $j=2, \cdots, m$, and $v^{(1)} \not\leq p_1$. Similarly, we can find $v^{(i)}$ of $\Sigma'(p_i)$ such that $v^{(i)} \leq p_j$ $(j \neq i)$ and $v^{(i)} \not\leq p_i$ for $i=2, \cdots, m$. Now let $v \equiv v^{(1)} \cup \cdots \cup v^{(m)}$. Then, since Σ is closed under finite join operation, v is contained in $\Sigma(a)$. Hence we have $v \leq p_i$ for a suitable prime p_i. This implies $v^{(i)} \leq p_i$, which is a contradiction. This completes the proof.

Theorem 8. *Suppose that* (*) *is a short decomposition of a with prime radicals, and let p be a prime element such that* $a \leq p \neq e$. *Then* $p = rad(q_i)$ *for some* q_i, *if and only if* $u(a, p)/p > u(a, p)$.

Proof. We have, by Theorem 6, $u(a, p) = q_1 \cap \cdots \cap q_k$, where $q_1, \cdots, q_k$ are those whose radicals p_i are less than p. This is a short decomposition of $u(a, p)$, and p is one of $p_1, \cdots, p_k$. Since an element x of $\Sigma = L$ is contained in $P'(u(a, p))$ if and only if $x \leq p$, we have $u(a, p)/p > u(a, p)$. Conversely, let $u(a, p)/p > u(a, p)$. Then the minimal primes of a are the minimal elements in the *po*-set $\{p_1, \cdots, p_n\}$. Hence $p_i \leq p$ for some p_i. We let $p_1, \cdots, p_k$ be the primes such that $p_i \leq p$ $(i=1, \cdots, k)$. Then $u(a, p) = q_1 \cap \cdots \cap q_k$, $\mathrm{rad}(q_i) = p_i$, and that is a short decomposition of $u(a, p)$. Now by the assumption $\Sigma(p)$ is contained in $P'(u(a, p)) = \vee_{i=1}^{k} \Sigma(p_i)$. Hence, we have by Lemma 15 $p \leq p_i$ for a suitable p_i $(1 \leq i \leq k)$. We obtain therefore $p = p_i$.

6. Artin-Rees property

In this section, we let L be a compactly generated integral multiplicative lattice with the compact generator system Σ.

DEFINITION 10. L is said to have the (*right*) *weak Artin-Rees property*, if for any a in L and any x in Σ, there exists a product $\mathfrak{P}$ of x such that $a \cap \mathfrak{P} \leq ax$.

Theorem 9. *Suppose that the a.c.c. holds for elements of L. If every element of L may be decomposed into a meet of a finite number of primary elements, then the weak Artin-Rees property holds for L.*

Proof. Let $a \in L$, and $x \in \Sigma$, and suppose that $ax = q_1 \cap \cdots \cap q_n$ is a primary decomposition of ax. If $a \leq q_i$ for every $i=1, \cdots, n$, we have that $a \cap x \leq a = ax$. Hence we can suppose that $a \not\leq q_i$ for $i=1, \cdots, m$, where $1 \leq m \leq n$. Then $ax = a \cap q_1 \cap \cdots \cap q_m$. Since there exists an element u of Σ such that $ux \leq q_i$ and $u \not\leq q_i$ $(1 \leq i \leq m)$, we obtain $x \leq \mathrm{rad}(q_i)$ $(1 \leq i \leq m)$. Hence we have that $\mathfrak{P}_i \equiv \mathfrak{P}_i(x, \cdots, x) \leq q_i$ for suitable product-forms $\mathfrak{P}_i$ $(1 \leq i \leq m)$. Hence $\mathfrak{P}' \equiv (\cdots((\mathfrak{P}_1 \cdot \mathfrak{P}_2)\mathfrak{P}_3 \cdots)\mathfrak{P}_m \leq q_1 \cap \cdots \cap q_m$. Therefore we obtain $a \cap \mathfrak{P}' \leq a \cap q_1 \cap \cdots \cap q_m = ax$, completing the proof.

Let Σ^* be the multiplicative monoid generated by Σ.

DEFINITION 11. L is called a *strictly upper semi-modular lattice related to* Σ^*, if the relations $a \cap u < b < a < a \cup u$ hold for $a, b \in L$, and $u \in \Sigma^*$, then there exists an element c of L such that $a \cap u < c \leq u$ and $(c \cup b) \cap a = b$.

This is a modification of the semi-modular lattice defined in [19, §45].

Lemma 16. *Let L be a strictly upper semi-modular lattice related to Σ^*, and let q be an irreducible element of L. If $q \cap u = a \cap u$ and $q < a$ for $a \in L$, $u \in \Sigma^*$, then $u \leq q$.*

Proof. Put $b = (q \cup u) \cap a$. Then $q \leq b$. If $b = q$, then since $q = (q \cup u) \cap a$ and $q < a$, we have $q = q \cup u$, $u \leq q$. Next we suppose that $q < b$. Now we have that $a \cap u \leq q < a \leq a \cup u$. If $a = a \cup u$, then $u \leq a$, $u = a \cap u = q \cap u$. This implies $u \leq q$. If $a \cap u = q$, then $q \cap u = q$, $q \leq u$. This implies $q \cup u = u$. Hence we have $b = a \cap u = q \cap u = q$, a contradiction. Now it remains to consider the case of $a \cap u < q < a < a \cup u$. Then there exists an element c of L such that $a \cap u < c \leq u$ and $q = (q \cup c) \cap a$. Since q is irredicible, we have $q = q \cup c$. Hence $c \leq q < a$, $c \leq a \cap u$. This contradicts $a \cap u < c$.

Lemma 17. *A non-void μ-system M meets $\Sigma(\mathfrak{P}(x, \cdots, x))$ for every element $x \in M$ and every product-form $\mathfrak{P}$.*

Proof. The proof will be given by induction with respect to the weight m of $\mathfrak{P}$. If $m = 1$, the lemma is evident. We suppose that the lemma has been proved for $\mathfrak{P}'$ with any weight $m' < m$. Now $\mathfrak{P}$ is expressible as $\mathfrak{P} = \mathfrak{P}_1 \cdot \mathfrak{P}_2$. Of course the weight of $\mathfrak{P}_i$ is strictly less than that of $\mathfrak{P}$. Hence by the induction hypothesis M meets $\Sigma(\mathfrak{P}_i)$; accordingly there exists u_i such that $u_i \in M$ and $u_i \leq \mathfrak{P}_i$ $(i = 1, 2)$. Since there exists an element u of M such that $u \leq u_1 u_2$, M meets $\Sigma(\mathfrak{P})$, as desired.

Theorem 10. *Let L be a strictly upper semi-modular lattice related to Σ^*, and suppose that the a. c. c. holds for elements of L. If the weak Artin-Rees property holds for L, every element of L is decomposed into a meet of a finite number of primary elements.*

Proof. Since L satisfies the a. c. c., it is sufficient to show that every irreducible element of L is primary. Suppose that q is irreducible, and let $xy \leq q$ but $x \not\leq q$ for two elements x, y in Σ. Put $a = x \cup q$. Then $a > q$ and $ay = (x \cup q)y = xy \cup qy \leq q$. Now let $\mathfrak{P} \equiv \mathfrak{P}(y, \cdots, y)$ be a product of y such that $a \cap \mathfrak{P} \leq ay \leq q$. Then we have $a \cap \mathfrak{P} \leq q \cap \mathfrak{P}$. Hence $a \cap \mathfrak{P} = q \cap \mathfrak{P}$. Since $q < a$, we have by Lemma 16 $\mathfrak{P} \leq q$. Next, we let M be an arbitrary μ-system containing y. Then by Lemma 17 $M \wedge \Sigma(\mathfrak{P})$ is not void. Since $\mathfrak{P} \leq q$, we have that $M \wedge \Sigma(\mathfrak{P}) \subseteq M \wedge \Sigma(q)$. Therefore M meets $\Sigma(q)$, that is, $y \leq \operatorname{rad}(q)$, as desired.

7. Applications

[**1**] Let R be a non-associative (not necessarily) ring with or without unity quantity. The word "ideals" will mean always "two-sided ideals" of R. Ideals of R will be denoted by $A, B, P, Q, \cdots$. For an element x of R, (x) will denote the principal ideal generated by x. (x) consists of the elements u such of that $u=\Sigma\mathfrak{P}(\cdots, x, \cdots)$, where $\mathfrak{P}(\cdots, x, \cdots)$ is a product with x as its factor, and Σ is a finite sum.

Now it can be proved that the set of all ideals of R forms a compactly generated integral multiplicative lattice with the compact generator system consisting of the principal ideals. The results in the preceding sections are accordingly applicable to the ideals of R.

Throughout [**1**], there is a complete parallelism between the theory of right-side and that of left-side. We shall therefore state the results for right-side only.

For any two ideals A and B, the (right) quotient A by B, denoted by A/B, is the set of the elements u in R such that $(u)B\subseteq A$ (Cf. [2], [8]). Then A/B is an ideal of R, and it can be proved easily that A/B coincides with the set-union of all the principal ideals (u) such that $(u)B\subseteq A$. An element x of R is said to be *(right) related* an ideal A, if and only if $A/(x)$ contains A properly. Otherwise x is said to be *(right) unrelated* to A. It is then easily seen that if x is related to A, every element in (x) is also related to A.

A family $\mathfrak{M}$ of principal ideals of R is called a *μ-system*, if there exists (z) of $\mathfrak{M}$ such that $(z)\subseteq(x)(y)$ for any two principal ideals (x) and (y) in $\mathfrak{M}$. The void set is also defined to be a μ-system. Let P be a prime ideal of R (Cf. [2]). It is then easily verified that the family of principal ideals $\mathfrak{M}_P=\{(x)\,|\,(x)$ is not contained in $P\}$ forms a μ-system. Conversely, if $\mathfrak{M}_P$ is a μ-system for an ideal P, then P is prime. Let A be an ideal of R, and let $\mathfrak{M}$ be a μ-system which does not contain any ideal $(x)\subseteq A$. Then we can show that the existence of the (maximal) prime ideal P such that P contains A and every principal ideal in P does not contained in $\mathfrak{M}$ (Cf. [16, §14]).

Let M be an M-system in the sense of Behrens [2]. If we make the family $\mathfrak{M}=\{(x)\,|\,x\in M\}$ of principal ideals, it is easily verified that $\mathfrak{M}$ is a μ-system. But, for any μ-system $\mathfrak{M}$, it can not be proved in general, that the set $\{x\,|\,(x)\in\mathfrak{M}\}$ is an M-system in the sense of Behrens. By Definition 2, we define the radical of an ideal A, which is denoted by rad(A), is the ideal generated by the set-union of principal ideals (x) with the property that every μ-system which contains (x) contains a principal ideal in A. Definition of a minimal prime ideal of an ideal is the same as in the case of an associative ring (Cf. [11]). Then by Corollary to Theorem 1, we obtain that the radical of an ideal A is the intersection of all the minimal prime ideals of A. Therefore we obtain that rad(A) coincides with the Behrens' radical $\mathfrak{r}(A)$.

In order that an ideal Q of R is (right) primary (Cf. [2]) it is necessary and sufficient that every element which is (right) related to Q is contained in $\mathfrak{r}(Q)$. Irredundant decomposition of an ideal of R is defined as usual. Let

$$A = Q_1 \cap \cdots \cap Q_n \tag{$*$}$$

be an irredundant decomposition of an ideal A into primary components Q_i.

The representation $(*)$ of A is called a short decomposition of A, if none of the meets of two (or more) of $Q_1, \cdots, Q_n$ are primary. By Theorem 2, we obtain the following statement.

1) *If an ideal A of R can be decomposed as an intersection of a finite number of primary ideals, A has a short decomposition. In any two short decompositions of A, the number of primary components as well as their radicals are necessarily the same.*

Let $\mathfrak{M}$ be a non-void μ-system. A family $\mathfrak{N}$ of principal ideals of R is called a *(right) $\mathfrak{M}$-ν-system* of R, if $\mathfrak{N}$ contains $\mathfrak{M}$ and if for every (u) in $\mathfrak{N}$ and every (x) in $\mathfrak{M}$, there exists an ideal (z) in $\mathfrak{N}$ such that $(z) \subseteq (u)(x)$. If $\mathfrak{M}$ is void, the $\mathfrak{M}$-ν-system is also void. Let $\mathfrak{M}$ be a μ-system such that every ideal in $\mathfrak{M}$ is not contained in an ideal A. A *(right) upper $\mathfrak{M}$-component* of A is defined to be the ideal generated by the set-union of all the principal ideals (x) having the property that every $\mathfrak{M}$-ν-system which contains (x) has an ideal in A. The upper $\mathfrak{M}$-component of A will be denoted by $U(A, \mathfrak{M})$. Let P be a prime ideal containing A. Then the (right) upper isolated P-component of A, which is denoted by $U(A, P)$, means $U(A, \mathfrak{M})$, where $\mathfrak{M} = \{(x) \mid (x) \text{ is not contained in } P\}$. If P is a minimal prime of A, $U(A, P)$ is called an isolated (right) primary component of A. Now let $\mathfrak{M}$ $(\neq \phi)$ be a μ-system which does not contain any ideal in A, and let $\mathfrak{N}^*$ be the (unique) maximal $\mathfrak{M}$-ν-system such that every ideal in $\mathfrak{N}^*$ is not contained in A. Then by Theorem 3 $U(A, \mathfrak{M})$ is the intersection of all the ideals B having the property that (1) B contains A and (2) $\{(x) \mid B/(x) = B\}$ contains $\mathfrak{M}$. Moreover $U(A, \mathfrak{M})$ is the ideal generated by the set-union of all the principal ideals, each of which is not in $\mathfrak{N}^*$.

A product-form $\mathfrak{Q}(X_1, \cdots, X_m) = (\cdots((X_1X_2)X_3)\cdots)X_m$ is called that it has a (right) *nested* type of weight m, where X_i are indeterminates over the ideal-m-lattice of R.

A non-associative ring R is called here an (N)-*ring* if it satisfies the following condition:

(N) *For every product-form $\mathfrak{P}$ of weight n, and for every ideals $A_1, \cdots, A_n$ (repetitions allowed) of R, there exists a product-form $\mathfrak{Q}$ with nested type of weight m such that*

$$\mathfrak{Q}(A_{i_1}, \cdots, A_{i_m}) \subseteq \mathfrak{P}(A_1, \cdots, A_n),$$

where $i_1 \leq \cdots \leq i_m$.

Any associative ring is evidently an (N)-ring. Any nilpotent Lie ring is also an (N)-ring. Now we have the following statement.

2) *Suppose that the a. c. c. holds for ideals of an (N)-ring R. Then the radical of every primary ideal of R is prime.* (by Theorem 4).

We now suppose that $(*)$ in this section is an irredundant decomposition of an ideal A of a ring R into primary ideals Q_i with prime radicals P_i. If the a. c. c. holds for ideals of R, the minimal primes belonging to A coincide with the minimal elements in the *po*-set $\{P_1, \cdots, P_n\}$. This is the immediate consequence of Theorem 5. In particular, we obtain the following:

3) *Assume that the a. c. c. holds for ideals of an (N)-ring R. If $(*)$ is a decomposition of an ideal A into primary ideals Q_i, the minimal primes belonging to A coincide with the minimal elements in the po-set consisting of the radicals of Q_i* (by Theorem 5).

In the rest of this paragraph, we let R be an (N)-ring with the a. c. c. for ideals of R. Then by Theorems 6, 7, 8, 9 and 10 we have the followings 4)$\sim$8).

4) *Suppose that $(*)$ is a decomposition of an ideal A of R into primary ideals Q_i with prime radicals P_i. If $P(\neq R)$ is a prime ideal such that $P_1 \subseteq P, \cdots, P_s \subseteq P$, $P_{s+1} \not\subseteq P, \cdots, P_n \not\subseteq P$, then*

$$U(A, P) = Q_1 \cap \cdots \cap Q_s .$$

5) *Suppose that $(*)$ is a short decomposition of A with (prime) radicals $P_i = rad(Q_i)$. If P is any minimal prime ideal belonging to A and $P \neq R$, then $U(A, P) = Q_i$ for some i, and $U(A, P)$ is primary.*

6) *Suppose that $(*)$ is a short decomposition of A, and let $P_1, \cdots, P_s$ be the minimal primes belonging to A. Then*

$$A = U(A, P_1) \cap \cdots \cap U(A, P_s) \cap Q_{s+1} \cap \cdots \cap Q_n .$$

7) *Suppose that $(*)$ is a short decomposition of A, and let P be a prime ideal such that $A \subseteq P \neq R$. Then $P = rad(Q_i)$ for some Q_i, if and only if $U(A, P)/P \supsetneqq (A, P)$.*

R is said to have the (*right*) *weak Artin-Rees property*, if for any ideal A and any principal ideal (x) of R, there exists a product $\mathfrak{P}$ of (x) such that $A \cap \mathfrak{P} \subseteq A(x)$. (Cf. [8]). Then we have

8) *In order that every ideal of R is decomposed into a meet of a finite number of primary ideals, it is necessary and sufficient that the weak Artin-Rees property holds for R.*

[2] Let G be a group. The set of all normal subgroups $A, B, N, \cdots$ of G is a commutative residuated *cm*-lattice under commutator-product $[A, B]$ and the set-inclusion relation. The *residual* of A by B, which is denoted by $A:B$, is defined as the set-union of the elements $u \in G$ such that $[(u), B] \subseteq A$, where (u) is the normal subgroup generated by $u \in G$, that is, $(u) = \{\Pi x_\rho^{-1} u^\rho x_\rho \mid x \in G,\ \rho \in Z$

(the integers)}. Then it can be proved that $A:B$ is a normal subgroup of G. It is easily be seen that the *cm*-lattice has the *zero element* 1 (the group identity) (Cf. [3]). Now we can show that the set of the normal subgroups of G is a compactly generated multiplicative lattice with the compact generator system consisting of normal subgroups, each of which is generated by a single element.

An element x of G is said to be *unrelated* to a normal subgroup N, if $N:(x) = N$. Otherwise, x is *related* to N. A family $\boldsymbol{M}$ consisting of normal subgroups with single generators is called a *μ-system,* if there exists (z) of $\boldsymbol{M}$ such that $(z) \subseteq [(x), (y)]$ for any two (x) and (y) or $\boldsymbol{M}$. The void set is also defined to be a μ-system. A normal subgroup P of G is said to be prime, if $[A, B] \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. Then it can be proved that P is prime if and only if $[(x), (y)] \subseteq P$ implies $(x) \subseteq P$ or $(y) \subseteq P$. If P is prime, the family $\{(x) \mid x \notin P\}$ forms a μ-system. Moreover a normal subgroup $P(\neq G)$ of G is prime if and only if $\{(x) \mid x$ is related to $P\}$ is a μ-system.

Let $\boldsymbol{M}$ be a μ-system which does not contain (x) such that $(x) \subseteq A$. Then there exists a normal subgroup P which is maximal in the family of normal subgroups B such that $B \supseteq A$ and $(b) \notin \boldsymbol{M}$ for every $b \in B$. P is necessariry prime.

A *radical* of normal subgroup N of G is the normal subgroup generated by the set-union of (x) with the property that the every μ-system containing (x) contains a subgroup in N. In symbol: $\mathrm{rad}(N)$. Minimal primes of a normal subgroup is defined in the obvious way. Then by Corollary to Theorem 1 we obtain that $\mathrm{rad}(N)$ is the intersection of all minimal primes of N.

A normal subgroup Q of G is called *primary,* if $[(x), (y)] \subseteq Q$ and $(y) \nsubseteq \mathrm{rad}(Q)$ imply that $(x) \subseteq Q$.

Let

$$N = Q_1 \cap \cdots \cap Q_n \tag{**}$$

be an irredundant decomposition of a normal subgroup N into primary normal subgroups Q_i. The representation (**) of N is called a short decomposition of N, if none of the meets of two (or more) of $Q_1, \cdots, Q_n$ are primary. By Theorem 2, we obtain the following statement.

1) *If a normal subgroup N of G can be decomposed as an intersection of a finite number of primary normal subgroups, then N has a short decomposition. In any two short decompositions of N, the number of primary components as well as their radicals are necessarily the same.*

Let $\boldsymbol{M}$ be a non-void μ-system. A family $\boldsymbol{N}$ of principal normal subgroups of G is called an *$\boldsymbol{M}$-ν-system,* if $\boldsymbol{N}$ contains $\boldsymbol{M}$ and if for every (u) in N and every (x) in $\boldsymbol{M}$ there exists (z) in $\boldsymbol{N}$ such that $(z) \subseteq [(u), (x)]$. If $\boldsymbol{M}$ is void, the $\boldsymbol{M}$-ν-system is also void. By using $\boldsymbol{M}$-ν-system, the *upper $\boldsymbol{M}$-component* $U(N, \boldsymbol{M})$ of N is defined in an obvious way. In particular, *upper isolated P-component* $U(N, P)$ of N is defined for any minimal prime of N. Now let $\boldsymbol{M}$

be a μ-system which does not contain a normal subgroup (of G) in N, and let $\boldsymbol{N}^*$ be the (unique) maximal $\boldsymbol{M}$-ν-system such that every normal subgroup in $\boldsymbol{N}^*$ does not contained in N. Then by Theorem 3 $U(N, \boldsymbol{M})$ is the intersection of all the normal subgroups H having the property that (1) $H \supseteq N$ and (2) $\{(a) \mid H:(a)=H\} \supseteq \boldsymbol{M}$. Moreover $U(N, \boldsymbol{M})$ is the normal subgroup generated by the set-union of all the normal subgroups such that each of which has a single generator and is not contained in $\boldsymbol{N}^*$.

A product-form $\mathfrak{Q}(X_1, \cdots, X_m)=(\cdots((X_1X_2)X_3))\cdots)X_m$ is called here that it has a *nested* type of weight m, where X_i are the indeterminates over the m-lattice of the normal subgroups of G.

A group G is called an (N)-*group* if it satisfies the following condition:

(N) *For every product-form $\mathfrak{P}$ of weight n, and for every normal subgroup $N_1, \cdots, N_n$ (repetitions allowed) of G, there exists a product-form $\mathfrak{Q}$ with nested type of weight m such that*

$$\mathfrak{Q}(N_{i_1}, \cdots, N_{i_m}) \subseteq \mathfrak{P}(N_1, \cdots, N_n),$$

where $i_1 \leq \cdots \leq i_m$.

Nilpotent groups are evidently (N)-groups.

Now we let G be an (N)-group with the a. c. c. for normal subgroups. Then by Theorems 4, 5, 6, 7, 8, 9 and 10 we obtain the following statements:

2) *The radical of any normal subgroup of G is prime.*

3) *If $(**)$ is an irredundant decomposition of a normal subgroup N of G into primary normal subgroups Q_i, the minimal primes belonging to N coincide with the minimal elements in the po-set consisting of the* $rad(Q_i)$.

4) *Suppose that $(**)$ is a decomposition of a normal subgroup N of G into primary normal subgroups Q_i with prime radicals P_i. If $P(\neq G)$ is a prime normal subgroup such that $P_1 \subseteq P, \cdots, P_s \subseteq P$, $P_{s+1} \not\subseteq P, \cdots, P_n \not\subseteq P$, then*

$$U(N, P) = Q_1 \cap \cdots \cap Q_s .$$

5) *Suppose that $(**)$ is a short decomposition of N with (prime) radicals $P_i = rad(Q_i)$. If P is any minimal prime belonging to N and $P \neq G$, then $U(N, P) = Q_i$ for some i, and $U(N, P)$ is primary.*

6) *Suppose that $(**)$ is a short decompolition of N, and let $P_1, \cdots, P_s$ be the minimal primes belonging to N such that $P_i \neq G$ $(i=1, \cdots, s)$. Then*

$$N = U(N, P_1) \cap \cdots \cap U(N, P_s) \cap Q_{s+1} \cap \cdots \cap Q_n .$$

7) *Suppose that $(**)$ is a short decomposition of N, and let P be a prime normal subgroup such that $N \subseteq P \neq G$. Then $P = rad(Q_i)$ for some Q_i, if and only if $U(N, P):P \supsetneqq U(N, P)$.*

G is said to have *the weak Artin-Rees property*, if for any normal subgroup N of G and for any normal subgroup (x) with single generator x, there exists a commutator-product $\mathfrak{P}$ of (x) such that $N \cap \mathfrak{P} \subseteq [N, (x)]$. Then we obtain

8) *In order that every normal subgroup of G is decomposed into a finite number of primary normal subgroups, it is necessary and sufficient that the weak Artin-Rees property holds for G.*

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References

[1] V.A. Andrunakievič and Ju. M. Rjabuhin: *Additive ideal theory of ideals in rings, modules and groupoids*, Doklady **163** (translated in English) (1966), 653–657.

[2] E.A. Behrens: *Zur additiven Idealtheorie in nichtassoziativen Ringen*, Math. Z. **64** (1956), 169–182.

[3] G. Birkhoff: Lattice Theory, rev. ed., Amer. Math. Soc. Colloquium Publications, 25, New York, 1948.

[4] R. Croisot: *Théorie noethérienne des idéaux dans les anneaux et les demigroupes non nécessariement commutatifs*, Séminaire P. Dubreil et C. Pisot 10, Exposé 22, 1956–57.

[5] R.P. Dilworth: Lattice Theory, Amer. Math. Soc. Proceedings of Symposia in Pure Mathematics II, 1961.

[6] R.P. Dilworth and Peter Crawley: *Decomposition theory for lattices without chain conditions*, Trans. Amer. Math. Soc. **96** (1960), 1–22.

[7] M.L. Dubreil-Jacotin L. Lesieur et R. Croisot: Leçons sur la Théorie de Trellis de Structures Algébriques Ordonnées et des Trellis Géométriques, Gauthier-Villars, Paris, 1953.

[8] Y. Kurata: *On an additive ideal theory in a non-associative ring*, Math. Z. **88** (1963), 129–135.

[9] L. Lesieur: *Sur les demi-groupes reticules satisfaisant à une condition de chaine*, Bull. Soc. Math. France **83** (1955), 161–193.

[10] P.J. McCarthy: *Primary decomposition in multiplicative lattices*, Math. Z. **90** (1965), 185–189.

[11] N.H. McCoy: *Prime ideals in general rings*, Amer. J. Math. **71** (1949), 823–833.

[12] K. Murata: *Additive ideal theory in multiplicative systems*, J. Inst. Polytec. Osaka City Univ. **10** (1959), 91–115.

[13] ———: *On isolated components of ideals in multiplicative systems*, J. Inst. Polytec. Osaka City Univ. **11** (1960), 1–9.

[14] ———: *On nilpotent-free multiplicative systems*, Osaka Math. J. **14** (1962), 53–70.

[15] D.C. Murdoch: *Contributions to noncommutative ideal theory*, Canad. J. Math. **4** (1952), 43–57.

[16] T. Nakayama and G. Azumaya: Algebra II (in Japanese), Tokyo, 1954.

[17] V. Rolf and P. Holzapfel: *Eine allgemeine Primärzerlegungstheorie*, Math. Nach. **41** (1969), 227–245.

[18] E. Strohmeier: *Homogeneous ideal decompositions in general graded rings with ACC*, Math. Ann. **181** (1969), 103–108.

[19] G. Szász: Einführung in die Verbandstheorie, Akadémiai Kiado, Budapest, 1962.