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ON MULTIPLY TRANSITIVE GROUPS IX

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1. Introduction

Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$, and let P be a Sylow 2-subgroup of a stabilizer of four points in G. In a previous paper [6] the following theorem has been established: If P fixes exactly six points, then G must be A_6.

The purpose of this paper is to prove the following

Theorem. *Let G be a 4-fold transitive group. If a Sylow 2-subgroup of a stabilizer of four points in G fixes exactly eleven points, then G must be M_{11}.*

Therefore, by a theorem of M. Hall [1. Theorem 5.8.1], this theorem implies the following

Corollary. *Let G be a 4-fold transitive group. If a Sylow 2-subgroup P of a stabilizer of four points in G is not identity, then P fixes exactly four, five or seven points.*

We shall follow the notations of T. Oyama [5] and [6].

2. Preliminary lemmas

Lemma 1. *Let R be a 2-subgroup of a group G and H a subgroup of G. If $R\leqq N_G(H)$ and $|H|$ is even, then there exists an involution a of H such that $R\leqq C_G(a)$.*

Proof. Since the number of Sylow 2-subgroups of H is odd, by assumption R normalizes some non-identity Sylow 2-subgroup Q of H. Since the number of central involutions of Q is also odd, R centralizes some involution of Q.

Lemma 2. *Let G be a permutation group and H a stabilizer of some points in G. Suppose that a subgroup U of H has the following property:*

(*) *If a subgroup V of H is conjugate to U in G, then it is conjugate to U in H. Then there is a subgroup N of $N_G(U)$ such that N fixes $I(H)$ as a set and $N^{I(H)}=N_G(H)^{I(H)}$.*

Proof. Let N be a subgroup of $N_G(U)$ consisting of all the elements of $N_G(U)$ which fix $I(H)$ as a set. Obviously $N^{I(H)} \leqq N_G(H)^{I(H)}$. Let x be any element of $N_G(H)$. Then U^x is a subgroup of H. By $(*)$, there is an element y of H such that $U^x = U^y$. Then $xy^{-1} \in N_G(U)$. Since xy^{-1} fixes $I(H)$ as a set, $xy^{-1} \in N$. Furthermore $(xy^{-1})^{I(H)} = x^{I(H)} \cdot (y^{-1})^{I(H)} = x^{I(H)}$. Hence $x^{I(H)} \in N^{I(H)}$. Thus $N^{I(H)} = N_G(H)^{I(H)}$.

3. Proof of the theorem

Let G be a 4–fold transitive group. By the theorem of M. Hall, if a stabilizer of four points in G is of odd order, then G must be one of the following groups: S_4, S_5, A_6, A_7 or M_{11}. Therefore to prove our theorem we may assume that a Sylow 2–subgroup of a stabilizer of four points in G is not identity.

Lemma 3. *Let G be a 4–fold transitive group on $\Omega = \{1, 2, \cdots, n\}$, and P a Sylow 2–subgroup of G_{1234}. Suppose that P is not identity and $N_G(P)^{I(P)} = M_{11}$. For a point t of a minimal orbit of P in $\Omega - I(P)$ let $P_t = Q$, $N_G(Q) = N$ and $I(Q) = \Delta$. Then a Sylow 2–subgroup R of N_{ijkl} satisfies the following conditions, where $\{i, j, k, l\} \subset \Delta$.*

(*a*) *$I(R) = I(P')$, where P' is some Sylow 2–subgroup of G_{ijkl}.*

(*b*) *R^Δ is a Sylow 2–subgroup of $(N^\Delta)_{ijkl}$.*

(*c*) *R^Δ is a non-identity semi-regular group.*

(*d*) *$N_N(R)^{I(R)} \leqq M_{11}$.*

Proof. (a), (b) and (c) follow from Lemma 1 in [6].

(d). Obviously $N_N(R)^{I(R)} \leqq N_G(G_{I(R)})^{I(R)}$. By (a) and Lemma 2 $N_G(G_{I(R)})^{I(R)} = N_G(P')^{I(P')}$. Hence $N_N(R)^{I(R)} \leqq M_{11}$.

In the following lemma we consider a permutation group G on $\Omega = \{1, 2, \cdots, n\}$, which is not necessarily 4–fold transitive.

Lemma 4. *Let P be a Sylow 2–subgroup of any stabilizer of four points in G. Then there is no group, which satisfies the following conditions.*

(*a*) *$|I(P)| = 11$ and $N_G(P)^{I(P)} \leqq M_{11}$.*

(*b*) *P is a non-identity semi-regular group.*

The proof will be given in various steps. Suppose by way of contradiction that G is a counterexample to Lemma 4.

(1) *P has only one involution.*

Proof. By the same argument as in Case I of [5] we have this assertion.

(2) *Any involution of G fixes exactly eleven points.*

Proof. Let x be an arbitrary involution of G. If $|I(x)| \geqq 4$, then $|I(x)|$

$=11$ by assumption. Since $|\Omega|$ is odd, $|I(x)|$ is odd and so $|I(x)|=1, 3$ or 11.

Suppose $|I(x)|=1$ or 3. We may assume that x is of the form

$$x=(1\ 2)(3\ 4)\cdots.$$

Since $x\in N_G(G_{1234})$, x normalizes some Sylow 2-subgroup P' of G_{1234}. Let $I(P')=\{1, 2, \cdots, 11\}$. By assumption $x^{I(P')}\in M_{11}$. Hence we may assume that x is of the form

$$x=(1\ 2)(3\ 4)(5\ 6)(7\ 8)(9)(10)(11)(i\,j)\cdots.$$

Thus $|I(x)|\neq 1$. Let a be a involution of P'. Then x commutes with a by (1). Suppose $x^{\Omega-I(P')}\neq a^{\Omega-I(P')}$. Then we may assume that x and a have two 2-cycles $(i\,j)(k\,l)$ and $(i\,k)(j\,l)$ respectively. Since $\langle x, a\rangle < N_G(G_{ijkl})$, $\langle x, a\rangle$ normalizes some Sylow 2-subgroup P'' of G_{ijkl}. Since $x^{I(P'')}$ is an involution of M_{11} and x fixes only three points 9, 10 and 11, $x^{I(P'')}$ fixes these three points. Therefore $(N_G(P'')^{I(P'')})_{9\,10\,11}\geqq\langle x, a\rangle^{I(P'')}$ and $x^{I(P'')}\neq a^{I(P'')}$. But this is a contradiction, because a stabilizer of three points in M_{11} is a quaternion group. Therefore $x^{\Omega-I(P')}=a^{\Omega-I(P')}$, and so $a=(1)(2)\cdots(11)(i\,j)\cdots$. Then $\langle a, x\rangle$ also normalizes some Sylow 2-subgroup P''' of G_{12ij}. In the same way we get $I(P''')\supset\{9, 10, 11\}$. Since $I(P''')\supset\{1, 2, 9, 10, 11, i, j\}$, $a^{I(P''')}=(1)(2)(9)(10)(11)(i\,j)\cdots$. By assumption (a) this is a contradiction.

Thus $|I(x)|=11$.

(3) $|\Omega|\geqq 27$.

Proof. Let x be an involution. By (2), we may assume that x is of the form

$$x=(1)(2)\cdots(11)(12\ 13)\cdots.$$

By Lemma 1, x commutes with some involution y of $G_{1\,2\,12\,13}$. By (2), $|I(y)|=11$. Since $x^{I(y)}\in M_{11}$ and $y^{I(x)}\in M_{11}$, we may assume that x and y are of the forms

$$y=(1)(2)(3)(4\ 5)(6\ 7)(8\ 9)(10\ 11)(12)(13)\cdots(19)\cdots,$$
$$x=(1)(2)\cdots(11)(12\ 13)(14\ 15)(16\ 17)(18\ 19)\cdots.$$

Then xy is also an involution. Hence $|I(xy)|=11$. Therefore xy must be of the following form

$$xy=(1)(2)(3)(4\ 5)(6\ 7)\cdots(18\ 19)(20)(21)\cdots(27)\cdots.$$

Thus $|\Omega|\geqq 27$.

(4) *$N_G(P)^{I(P)}$ is one of the following:*

Case I. $N_G(P)^{I(P)}$ *is transitive.* $N_G(P)^{I(P)}=M_{11}$ *or* $LF_2(11)$.

Case II. $N_G(P)^{I(P)}$ *has exactly two orbits, say* Δ *and* Γ.

(*i*) $|\Delta|=1$ *and* $|\Gamma|=10$. $N_G(P)^{I(P)}=M_{10}$ *or* M'_{10}, *where* M'_{10} *is a commutator subgroup of* M_{10}.

(*ii*) $|\Delta|=2$ *and* $|\Gamma|=9$. $N_G(P)^{I(P)}=N(M_9)$ *or* $N(M_9^*)$, *where* $N(M_9)=N_{M_{11}}(M_9)$ *and* $|N(M_9): N(M_9^*)|=2$.

(*iii*) $|\Delta|=5$ *and* $|\Gamma|=6$. $N_G(P)^{I(P)}=S_5\cdot S_6^*$, *where* $S_5\cdot S_6^*$ *is isomorphic to* S_5.

Proof. Let $I(P)=\{1, 2, \cdots, 11\}$. Then we may assume that an involution a of P is of the form

$$a=(1)(2)\cdots(11)(ij)\cdots .$$

For any two points i_1, i_2 in $I(P)$ a normalizes $G_{i_1 i i j k}$. By Lemma 1, there is an involution $x_{i_1 i_2}$ of $G_{i_1 i_2 i j}$ such that $x_{i_1 i_2}$ commutes with a. We denote the restriction of $x_{i_1 i_2}$ on $I(P)$ by $a_{i_1 i_2}$. By assumption (a) $a_{i_1 i_2}$ fixing a point i_3, is of the form

$$a_{i_1 i_2}=(i_1)(i_2)(i_3)(i_4\, i_5)(i_6\, i_7)(i_8\, i_9)(i_{10}\, i_{11}) .$$

Let $T=\langle\{a_{i_1 i_2} \mid \{i_1, i_2\}\subset I(P)\}\rangle$. Then $T\leqq N_G(a)^{I(P)}$. Since a is a unique involution of P, by Lemma 2 $N_G(a)^{I(P)}=N_G(G_{I(P)})^{I(P)}=N_G(P)^{I(P)}$. Therefore $T\leqq N_G(P)^{I(P)}\leqq M_{11}$.

Case I. Let $N_G(P)^{I(P)}$ be transitive. Since there exists an involution in T fixing three points, by a theorem of Galois [7. Theorem 11.6] $N_G(P)^{I(P)}$ is nonsolvable. Since a nonsolvable transitive group of degree 11 in M_{11} is M_{11} or $LF_2(11)$ (see [2]),

$$N_G(P)^{I(P)}=M_{12} \quad \text{or} \quad LF_2(11) .$$

Case II. Let $N_G(P)^{I(P)}$ be intransitive. Since $T\leqq N_G(P)^{I(P)}$, T is also intransitive. Therefore we denote one of the T-orbits by Δ.

i) Suppose $|\Delta|=1$. Let $\Delta=\{1\}$ and $\Gamma=\{2, 3, \cdots, 11\}$. For any two points i_1, i_2 in Γ there is an involution $a_{i_1 i_2}$ of the following form

$$a_{i_1 i_2}=(1)(i_1)(i_2)(i_3\, i_4)(i_5\, i_6)(i_7\, i_8)(i_9\, i_{10}) .$$

By a lemma of D. Livingstone and A. Wagner [3. Lemma 6] T is doubly transitive on Γ. Since $T\leqq M_{10}$, $|T|=10\cdot 9\cdot 2k$, where $k=1$, 2 or 4. By a theorem of G. Frobenius [4. Proposition 14.5],

$$\sum_{x\in T}\alpha_2(x)=\frac{|T|}{2}=10\cdot 9\cdot k .$$

On the other hand since any two points j_1, j_2 in Γ determin uniquely an involution $a_{j_1 j_2}$ and conversely any involution x' of T determins exactly two points of Γ, which are fixed by x', the number of involutions is $\binom{10}{2}$. Therefore

$$\sum_{x'} \alpha_2(x') = \binom{10}{2} 4 = 10\cdot 9\cdot 2\,,$$

where x' ranges over all involutions of T. Since $\sum_{x} \alpha_2(x) \geqq \sum_{x'}' \alpha_2(x')$, $k \geqq 2$. Thus $T = M_{10}$ or M'_{10}, where M'_{10} is a commutator subgroup of M_{10}. Since $N_G(P)^{I(P)}$ is intransitive and $T \leqq N_G(P)^{I(P)}$, $N_G(P)^{I(P)} = M_{10}$ or M'_{10}.

ii) Suppose $|\Delta| = 2$. Let $\Delta = \{1, 2\}$ and $\Gamma = \{3, 4, \cdots, 11\}$. For any point i_1 of Γ there is an involution a_{1i_1} of the form

$$a_{1i_1} = (1)(2)(i_1)(i_2\, i_3)(i_4\, i_5)(i_6\, i_7)(i_8\, i_9)\,.$$

By Lemma 6 of [3] T_{12} is transitive on Γ. Since $T_{12} \leqq M_9$, $|T_{12}| = 9\cdot 2k$, where $k = 1, 2$ or 4. Since T contains an involution $a_{34} = (1\ 2)(3)(4)\cdots$, $T = T_{12} + T_{12}a_{34}$ and so $|T| = 2\cdot 9\cdot 2k$. From the theorem of G. Frobenius

$$\sum_{x \in T} \alpha_1(x^\Gamma) = 9\cdot 4k\,,$$

$$\sum_{x' \in T_{12}}' \alpha_1(x'^\Gamma) = 9\cdot 2k\,,$$

On the other hand since two points j_1, j_2 in Γ determine uniquely an involution $a_{j_1 j_2}$, which fixes three points of Γ, the number of involutions of $T_{12}a_{34}$ is $\binom{9}{2}\cdot \frac{1}{3}$. Hence

$$\sum_{x''}'' \alpha_1(x''^\Gamma) = \binom{9}{2}\cdot\frac{1}{3}\cdot 3 = 9\cdot 4\,,$$

where x'' ranges over all involutions of $T_{12}a_{34}$. Since

$$\sum_{x} \alpha_1(x^\Gamma) \geqq \sum_{x'}' \alpha_1(x'^\Gamma) + \sum_{x''}'' \alpha_1(x''^\Gamma)\,,$$

$$9\cdot 4k \geqq 9\cdot 2k + 9\cdot 4\,.$$

Hence $k \geqq 2$ or 4. Thus $T = N_{M_{11}}(M_9)$ or $N(M_9^*)$, where $N(M_9^*)$ is the following group: The index of $N(M_9^*)$ in $N_{M_{11}}(M_9)$ is 2 and $N(M_9^*)$-orbits are Δ and Γ. Similarly to i) $N_G(P)^{I(P)} = N(M_9)$ or $N(M_9^*)$.

iii) Let $|\Delta| = 3$ and $I(P) - \Delta = \Gamma$. For any two points i_1, i_2 in Γ since $|\Gamma|$ is even, there is an involution such that it's restriction on Γ fixes exactly these two points. Therefore again by Lemma 6 of [3], T^Γ is doubly transitive and so $|T| = 8\cdot 7\cdot k$. But this is impossible since $7 \nmid |M_{11}|$. Therefore there is no such T that $|\Delta| = 3$.

iv) Let $|\Delta|=4$ and $I(P)-\Delta=\Gamma$. From the results above the length of a T-orbit in Γ is not 1, 2 or 3. Therefore T^{Γ} is transitive. Since $|\Gamma|=7$, in the same way as in iii) we have a contradiction. Thus $|\Delta|\neq 4$.

v) Suppose $|\Delta|=5$. Let $\Delta=\{1, 2, \cdots, 5\}$ and $\Gamma=\{6, 7, \cdots, 11\}$. For any two points i_1, i_2 of Γ since $|\Gamma|$ is even, there is an involution such that it's restriction on Γ fixes exactly these two points. Therefore again by Lemma 6 of [3], T^{Γ} is doubly transitive. Since $T\leqq M_{11}$, $T_{\Delta}=T_{\Gamma}=\{1\}$. Hence $|T|=|T^{\Delta}|=|T^{\Gamma}|\geqq 6\cdot 5\cdot 2k$. Since $|\Delta|=5$, $|T^{\Delta}|=60$ or 120, namely $T^{\Delta}=A_5$ or S_5. On the other hand T^{Δ} has a transposition $(1)(2)(j_1)(j_2 j_3)$. Therefore $T^{\Delta}=S_5$. Thus T is isomorphic to S_5. We denote this group by $S_5\cdot S_6^*$. Similarly to i) $N_G(P)^{I(P)}=S_5\cdot S_6^*$.

vi) If $|\Delta|\geqq 6$, then $|I(P)-\Delta|\leqq 5$. Considering the length of T-orbit in $I(P)-\Delta$, we have that $N_G(P)^{I(P)}$ is one of the groups above.

REMARK. Every involution $x_{i_1 i_2}$ has the following property: $x_{i_1 i_2}$ commutes with a and fixes two points i, j, where (ij) is a 2–cycle of a. Therefore from now on we denote T by $\mathcal{F}_{ij}(a)$ or $\mathcal{F}$.

(5) *P is cyclic or a generalized quaternion group.*

Proof. This follows immediately from (1).

(6) *If P is cyclic, then the automorphism group A(P) of P is a 2–group. If P is a quaternion group, then $A(P)=S_4$. If P is a generalized quaternion group and $|P|>8$, then A(P) is a 2–group.*

Proof. For a proof see [8. IV, § 3].

(7)* *Let b be an involution of $C_G(P)\cdot N_G(P)_{I(P)}-P$ and $|P|\geqq 4$. If there is an involution c of $C_G(P)\cdot N_G(P)_{I(P)}-P$ such that c commutes with b and $b^{I(P)}\neq c^{I(P)}$, then $b\notin C_G(P)$.*

Proof. Let R be a Sylow 2–subgroup of $C_G(P)$. Then $R^{I(P)}$ is a Sylow 2–subgroup of $C_G(P)^{I(P)}$. Set $S=R\cdot P$. Then S is a 2–group and $S^{I(P)}=R^{I(P)}$. Furthermore $S_{I(P)}=(R\cdot P)_{I(P)}=P$ is a Sylow 2–subgroup of $N_G(P)_{I(P)}$. Since

$$\frac{|C_G(P)\cdot N_G(P)_{I(P)}|}{|S|}=\frac{|C_G(P)^{I(P)}|\cdot|N_G(P)_{I(P)}|}{|R^{I(P)}|\cdot|P|},$$

S is also a Sylow 2–subgroup of $C_G(P)\cdot N_G(P)_{I(P)}$. Let S' be an arbitrary Sylow 2–subgroup of $C_G(P)\cdot N_G(P)_{I(P)}$. Then $S^x=S'$, where x is some element of $C_G(P)\cdot N_G(P)_{I(P)}$. Since $C_G(P)$ is a normal subgroup of $C_G(P)\cdot N_G(P)_{I(P)}$,

* (7) and (8) are due to Professor H. Nagao. The auther is grateful to Professor H. Nagao for communicating these results.

$R^x = R'$ is a Sylow 2–subgroup of $C_G(P)$ contained in S'. Since $I(P)^x = I(P)$ and $R^{I(P)} = S^{I(P)}$, $R'^{I(P)} = S'^{I(P)}$. Thus an arbitrary Sylow 2–subgroup S' of $C_G(P) \cdot N_G(P)_{I(P)}$ contains a Sylow 2-subgroup R' of $C_G(P)$ such that $R'^{I(P)} = S'^{I(P)}$.

Suppose by way of contradiction that b belongs to $C_G(P)$. Since $|P| \geqq 4$, P has an element of order 4 by (5). If $c \in C_G(P)$, then c commutes with an element of P, whose order is at least 4. If $c \notin C_G(P)$, then above remark yields that a Sylow 2–subgroup of $C_G(P) \cdot N_G(P)_{I(P)}$ containing b and c has an element c' of $C_G(P)$ such that $c'^{I(P)} = c^{I(P)}$. Then $cc' \in P$ but $cc' \notin C_G(P)$. Since $c' \in C_G(P)$, c' commutes with cc', and so c commutes with cc'. Since cc' does not belong to the center of P, the order of cc' is at least 4. In any case, c commutes with some element y of P, where $|y| \geqq 4$. Since $b \in C_G(P)$, b also commutes with y. Since b commutes with c, $I(b) \neq I(c)$ by (1). Hence $c^{I(b)}$ fixes exactly three points, namely $|I(b) \cap I(c)| = 3$. Since $(I(b) \cap I(c))^y = I(b) \cap I(c)$ and y has no 2–cycle, y fixes $I(b) \cap I(c)$ pointwise. Thus $I(P) = I(y) \supset I(b) \cap I(c)$. Therefore $b^{I(P)}$ and $c^{I(P)}$ fix the same three points. But this is impossible since $b^{I(P)} \neq c^{I(P)}$ and the stabilizer of three points in M_{11} has only one involution. Therefore $b \notin C_G(P)$.

(8) *If* $N_G(P)^{I(P)} = M_{11}$, *then* $|P| = 2$.

Proof. Since $N_G(P)/C_G(P) \leqq A(P)$ and $N_G(P)/N_G(P)_{I(P)} \simeq N_G(P)^{I(P)} = M_{11}$, $N_G(P)_{I(P)} \ngeqq C_G(P)$ by (6). Hence $\{1\} \lneqq (C_G(P) \cdot N_G(P)_{I(P)})/N_G(P)_{I(P)} \trianglelefteq N_G(P)/N_G(P)_{I(P)} \simeq M_{11}$. Since M_{11} is a simple group, $(C_G(P) \cdot N_G(P)_{I(P)})/N_G(P)_{I(P)} = N_G(P)/N_G(P)_{I(P)}$ and so $C_G(P) \cdot N_G(P)_{I(P)} = N_G(P)$. Furthermore from this relation we get $M_{11} = N_G(P)^{I(P)} = (C_G(P) \cdot N_G(P)_{I(P)})^{I(P)} = C_G(P)^{I(P)}$.

Suppose by way of contradiction that $|P| \geqq 4$. Let a be an involution of P and I(P)$=\{1, 2, \cdots, 11\}$. We may assume that a is of the form

$$a = (1)(2)\cdots(11)(12\ 13)\cdots .$$

First assume that P is cyclic. Then $C_G(P)_{I(P)} \geqq P$. From $N_G(P) = C_G(P) \cdot N_G(P)_{I(P)}$ we get $N_G(P)/C_G(P) \simeq N_G(P)_{I(P)}/C_G(P)_{I(P)}$. Since P is a Sylow 2–subgroup of $N_G(P)_{I(P)}$ and $C_G(P)_{I(P)} \geqq P$, the order of $N_G(P)/C_G(P)$ is odd. On the other hand by (6), $A(P)$ is a 2–group. Therefore $|N_G(P)/C_G(P)| = 1$. Thus $N_G(P) = C_G(P)$.

Now since a normalizes $G_{1\,2\,12\,13}$, there is an involution b of $G_{1\,2\,12\,13}$ commuting with a by Lemma 1. We may assume that b is of the form

$$b = (1)(2)(3)(4\ 5)(6\ 7)(8\ 9)(10\ 11)(12)(13)\cdots .$$

Since $\langle a, b \rangle < N_G(G_{4\,5\,12\,13})$, there is also an involution c of $G_{4\,5\,12\,13}$ commuting with both a and b by Lemma 1. Since $\langle b, c \rangle < N_G(G_{I(P)})$, $\langle b, c \rangle$ normalizes some Sylow 2–subgroup P' of $G_{I(P)}$. Obviously $I(P) = I(P')$, $b^{I(P')} \neq c^{I(P')}$ and $N_G(P')$

$=C_G(P')$. Hence both b and c belong to $C_G(P')-P'$, which is a contradiction by (7).

Next assume that P is a generalized quaternion group. Since $C_G(P)^{I(P)}=M_{11}$, there are two 2–elements d and f of $C_G(P)$ such that $d^{I(P)}$ and $f^{I(P)}$ are involutions, $d^{I(P)}$ commutes with $f^{I(P)}$ and $d^{I(P)}\neq f^{I(P)}$. Let $I(d^{I(P)})=\{i, j, k\}$. Let Q be a Sylow 2–subgroup of $C_G(P)_{ijk}$ containing d. Since $Q_{I(P)}=Q\cap(P\cap C_G(P))=\langle a\rangle$, $Q^{I(P)}\cong Q/Q_{I(P)}=Q/\langle a\rangle$. On the other hand from $C_G(P)^{I(P)}=M_{11}$, $Q^{I(P)}$ is a quaternion group. Suppose that d is not an involution. Then a is only one involution of Q. Therefore Q is cyclic or a generalized quaternion group. Hence $Q/\langle a\rangle$ is cyclic or a dihedral group, which is a contradiction. Therefore d is an involution of $C_G(P)$. The same is true for f and af. But this is impossible by (7).

Thus $|P|=2$.

(9) *If $N_G(P)^{I(P)}$ is $LF_2(11)$, M_{10} or M'_{10}, then P is a generalized quaternion group.*

Proof. Let a be an involution of P, and $I(P)=\{1, 2, \cdots, n\}$. In the following proof if $N_G(P)^{I(P)}=M_{10}$ or M'_{10}, then we assume that it's orbits are $\{1\}$ and $\{2, 3, \cdots, 11\}$. We may assume that a is of the form

$$a=(1)(2)\cdots(11)(12\ 13)\cdots.$$

Since $a\in N_G(G_{1\,2\,12\,13})$, there is an involution b of $G_{1\,2\,12\,13}$ commuting with a. We may assume that b is of the form

$$b=(1)(2)(3)(4\ 5)(6\ 7)(8\ 9)(10\ 11)(12)(13)\cdots(19)\cdots.$$

Hence

$$a=(1)(2)\cdots(11)(12\ 13)(14\ 15)(16\ 17)(18\ 19)\cdots.$$

Since $\langle a, b\rangle<N_G(G_{4\,5\,12\,13})$, there is an involution c of $G_{4\,5\,12\,13}$ commuting with both a and b. We may assume that c is of the form

$$c=(1)(2\ 3)(4)(5)(6\ 7)(8\ 10)(9\ 11)(12)(13)(14\ 15)(16\ 18)(17\ 19)\cdots.$$

First assume that $N_G(P)^{I(P)}=LF_2(11)$. Then $\mathscr{F}\leqq LF_2(11)$, where $\mathscr{F}$ is one of the groups obtained in the proof of (4). Comparing the orders of these groups of (4) we have $\mathscr{F}=LF_2(11)$. Since $c^{I(P)}\in LF_2(11)$, there is a 2–element c' such that $c^{I(P)}=c'^{I(P)}$ and $c'^{I(P)}\in\mathscr{F}_{16\,17}(a)$. Since $I(c'^2)\supset\{1, 2, \cdots, 11, 16, 17\}$, c' is an involution. Next assume that $N_G(P)^{I(P)}=M_{10}$ or M'_{10}. Similarly $\mathscr{F}=M_{10}$ or M'_{10}, and so we get the same element c'. Thus c' is an involution of the form

$$c'=(1)(2\ 3)(4)(5)(6\ 7)(8\ 10)(9\ 11)(16)(17)\cdots.$$

Since

$$bc' = (1)(2\ 3)(4\ 5)(6)(7)(8\ 11)(9\ 10)(16)(17)\cdots,$$

the order of bc' is also $2k$, where k is odd. Therefore $(bc')^k$ is a central involution of a dihedral group $\langle b, c'\rangle$. Thus we get an involution

$$c'' = b(bc')^k = (1)(2\ 3)(4)(5)(6\ 7)(8\ 10)(9\ 11)(16)(17)\cdots,$$

commuting with both a and b. Then $cc'' \in G_{I(P)}$ and $(cc'')^{I(b)}$ is of order 4. Thus $G_{I(P)}$ has an element of order 4. Hence $|P| \geqq 4$.

Suppose that P is cyclic. If $N_G(P)^{I(P)} = LF_2(11)$ or M'_{10}, then $N_G(P)^{I(P)}$ is a simple group. Since $|P| \geqq 4$, by the same argument as in (8) we have a contradiction. If $N_G(P)^{I(P)} = M_{10}$, then M'_{10} is only one non-identity normal subgroup of M_{10}. Therefore similary to (8) we have $C_G(P)^{I(P)} \geqq M'_{10}$ and $C_G(P) \geqq N_G(P)_{I(P)}$. Thus b and c belong to $C_G(P)$. But this is a contradiction by (7).

Thus P must be a generalized quaternion group.

(10) *If $N_G(P)^{I(P)}$ is $S_5 \cdot S_6^*$, $N(M_9)$ or $N(M_9^*)$, then P is a generalized quaternion group whose order is at least* 16.

Proof. Let $I(P) = \{1, 2, \cdots, 11\}$. We may assume that if $N_G(P)^{I(P)}$ is $S_5 \cdot S_6^*$, then $N_G(P)^{I(P)}$-orbits are $\{1, 2, \cdots, 5\}$ and $\{6, 7, \cdots, 11\}$, and if $N_G(P)^{I(P)}$ is $N(M_9)$ or $N(M_9^*)$, then $N_G(P)^{I(P)}$-orbits are $\{1, 2\}$ and $\{3, 4, \cdots, 11\}$. Let an involution a of P be of the form

$$a = (1)(2)\cdots(11)(12\ 13)(14\ 15)(16\ 17)(18\ 19)\cdots.$$

Since $a \in N_G(G_{3\,4\,12\,13})$, there is an involution b of $G_{3\,4\,12\,13}$ commuting with a. By assumption on $N_G(P)^{I(P)}$-orbits we may assume that b is of the form

$$b = (1\ 2)(3)(4)(5)(6\ 7)(8\ 9)(10\ 11)(12)(13)\cdots(19)\cdots.$$

Since $\langle a, b\rangle < N_G(G_{6\,7\,12\,13})$, there is an involution c of $G_{6\,7\,12\,13}$ commuting with both a and b. In the same way c is of the form

$$c = (1\ 2)(3)(4\ 5)(6)(7)(8\ 10)(9\ 11)(12)(13)(14\ 15)(16\ 18)(17\ 19)\cdots.$$

On the other hand since $\langle a, b\rangle < N_G(G_{6\,7\,16\,17})$, there is an involution d of $G_{6\,7\,16\,17}$ commuting with both a and b. In the case $N_G(P)^{I(P)} = N(M_9)$ or $N(M_9^*)$, by assumption on $N_G(P)^{I(P)}$-orbits $d^{I(P)} = (1\ 2)(6)(7)\cdots$. Since $c^{I(P)} = (1\ 2)(6)(7)\cdots$, $d^{I(P)} = c^{I(P)}$. In the case $N_G(P)^{I(P)} = S_5 \cdot S_6^*$, since the restriction of c on the orbit $\{6, 7, \cdots, 11\}$ is $(6)(7)(8\ 10)(9\ 11)$, $S_5 \cdot S_6^*$ has no element of a form $(6)(7)(8\ 9)(10\ 11)\cdots$. Hence the restriction of d on $\{6, 7, \cdots, 11\}$ is the same form as c. Therefore $c^{I(P)} = d^{I(P)}$. Thus in both cases $c^{I(P)} = d^{I(P)}$. On the other hand since $c^{I(b)} = (3)(4\ 5)(12)(13)(14\ 15)(16\ 18)(17\ 19)$ and $d^{I(b)} = (3)(4\ 5)(16)(17)\cdots$, $(cd)^{I(b)}$ is of order 4. Thus d is of the form

$$d = (1\ 2)(3)(4\ 5)(6)(7)(8\ 10)(9\ 11)(12\ 14)(13\ 15)(16)(17)(18\ 19)\cdots .$$

Hence

$$f = cd = (1)(2)\cdots(11)(12\ 14\ 13\ 15)(16\ 19\ 17\ 18)\cdots .$$

Next since $\langle a, b\rangle < N_G(G_{8\,9\,12\,13})$, there is an involution c' of $G_{8\,9\,12\,13}$ commuting with both a and b. By assumption on $N_G(P)^{I(P)}$-orbits $c'^{I(P)}=(1\ 2)(3)(4\ 5)(8)(9)\cdots$ or $c'^{I(P)}=(1\ 2)(4)(3\ 5)(8)(9)\cdots$. But $c'^{I(P)}\neq(1\ 2)(3)(4\ 5)(8)(9)\cdots$, since $c^{I(P)}=(1\ 2)(3)(4\ 5)(6)(7)(8\ 10)(9\ 11)$. Therefore c' is of the form

$$c' = (1\ 2)(4)(3\ 5)(8)(9)(6\ 10)(7\ 11)(12)(13)\cdots .$$

Since $(cc')^{I(b)}=(3\ 5\ 4)(12)(13)\cdots$, $|(cc')^{I(b)}|=3$. Therefore

$$c' = (1\ 2)(4)(3\ 5)(8)(9)(6\ 10)(7\ 11)(12)(13)(14\ 16)(15\ 17)(18\ 19)\cdots \text{ or}$$
$$c' = (1\ 2)(4)(3\ 5)(8)(9)(6\ 10)(7\ 11)(12)(13)(14\ 18)(15\ 19)(16\ 17)\cdots .$$

Let c' be of the first form. Then $c'fc'=f'$ is of the form

$$f' = (1)(2)\cdots(11)(12\ 16\ 13\ 17)(14\ 18\ 15\ 19)\cdots .$$

Let c' be of the second form. Then $c'fc'=f'$ is of the form

$$f' = (1)(2)\cdots(11)(12\ 18\ 13\ 19)(14\ 17\ 15\ 16)\dots .$$

Thus in any case $H=\langle f, f'\rangle$ is a subgroup of $G_{I(P)}$, and a Sylow 2–subgroup P' of H is a quaternion group, because the restrictions of P' and H on $\{12, 13, \cdots, 19\}$ have the same form. From now on we may assume that c' is of the first form.

Suppose $|P|=8$. Then P' is a Sylow 2–subgroup of $G_{I(P)}$. Since $\langle b, c'\rangle < N_G(H)$, there is a Sylow 2–subgroup P'' of H such that $\langle b, c'\rangle < N_G(P'')$. Since $b\in C_G(H)$, $b\in C_G(P'')$. By the conjugacy of Sylow 2–subgroups of $G_{I(P)}$, $N_G(P'')^{I(P'')}=S_5\cdot S_6^*$, $N(M_9)$ or $N(M_9^*)$.

First assume that $N_G(P'')^{I(P'')}=S_5\cdot S_6^*$. Since $N_G(P'')^{I(P'')}\trianglerighteq C_G(P'')^{I(P'')}$, $C_G(P'')^{I(P'')}\cong S_5$, A_5 or $\{1\}$. On the other hand $C_G(P'')^{I(P'')}$ has an involution $b^{I(P'')}$, whose restriction on $\{1, 2, \cdots, 5\}$ is a transposition. Therefore $C_G(P'')^{I(P'')} =N_G(P'')^{I(P'')}\cong S_5$. Hence $N_G(P'')=C_G(P'')\cdot N_G(P'')^{I(P'')}$. Thus the involution c' belongs to $C_G(P'')\cdot N_G(P'')_{I(P'')}-P''$ and commutes with b, which is impossible by (7).

Next assume that $N_G(P'')^{I(P'')}=N(M_9)$ or $N(M_9^*)$. Since $\langle b, c'\rangle^{I(P'')} < (N_G(P'')^{I(P'')})_{1\,2\,4}$, $N_G(P'')^{I(P'')}$ has the following element

$$x = (1)(2)(4)(3\ 8\ 5\ 9)(6\ 10\ 11\ 7) .$$

Then $(b^{I(P'')})^x=c'^{I(P'')}$. Since $b\in C_G(P'')$ and $N_G(P'')^{I(P'')}\trianglerighteq C_G(P'')^{I(P'')}$,

$c'^{I(P'')}\in C_G(P'')^{I(P'')}$. Therefore there is an element $y\in C_G(P'')$ such that $c'^{I(P'')}=y^{I(P'')}$. Then $yc'\in N_G(P'')_{I(P'')}$, and so $c'\in y^{-1}N_G(P'')_{I(P'')}$. Thus $c'\in C_G(P'')\cdot N_G(P'')_{I(P'')}$, which is a contradiction by (7).

Thus we have $|P|\geqq 16$.

(11) *The case $N_G(P)^{I(P)}=S_5\cdot S_6^*$ does not occur. If $N_G(P)^{I(P)}=LF_2(11)$, M_{10} or M'_{10}, then P is a quaternion group.*

Proof. Let $N_G(P)^{I(P)}=S_5\cdot S_6^*$, $LF_2(11)$, M_{10} or M'_{10}. By (9) and (10) P is a generalized quaternion group. Since $N_G(P)/C_G(P)$ is a subgroup of $A(P)$ and $N_G(P)^{I(P)}$ is a simple group or $N_G(P)^{I(P)}$ has a simple normal subgroup of index 2, $N_G(P)^{I(P)}/C_G(P)^{I(P)}$ is of order 1 or 2 by (6). Hence $C_G(P)$ has 2–element x such that $x^{I(P)}$ is an involution.

If x is an involution, then x fixes eight points of $\Omega-I(P)$. Since P is semiregular and $x\in C_G(P)$, $|P|=8$.

If x is not an involubiton, then $x^2=a$, where a is an involution of P. Let b and c be the generators of P such that $b^{2^k}=c^2=a$. Set $y=b^{2^{k-1}}$. Then y is of order 4. Since $x\in C_G(P)$, $(xy)^2=x^2y^2=a\cdot a=1$. Thus xy is an involution commuting with b. Since xy fixes eight points of $\Omega-I(P)$, the order of b is at most 8. If b is of order 8, then $b^{I(xy)}$ has a 8–cycle and three fixed points. But M_{11} has no such element. Therefore b is of order 4. Thus $|P|=8$.

In particular by (10) there is no group such that $N_G(P)^{I(P)}=S_5\cdot S_6^*$

(12) *The case $N_G(P)^{I(P)}=M_{10}$ or M'_{10} does not occur.*

Proof. Suppose by way of contradiction that $N_G(P)^{I(P)}=M_{10}$ or M'_{10}. In the proof of (11) we have showed that $C_G(P)^{I(P)}\geqq M'_{10}$. Hence let x be a 2–element of $C_G(P)$ such that $x^{I(P)}$ is an involution.

Suppose that x is not an involution. Since $C_G(P)^{I(P)}\geqq M'_{10}$, there is a 2–element y of $C_G(P)$ such that $(y^2)^{I(P)}=x^{I(P)}$. Then a Sylow 2–subgroup of $\langle x, y\rangle$ containing x has an element z such that $z^{I(P)}=y^{I(P)}$. Since z^4 and xz^2 are 2–elements of $G_{I(P)}$ centralizing P, $z^4=1$ or a and $xz^2=1$ or a, where a is an involution of P. If $z^4=1$, then $xz^2\neq 1$ because x is not an involution. Therefore $xz^2=a$. Then $z^4=(x^{-1}a)^2=x^{-2}a^2=x^{-2}=1$, which is also a contradiction. Therefore $z^4=a$. By (11) P has an element b of order 4. Then $(bz^2)^2=b^2z^4=a\cdot a=1$. Thus bz^2 is an involution commuting with z. Since $z^4=a$, z is of order 8. Then $z^{I(P)}$ has two 4–cycles and three fixed points, hence $z^{\Omega-I(P)}$ has only 8–cycles. Since bz^2 fixes three points in $I(P)$, bz^2 fixes eight points in $\Omega-I(P)$. Thus $z^{I(bz^2)}$ has one 8–cycle and three fixed points. But M_{11} has no such element. Therefore x must be an involution.

Now since $C_G(P)^{I(P)}\geqq M'_{10}$, there are two 2–elements u and v in $C_G(P)$ such that $u^{I(P)}$, $v^{I(P)}$ and $(uv)^{I(P)}$ are all different involutions. Then by the above proof u, v and uv are involutions. Thus u commutes with v. But this is a contradiction by (7).

This contradiction shows that there is no group such that $N_G(P)^{(P)}=M_{10}$ or M'_{10}.

(13) *If* $N_G(P)^{I(P)}=M_{11}$, *then there are four points* i, j, k *and* l *of* Ω *such that* $N_G(P')^{I(P')}=N(M_9)$ *or* $N(M_9^*)$, *where* P' *is a Sylow* 2*-subgroup of* G_{ijkl}.

Proof. Let $I(P)=\{1, 2, \cdots, 11\}$, and a be an involution of P. By (8) $|P|=2$. We may assume that a is of the form

$$a=(1)(2)\cdots(11)(12\ 13)(14\ 15)(16\ 17)(18\ 19)(20\ 21)(22\ 23)(24\ 25)(26\ 27)\dots .$$

Since $a\in N_G(G_{1\,2\,12\,13})$, there is an involution b of $G_{1\,2\,12\,13}$ commuting with a. Since $|I(ab)|=11$, we may assume that b is of the form

$$b=(1)(2)(3)(4\ 5)(6\ 7)(8\ 9)(10\ 11)(12)(13)\cdots(19)(20\ 21)(22\ 23)(24\ 25)(26\ 27)\cdots .$$

Since $\langle a, b\rangle < N_G(G_{4\,5\,12\,13})$, there is an involution c of $G_{4\,5\,12\,13}$ commuting with both a and b. We may assume that c is of the form

$$c=(1\ 2)(3)(4)(5)(6\ 7)\ (8\ 10)(9\ 11)(12)(13)(14\ 15)(16\ 18)(17\ 19)(20)(21)(22\ 23)(24\ 26)(25\ 27)\cdots .$$

Since there is a Sylow 2-subgroup of $N_G(P)^{I(P)}$ such that it contains $\langle b, c\rangle^{I(P)}$ and two elements $(1)(2)(3)(4\ 6\ 5\ 7)(8\ 10\ 9\ 11)$, $(1)(2)(3)(4\ 10\ 5\ 11)(6\ 8\ 7\ 9)$, there is a 2-group of $N_G(P)$ containing $\langle a, b, c\rangle$ and the following two elements

$$x=(1)(2)(3)(4\ 6\ 5\ 7)(8\ 10\ 9\ 11)\cdots ,$$
$$y=(1)(2)(3)(4\ 10\ 5\ 11)(6\ 8\ 7\ 9)\cdots .$$

Since $x^2b\in P$, $x^2b=1$ or a. Set $\Delta=\{12, 13, \cdots, 19\}$ and $\Gamma=\{20, 21, \cdots, 27\}$. If $x^2=b$, then $x^\Delta=1$ or x^Δ has four 2-cycles. In the later case since $\langle x, a\rangle < N_G(G_{I(b)})$ and $I(x^{I(b)})=I(a^{I(b)})=\{1, 2, 3\}$, $x^{I(b)}=a^{I(b)}$. Thus xa fixes Δ pointwise. If $x^2b=a$, then $x^2=ab$. In the same way x or xa fixes Γ pointwise. Therefore if necessary we take xa instead of x, we may assume that x fixes Δ or Γ pointwise. The same is true for y.

Suppose that x fixes Δ pointwise and y fixes Γ pointwise. Since both x and y are of order 4, x^Γ has two 4-cycles and y^Δ has two 4-cycles. Since $(y^{-1}xy)^{I(P)}=(x^{-1})^{I(P)}$, $x^\Delta=1$ and $y^\Gamma=1$, $y^{-1}xyx$ fixes $\{1, 2, \cdots, 19\}$ pointwise and has 2-cycles on Γ. This is a contradiction by (2).

Therefore x and y fixes the same eleven points. Let P' be a Sylow 2-subgroup of $G_{I(\langle x, y\rangle)}$ containing $Q=\langle x, y\rangle$. Then P' is a generalized quaternion group. By (8), (11) and (12) $N_G(P')^{I(P')}=N(M_9)$, $N(M_9^*)$ or $LF_2(11)$.

Suppose that $N_G(P')^{I(P')}=LF_2(11)$. By (11) $P'=Q$. Simiraly to the proof in (11) $N_G(P')^{I(P')}=C_G(P')^{I(P')}$. Since $c\in N_G(P')$, $c\in C_G(P')\cdot N_G(P')_{I(P')}$. On the other hand $a\in C_G(P')$ and a commutes with c. Since $a^{I(P')}\neq c^{I(P')}$, we have a contradiction by (7).

Therefore $N_G(P')^{I(P')}=N(M_9)$ or $N(M_9^*)$

(14) *The case* $N_G(P)^{I(P)}=LF_2(11)$ *does not occur.*

Proof. Suppose by way of contradiction that $N_G(P)^{I(G)}=LF_2(11)$. By (11) P is a quaternion group. Let R be a Sylow 2–subgroup of $N_G(P)$. Then the lengthes of R-orbits on $I(P)$ are at most 4, but on $\Omega-I(P)$ these lengthes are at least 8. Therefore a 2–group R', which contains R as a normal subgroup, fixes $I(P)$. Hence R' normalizes some Sylow 2–subgroup P' of $G_{I(P)}$. By the conjugacy of Sylow 2–subgroup of $G_{I(P)}$ $|N_G(P)|=|N_G(P')|$. Since R is a Sylow 2–subgroup of $N_G(P)$, $|R'|\leqq|R|$. Hence $R'=R$. This shows that R is a Sylow 2–subgroup of G. Since $R^{I(P)}$ is a Sylow 2–subgroup of $N_G(P)^{I(P)}=LF_2(11)$, there are exactly three $R^{I(P)}$-orbits of length 2. Suppose that there is a Sylow 2–subgroup P'' of G_{ijkl} such that $N_G(P'')^{I(P'')}\neq LF_2(11)$, where $i. j$, k and l are some points in Ω. By (13), we may assume that $N_G(P'')^{I(P'')}=N(M_9)$ or $N(M_9^*)$. Then by (10) $|P|\geqq 16$. In the same way a Sylow 2–subgroup R'' of $N_G(P'')$ is also a Sylow 2–subgroup of G. Since $N_G(P'')^{I(P'')}=N(M_9)$ or $N(M_9^*)$, there is only one R''-orbit of length 2, which contradicts the conjugacy of Sylow 2–subgroups. Thus for any points i, j, k and l $N_G(P'')^{I(P'')}=LF_2(11)$, where P'' is a Sylow 2–subgroup of G_{ijkl}.

Now by (11) P has an element x of order 4. For a 4–cycle $(i_1 i_2 i_3 i_4)$ of x $x\in N_G(G_{i_1i_2i_3i_4})$. Therefore x normalizes some Sylow 2–subgroup P''' of $G_{i_1i_2i_3i_4}$. Then $N_G(P''')^{I(P''')}$ has the element $x^{I(P''')}$ of order 4. Hence $N_G(P''')^{I(P''')}\neq LF_2(11)$, which is a contradiction.

Thus there is no group such that $N_G(P)^{I(P)}=LF_2(11)$.

(15) *If* $N_G(P)^{I(P)}=N(M_9)$ or $N(M_9^*)$, *then* G *has two orbits, say* Γ_1 *and* Γ_2. *The length of* Γ_1 *is odd and the length of* Γ_2 *is* 2.

Proof. By (10), $|P|\geqq 16$. Let R be a Sylow 2–subgroup of $N_G(P)$. Then the lengthes of R-orbits in $I(P)$ are at most 8 and in $\Omega-I(P)$ these lengthes are at least 16. Therefore in the same way as in (14) R is a Sylow 2–subgroup of G. Let $N_G(P)^{I(P)}$-orbit of length 2 be $\{1, 2\}$. Since R fixes exactly one point i, which does not belong to $\{1, 2\}$, R is also a Sylow 2–subgroup of G_i. Since R_1 is a Sylow 2–subgroup of $N_G(P)_1$, in the same way R_1 is also a Sylow 2–subgroup of G_1. If G is transitive, then G_i is conjugate to G_1. Hence R is conjugate to R_1, which is a contradiction. Thus G is intransitive.

Let three points i_1, i_2 and i_3 belong to different orbits. For a point i_4 in $\Omega-\{i_1, i_2, i_3\}$ let P' be a Sylow 2–subgroup of $G_{i_1i_2i_3i_4}$. Since $N_G(P')^{I(P')}$ is M_{11},

$N(M_9)$ or $N(M_9^*)$, at least two points of $\{i_1, i_2, i_3\}$ belong to the same orbit of $N_G(P)^{I(P)}$, which is a contradiction. Therefore G has exactly two orbits, say Γ_1 and Γ_2. Since $|\Omega|$ is odd, we may assume that $|\Gamma_1|$ is odd and $|\Gamma_2|$ is even.

Suppose that $|\Gamma_2| \gneqq 2$. Then for three points j_1, j_2 and j_3 of Γ_2 and a point j_4 of Γ_1 let P'' be a Sylow 2-subgroup of $G_{j_1 j_2 j_3 j_4}$. Since $I(P'')\cap\Gamma_1 \ni j_4$ and $I(P'')\cap\Gamma_2 \ni j_1$, $N_G(P'')^{I(P'')}$ is intransitive. Hence $N_G(P'')^{I(P'')}$ is $N(M_9)$ or $N(M_9^*)$. Since the lengthes of Γ_2 and P''-orbits in $\Omega - I(P'')$ are even, $|\Gamma_2 \cap I(P'')|$ is even or 0. On the other hand the length of a $N_G(P'')^{I(P'')}$-orbit is 2 or 9. Hence $|\Gamma_2\cap I(P'')|=0$ or 2. But $\Gamma_2 \cap I(P'') \supset \{i_1, i_2, i_3\}$, which is a contradiction. Therefore $|\Gamma_2|=2$.

(16) *The case $N_G(P)^{I(P)}=N(M_9)$ or $N(M_9^*)$ does not occur.*

Proof. Suppose by way of contradiction that $N_G(P)^{I(P)}=N(M_9)$ or $N(M_9^*)$. Then by (15) G has two orbits, say Γ_1 and Γ_2. Let $\Gamma_1=\{3, 4, \cdots, n\}$ and $\Gamma_2=\{1, 2\}$. Set $G^{\Gamma_1}=\bar{G}$, then $\bar{G}$ is transitive. Let P' be a Sylow 2-subgroup of $G_{1 i_1 i_2 i_3}$, where $\{i_1, i_2, i_3\}\subset\Gamma_1$. Since $I(P')\ni 1$, $I(P')\supset\Gamma_2$. Hence $N_G(P')^{I(P')}$ is intransitive, and so $N_G(P')^{I(P')}=N(M_9)$ or $N(M_9^*)$. We may assume that $I(P')=\{1, 2, 3, \cdots, 11\}$.

Now let

$$a_1 = (1)(2)(3)(4\ 6\ 5\ 7)(8\ 10\ 9\ 11)\,,$$
$$a_2 = (1)(2)(3)(4\ 10\ 5\ 11)(6\ 8\ 7\ 9)\,,$$
$$a_3 = (1\ 2)(3)(4)(5)(6\ 7)(8\ 10)(9\ 11)\,,$$
$$a_4 = (1)(2)(3\ 4\ 5)(6\ 10\ 9)(7\ 8\ 11)\,.$$

Then we may assume that if $N_G(P')^{I(P')}=N(M_9)$ then $N_G(P')^{I(P')}=\langle a_2, a_3, a_4\rangle$, and if $N_G(P')^{I(P')}=N(M_9^*)$ then $N_G(P')^{I(P')}=\langle a_1, a_3, a_4\rangle$ (see [1], P. 83). Let a be an involution of P'. Then a is of the form

$$a = (1)(2)(3)\cdots(11)(ij)\cdots\,.$$

Since $a\in N_G(G_{34ij})$, an involution b of G_{34ij} commuting with a is of the form

$$b = (1\ 2)(3)(4)(5)(6\ 7)(8\ 10)(9\ 11)(i)(j)\cdots\,.$$

Since $\{1, 2\}$ is a G-orbit and $I(b)\cap\{1, 2\}=\phi$, every element $(\neq 1)$ of a Sylow 2-subgroup of $G_{I(b)}$ has a 2-cycle $(1\ 2)$. By (10) $N_G(G_{I(b)})^{I(b)}=M_{11}$. Since M_{11} is 4-fold transitive, 4, 5 and i belong to the same G_3-orbit. Since (ij) is an arbitrary 2-cycle of a, $\{4, 5, 12, 13, \cdots, n\}$ is contained in a G_3-orbit. On the other hand for any point i' of $\{6, 7, \cdots, 11\}$ since $a\in N_G(G_{3i'ij})$, an involution b' of $G_{3i'ij}$ commuting with a is of the form

$$b' = (1\ 2)(3)(i')(i)(j)\cdots\,.$$

In the same way i' and i belong to the same G_3-orbit. Thus $\bar{G}_3$ is transitive, and so $\bar{G}$ is doubly transitive. Furthermore since $N_G(G_{I(b)})^{I(b)}=M_{11}$, 5 and i belong to the same G_{34}-orbit. Since $(i\,j)$ is an arbitrary 2–cycle of a, $\{5, 12, 13, \cdots, n\}$ is contained in a G_{34}-orbit. Set $T_5=\{5, 12, 13, \cdots, n\}$, $T_6=\{6, 7\}$, $T_8=\{8, 10\}$ and $T_9=\{9, 11\}$. Then $\bar{G}_{34}$-orbits consist of some T_i's.

Suppose that T_5 is a G_{34}-orbit. For any two points j_1 in $T_6 \cup T_8 \cup T_9$ and k_1 in $\{12, 13, \cdots, n\}$ let P'' be a Sylow 2–subgroup of $G_{34\,j_1k_1}$. If $I(P'')\not\supset\Gamma_2$, then $|P''|=2$. By (10) $N_G(P'')^{I(P'')}=M_{11}$. Since $I(P'')\supseteq\{3, 4, j_1, k_1\}$, j_1 and k_1 belong to the same G_{34}-orbit. But $j_1 \notin T_5$ and $k_1 \in T_5$, which is a contradiction. Therefore $I(P'')\supset\{1, 2, 3, 4, j_1, k_1\}$. Suppose that $I(P'')$ does not contain some point j_2 of $T_6 \cup T_8 \cup T_9-\{j_1\}$. Then j_2 belongs to a P''-orbit of at least length 16, which contains some point of T_5. This is impossible since $G_{34}>P''$. Thus $I(P'')=\{1, 2, 3, 4, 6, 7, 8, 9, 10, 11, k_1\}$. Set $\Delta=I(P'')-\{k_1\}$. Since k_1 is an arbitrary point in $\{12, 13, \cdots, n\}$ and $I(P')=\Delta\cup\{5\}$, by the conjugacy of Sylow 2–subgroups of G_Δ, G_Δ is transitive on $\Omega-\Delta$. On the other hand since $a\in N_G(G_{671213})$, an involution c of G_{671213} commuting with a is of the form

$$c=(1\ 2)(3)(4\ 5)(6)(7)(8\ 11)(9\ 10)(12)(13)\cdots.$$

Since $\bar{G}$ is doubly transitive, $\{7, 12, 13, \cdots, n\}$ is a G_{36}-orbit. Since $G_{36}>G_\Delta$ and $\bar{G}_\Delta$ is transitive on $\{5, 12, 13, \cdots, n\}$, 5 must belong to the G_{36}-orbit $\{7, 12, 13, \cdots, n\}$, which is a contradiction.

Therefore there is a G_{35}-orbit containing T_5 and some T_i $(i\neq5)$. We may assume that $T_6\cup T_5$ is contained in a G_{34}-orbit. Now a Sylow 2–subgroup of G_{345} containing P' fixes no point in $\Gamma_1-\{3, 4, 5\}$. Since 5 and 6 belong to the same G_{34}-orbit, a Sylow 2–subgroup of G_{346} containing P' fixes no point in $\Gamma_1-\{3, 4, 6\}$. On the other hand since a Sylow 2–subgroup of $N_G(P')_{346}$ is also a Sylow 2–subgroup of G_{346}, a Sylow 2–subgroup of G_{346} containing P' fixes $\{5, 7, 8, \cdots, 11\}$ pointwise, which is a contradiction.

Thus there is no group such that $N_G(P)^{I(P)}=N(M_9)$ or $N(M_9^*)$.

By (11), (12), (14) and (16), $N_G(P)^{I(P)}=M_{11}$. But this is a contradiction by (13) and (16)

Thus we complete the proof of Lemma 4.

Proof of the theorem. Suppose that there is a group G different from M_{11}. Then a Sylow 2–subgroup P of G_{1234} is not identity. Set $P_t=Q$, where t is a point of a minimal P-orbit in $\Omega-I(P)$. Then by Lemma 3 $N_G(Q)^{I(Q)}$ satisfies the conditions (*a*) and (*b*) of Lemma 4. Hence we have a contradiction by Lemma 4. Thus there is no group different from M_{11}.

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