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ON CENTRALIZERS IN SEPARABLE EXTENTIONS

Dedicated to Professor Keizo Asano on his 60th birthday

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Introduction

In the previous papers [5], [6] and [9], we studied the structure of a special type of separable extension, called H-separable extension, which is a ring extension $\Gamma\subset\Lambda$ such that $\Lambda\otimes_{\Gamma}\Lambda$ is isomorphic to a direct summand of a finite direct sum of the copies of Λ as two sided Λ-module. Such an extension becomes necessarily a separable extension, that is, a ring extension $\Gamma\subset\Lambda$ such that the map π of $\Lambda\otimes_{\Gamma}\Lambda$ to Λ with $\pi(x\otimes y)=xy$ splits as two sided Λ-module.

Recently K. Hirata has characterized the structure of an H-separable extension in case Λ is isomorphic to a direct summand of a finite direct sum of the copies of Γ as two sided Γ-module. He called that Λ is centrally projective over Γ in this case. He proved that in such case $V_{\Lambda}(\Gamma)$ is a central separable C-algebra and $\Lambda\cong\Gamma\otimes_{C}V_{\Lambda}(\Gamma)$, where C is the center of Λ, and that there exists a one to one correspondence between the set of H-separable subextensions of Γ in Λ and the set of central separable C-subalgebras of $\Delta=V_{\Lambda}(\Gamma)$ by corresponding the centralizers of each of them. (See [6].)

In §1 we shall concider the H-separable extension in a weaker condition, ${}_{\Gamma}\Gamma_{\Gamma}\langle\oplus_{\Gamma}\Lambda_{\Gamma}$ which means that Γ is a direct summand of Λ as two sided Γ-module. In this case we can find a one to one correspondence between the set of separable subextension B of Γ such that ${}_{B}B_{B}\langle\oplus_{B}\Delta\Gamma_{B}$ and the set of separable C'-subalgebras of Δ by corresponding their centralizers in Λ, where C' is the center of Δ (Theorem 1.2). This theorem generalizes a result of T. Kanzaki Theorem 2 [7] in the case of central separable algebras to the case of ring extensions. In §2 we extend this correspondence to a larger class of subextensions and class of subalgebras of Δ. In §3 we concider a relation between separable extensions and quasi-Frobenius extensions.

In this paper we always assume that a ring has the identity element and subrings have the common identity.

1. Centralizers in H-separable extensions

Throughout this paper whenever we consider the ring extensions $\Gamma\subset B\subset\Lambda$, we denote by Δ, D and C respectively $V_\Lambda(\Gamma)$, $V_\Lambda(B)$ and $V_\Lambda(\Lambda)$, where $V_\Lambda(X) =\{\lambda\in\Lambda \mid \lambda x=x\lambda$ for all $x\in X\}$ with $X\subset\Lambda$.

Let M be a left Λ-module (or left Λ and right Γ-bimodule) and N a left Λ-submodule (or a Λ-Γ-submodule). When we denote ${}_\Lambda N<\oplus {}_\Lambda M$ (or ${}_\Lambda M_\Gamma<\oplus {}_\Lambda M_\Gamma$), we mean that N is a direct summand of M as left Λ-module (or Λ-Γ-bimodule). When we denote ${}_\Lambda M<\oplus_\Lambda(N\oplus\cdots\oplus N)$ (or ${}_\Lambda M_\Gamma<\oplus_\Lambda(N\oplus\cdots\oplus N)_\Gamma$), we mean that M is isomorphic to a direct summand of $N\oplus\cdots\oplus N$ as left Λ-module (or Λ-Γ-bimodule).

As usual we denote by M^Γ the subset $\{m\in M \mid \lambda m=m\lambda$ for any $\lambda\in\Gamma\}$ for any two sided Λ-module M and any subring Γ of Λ

First, we shall give a characterization of centrally projective H-separable extension.

Theorem 1.1. *For any H-separable extension $\Lambda\supset\Gamma$ the following conditions are equivalent;*

1) *Λ is centrally projective over Γ' in the sence of K. Hirata* (i.e., ${}_{\Gamma'}\Lambda_{\Gamma'}<\oplus {}_{\Gamma'}(\Gamma'\oplus\cdots\oplus\Gamma')_{\Gamma'}$), *where $\Gamma'=V_\Lambda(V_\Lambda(\Gamma))$.*

2) *Δ is a central separable C-algebra and $\Lambda\cong\Gamma'\otimes_C\Delta$.*

3) $\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda)$ *is an H-separable extension of Λ regarding Λ as a subring of* $\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda)$ *by right multiplications of elements of Λ.*

Proof. If Λ is H-separable over Γ, Λ is also H-separable over Γ' (See Theorem 1.3′ [9]). Hence the equivalence 1)$\Leftrightarrow$2) is a direct consequence of the results of K. Hirata (See the introduction of this paper). On the other hand, for any H-separable extension $\Lambda\supset\Gamma$ the map η: $\Delta\otimes_C\Lambda\to\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda)$ such that $\eta(d\otimes y)(x)=d\otimes y$ for $d\otimes y\in\Delta\otimes_C\Lambda$ and $x\in\Lambda$ is a Λ-Λ-isomorphism by Prop. 3.3 [6]. Then 2)$\Rightarrow$3) is clear by Prop. 1.7 [9], since Δ is central separable over C. Suppose 3). This implies that $\Delta\otimes_C\Lambda$ is H-separable over Λ. Since Δ is finitely generated projective over C, ${}_\Lambda\Lambda_\Lambda<\oplus_\Lambda\Delta\otimes_C\Lambda_\Lambda$. Then Δ is separable C'-algebra by Prop. 4.7 [6], since $(\Delta\otimes_C\Lambda)^\Lambda\cong(\mathrm{Hom}({}_\Gamma\Lambda, {}_\Gamma\Lambda))^\Lambda =\mathrm{Hom}({}_\Gamma\Lambda_\Lambda, {}_\Gamma\Lambda_\Lambda)\cong\Delta$, where C' is the center of $\Delta\otimes_C\Lambda$. Clearly $C'\subset(\Delta\otimes_C\Lambda)^\Lambda =\Delta\otimes 1$ and we see that C' is equal to the center of Δ. Since ${}_\Lambda\Lambda_\Lambda<\oplus_\Lambda\Delta\otimes_C\Lambda_\Lambda$, $V_{\Delta\otimes\Lambda}(\Delta\otimes 1)=V_{\Delta\otimes\Lambda}(V_{\Delta\otimes\Lambda}(\Lambda))=C\otimes_C\Lambda\cong\Lambda$ by Prop. 1.2 [9]. Then $\Delta\otimes_C\Lambda \cong\Delta\otimes_{C'}(\Delta\otimes_C\Lambda)^{\Delta\otimes 1}\cong\Delta\otimes_{C'}\Lambda$. Then we have $\Lambda=V_\Lambda(C)\cong\mathrm{Hom}({}_\Delta\Delta\otimes_C\Lambda_\Lambda, {}_\Delta\Lambda_\Lambda)\cong\mathrm{Hom}({}_\Delta\Delta\otimes_{C'}\Lambda_\Lambda, {}_\Delta\Lambda_\Lambda)\cong V_\Lambda(C')$. Hence we have $V_\Lambda(C')=\Lambda$, and $C'\subset C$. Since $C\subset C'$, we see $C=C'$, and Δ is a central separable C-algebra. $\Lambda=\Gamma'\otimes_C\Delta$ is due to Theorem 3.1 [1].

As an immediate consequence of the above theorem we have.

Corollary 1.1. *Let Λ be an H-separable extension of Γ. Then Λ is centrally projective over Γ if and only if Hom $({}_\Gamma\Lambda, {}_\Gamma\Lambda)$ is an H-separable extension of Λ and $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$.*

The next lemma is a generalization of Cor. 4.2 [6].

Lemma 1.1. *Let* $\Gamma\subset B\subset\Lambda$ *be ring extensions such that* ${}_B B\otimes_\Gamma\Lambda_\Lambda<\oplus\,{}_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$. *Then if* ${}_\Gamma\Gamma_\Gamma<\oplus\,{}_\Gamma B_\Gamma$, Δ *is separable over* D, *if* ${}_\Gamma B_\Gamma<\oplus\,{}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$, Δ *is H-separable over D.*

Proof. ${}_B B\otimes_\Gamma\Lambda_\Lambda<\oplus\,{}_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$ implies that ${}_D\Delta$ is finitely generated projective and ${}_B B\otimes_\Gamma\Lambda_\Lambda\cong{}_B\mathrm{Hom}\,({}_D\Delta,\ {}_D\Lambda)$ (See Prop. 2.1 [6] or Theorem 1.2 [5]). Hence we have Δ-Δ-isomorphisms

$$\begin{aligned}\Delta\otimes_D\Delta&\cong\mathrm{Hom}\,({}_\Gamma\Lambda_\Lambda,\ {}_\Gamma\Lambda_\Lambda)\otimes_D\Delta\cong\mathrm{Hom}\,({}_\Gamma\mathrm{Hom}\,({}_D\Delta,\ {}_D\Lambda)_\Lambda,\ {}_\Gamma\Lambda_\Lambda)\\&\cong\mathrm{Hom}\,({}_\Gamma B\otimes_\Gamma\Lambda_\Lambda,\ {}_\Gamma\Lambda_\Lambda)\cong\mathrm{Hom}\,({}_\Gamma B_\Gamma,\ {}_\Gamma\mathrm{Hom}\,(\Lambda_\Lambda,\ \Lambda_\Lambda)_\Gamma)\cong\mathrm{Hom}\,({}_\Gamma B_\Gamma,\ {}_\Gamma\Lambda_\Gamma)\end{aligned}$$

the composition η of which is such that $\eta(d\otimes e)(b)=dbe$ for any $d,\ e\in\Delta$ and $b\in B$. Then we have a commutative diagram of Δ-Δ-maps

$$\begin{array}{ccl}\pi:\ \Delta\otimes_C\Delta & \longrightarrow & \Delta \quad (\pi(d\otimes e)=de \text{ for } \forall d,\ e\in\Delta)\\ \downarrow & & \downarrow\\ i^*:\ \mathrm{Hom}\,({}_\Gamma B_\Gamma,\ {}_\Gamma\Lambda_\Gamma) & \rightarrow & \mathrm{Hom}\,({}_\Gamma\Gamma_\Gamma,\ {}_\Gamma\Lambda_\Gamma)\end{array}$$

where i^* is induced by the inclusion map $i\colon\Gamma\subset\Lambda$. Hence if ${}_\Gamma\Gamma_\Gamma<\oplus\,{}_\Gamma B_\Gamma$, i^* and π split as Δ-Δ-map, and Δ is separable over D. If ${}_\Gamma B_\Gamma<\oplus\,{}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$, ${}_\Delta\Delta\otimes_D\Delta_\Delta\cong{}_\Delta\mathrm{Hom}\,({}_\Gamma B_\Gamma,\ {}_\Gamma\Lambda_\Gamma)_\Delta<\oplus\,{}_\Delta(\sum^\oplus\mathrm{Hom}\,({}_\Gamma\Gamma_\Gamma,\ {}_\Gamma\Lambda_\Gamma))_\Delta\cong{}_\Delta(\sum^\oplus\Delta)_\Delta$, and Δ is H-separable over D.

Proposition 1.1. *Let* $\Gamma\subset B\subset\Lambda$ *be any ring extensions. Then, if B is separable over* Γ, ${}_D D_D<\oplus\,{}_D\Delta_D$, *and if B is H-separable over* Γ, ${}_D\Delta_D<\oplus\,{}_D(D\oplus\cdots\oplus D)_D$ *(See Prop.* 4.1 [6]).

Proof. Suppose B is separable over Γ. Then there exists $\sum x_i\otimes y_i$ in $B\otimes_\Gamma B$ such that $\sum xx_i\otimes y_i=\sum x_i\otimes y_ix$ for all $x\in B$ and $\sum x_iy_i=1$. Then the map $f^*\colon\Delta\to\mathrm{D}$ such that $f^*(d)=\sum x_idy_i$ for every $d\in\Delta$ is a D-D-map. If $d\in D$, then $f^*(d)=\sum x_idy_i=d\sum x_iy_i=d$. Hence f^* splits as two sided D-homomorphism. Thus ${}_D D_D<\oplus\,{}_D\Delta_D$. Suppose ${}_B B\otimes_\Gamma B_B<\oplus\,{}_B(B\oplus\cdots\oplus B)_B$. Then, ${}_D\Delta_D\cong{}_D\mathrm{Hom}\,({}_B B\otimes_\Gamma B_B,\ {}_B\Lambda_B)_D<\oplus\,{}_D(\sum^\oplus\mathrm{Hom}\,({}_B B_B,\ {}_B\Lambda_B))_D\cong{}_D(\sum^\oplus D)_D$.

To obtain the next proposition we prepare the following lemma

Lemma 1.2. *Let* $\Gamma\subset B\subset\Lambda$ *be ring extensions such that* ${}_B B\otimes_\Gamma\Lambda_\Lambda<\oplus\,{}_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$ *and* $V_\Lambda(V_\Lambda(B))=B$. *Then we have*

1) ${}_B B\otimes_{\Gamma'}\Lambda_\Lambda<\oplus\,{}_B(\Lambda\oplus\cdots\oplus)_\Lambda$.

2) *If furthermore B is right finitely generated projective over* Γ, *the map* $\theta\colon B\otimes_\Gamma B\to Hom\,({}_D\Delta_D,\ {}_D\Lambda_D)$ *such that* $\theta(a\otimes b)(d)=adb$ *for* $a\otimes b\in B\otimes_\Gamma B$ *and* $d\in\Delta$ *is a B-B-isomorphism, where* ${}_B Hom\,({}_D\Delta_D,\ {}_D\Lambda_D)_B$ *is induced by* ${}_{B-D}\Lambda_{B-D}$.

Proof. 1). If ${}_B B\otimes_\Gamma\Lambda_\Lambda<\oplus\,{}_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$; then we have the B-Λ-isomorphism η of $B\otimes_\Gamma\Lambda$ to $\mathrm{Hom}\,({}_D\Delta,\ {}_D\Lambda)$ with $\eta(b\otimes\lambda)(d)=bd\lambda$ and Δ is left

D-finitely generated projective. Then η is factored through $B\otimes_{\Gamma}\Lambda\rightarrow B\otimes_{\Gamma'}\Lambda\rightarrow \mathrm{Hom}\,({}_D\Delta,\ {}_D\Lambda)$ where the first map is the canonical epimorphism and the latter map is similar to η. Hence we have the B-Λ-isomorphism $\eta'\ {}_BB\otimes_{\Gamma'}\Lambda\rightarrow \mathrm{Hom}\,({}_D\Delta,\ {}_D\Lambda)$ with $\eta'(b\otimes\lambda)(d)=bd\lambda$. This and together with the finitely generated projectivity of ${}_D\Delta$ imply that ${}_BB\otimes_{\Gamma'}\Lambda_{\Lambda}<\oplus_B(\Lambda\oplus\cdots\oplus\Lambda)_{\Lambda}$ by the remark stated after Theorem 1.2 [5] (See page 110 [5]).

2) Since B_{Γ} is finitely generated projective, we have

$$ {}_BB\otimes_{\Gamma}B_B\cong{}_BB\otimes_{\Gamma}\mathrm{Hom}\,({}_{\Lambda}\Lambda_D,\ {}_{\Lambda}\Lambda_D)_B \quad (\text{Since } B=V_{\Lambda}(D)\cong\mathrm{Hom}\,({}_{\Lambda}\Lambda_D,\ {}_{\Lambda}\Lambda_D)) $$
$$ \cong{}_B\mathrm{Hom}\,({}_{\Lambda}\mathrm{Hom}\,(B_{\Gamma},\ \Lambda_{\Gamma})_D,\ {}_{\Lambda}\Lambda_D)_B $$

On the other hand,

$$ {}_{\Lambda}\Lambda\otimes_D\Delta_{\Delta}\cong{}_{\Lambda}\mathrm{Hom}\,(\Lambda_{\Lambda},\ \Lambda_{\Lambda})\otimes_D\Delta_{\Delta}\cong{}_{\Lambda}\mathrm{Hom}\,(\mathrm{Hom}\,({}_D\Delta,\ {}_D\Lambda)_{\Lambda},\ \Lambda_{\Lambda})_{\Delta} $$
$$ \cong{}_{\Lambda}\mathrm{Hom}\,(B\otimes_{\Gamma}\Lambda_{\Lambda},\ \Lambda_{\Lambda})_{\Delta}\cong{}_{\Lambda}\mathrm{Hom}\,(B_{\Gamma},\ \mathrm{Hom}(\Lambda_{\Lambda},\ \Lambda_{\Lambda})_{\Gamma})_{\Delta} $$
$$ \cong{}_{\Lambda}\mathrm{Hom}\,(B_{\Gamma},\ \Lambda_{\Gamma})_{\Delta} \quad (\text{See Prop. 2.1 [6]})\,. $$

Hence

$$ {}_BB\otimes_{\Gamma}B_B\cong{}_B\mathrm{Hom}\,({}_{\Lambda}\Lambda\otimes_D\Delta_D,\ {}_{\Lambda}\Lambda_D)_B\cong{}_B\mathrm{Hom}\,({}_D\Delta_D,\ {}_D\mathrm{Hom}\,({}_{\Lambda}\Lambda_{\Lambda},\ {}_{\Delta}\Lambda)_D)_B $$
$$ \cong{}_B\mathrm{Hom}\,({}_D\Delta_D,\ {}_D\Lambda_D)_B $$

The composition of the above maps is exactly θ. Thus we have completed the proof of this lemma.

Proposition 1.2. *Let $\Gamma\subset B\subset\Lambda$ be ring extensions wihich satisfy the condition in Lemma* 1.2. *Then, we have*

1) *Δ is a separable extension of D if and only if ${}_{\Gamma'}\Gamma'{}_{\Gamma'}<\oplus_{\Gamma'}B_{\Gamma'}$, and Δ is an H-separable extension of D if and only if ${}_{\Gamma'}B_{\Gamma'}<\oplus_{\Gamma'}(\Gamma'\oplus\cdots\oplus\Gamma')_{\Gamma'}$*

2) *If furthermore B is right Γ-finitely generated projective, then B is a separable extension of Γ if and only if ${}_DD_D<\oplus_D\Delta_D$, and B is an H-separable extension of Γ if and only if ${}_D\Delta_D<\oplus_D(D\oplus\cdots\oplus D)_D$.*

Proof. Since the ring extensions $\Gamma'\subset B\subset\Lambda$ satisfy the same condition by Lemma 1.2, the 'if' parts of both statements of 1) have been proved by Lemma 1.1. All the 'only if' parts of both 1) and 2) have been proved in Proposition 1.1. Hence we need to prove only the 'if' parts of 2). Since we have obtained the B-B-isomorphism θ of $B\otimes_{\Gamma}B$ to $\mathrm{Hom}\,({}_D\Delta_D,\ {}_D\Lambda_D)$, we have a commutative diagram of B-B-maps

$$ \begin{array}{ccc} \pi:\ B\otimes_{\Gamma}B & \longrightarrow & B \quad (\pi(x\otimes y)=xy \text{ for any } x,\ y\in B) \\ \downarrow & & \downarrow \\ i^*:\ \mathrm{Hom}\,({}_D\Delta_D,\ {}_D\Lambda_D) & \rightarrow & \mathrm{Hom}\,({}_DD_D,\ {}_D\Lambda_D) \end{array} $$

where i^* is induced by the inclusion map i: $D\subset\Delta$, and all the vertical maps are isomorphisms. Hence if ${}_D D_D \langle \oplus_D \Delta_D$, i^* and π split as B-B-map, and B is separable over Γ. If ${}_D\Delta_D \langle \oplus_D (D \oplus \cdots \oplus D)_D$, ${}_B\mathrm{Hom}\,({}_D\Delta_D, {}_D\Lambda_D)_B \langle \oplus {}_B(\sum^{\oplus}\mathrm{Hom}\,({}_D D_D, {}_D\Lambda_D))_B$. Then, ${}_B B \otimes_\Gamma B_B \langle \oplus_B (B \oplus \cdots \oplus B)_B$, and B is H-separable over Γ.

Corollary 1.2. *Let Λ be an H-separable extension of Γ. Then Δ is separable (resp. central separable) C-algebra if and only if ${}_{\Gamma'}\Gamma'_{\Gamma'} \langle \oplus_{\Gamma'}\Lambda_{\Gamma'}$ (resp. ${}_{\Gamma'}\Lambda_{\Gamma'} \langle \oplus_{\Gamma'}(\Gamma' \oplus \cdots \oplus \Gamma')_{\Gamma'}$).*

Proof. Since Δ is a finitely generated C-module, Δ is central separable over C if and only if Δ is H-separable over C by Cor. 1.2 [9] on page 270. On the other hand, since Λ is H-separable over Γ, Λ is H-separable over Γ' by Theorem 1.3′ [9]. Hence the proof is immediate from Proposition 1.2.

Corollary 1.3. *Let Λ be an H-separable extension of Γ and B an intermediate subring of Λ such that ${}_B B_\Gamma \langle \oplus_B \Lambda_\Gamma$. Then, Δ is separable (resp. H-separable) over D if and only if ${}_{\Gamma'}\Gamma'_{\Gamma'} \langle \oplus_{\Gamma'} B_{\Gamma'}$ (resp. ${}_{\Gamma'}B_{\Gamma'} \langle \oplus_{\Gamma'}(\Gamma' \oplus \cdots \oplus \Gamma')_{\Gamma'}$). If furthermore B_Γ is finitely generated projective, B is separable (resp. H-separable) over Γ if and only if ${}_D D_D \langle \oplus_D \Delta_D$ (resp. ${}_D\Delta_D \langle \oplus_D (D \oplus \cdots \oplus D)_D$).*

Proof. ${}_B B_\Gamma \langle \oplus_B \Lambda_\Gamma$ implies ${}_B B \otimes_\Gamma \Lambda_\Lambda \langle \oplus_B \Lambda \otimes_\Gamma \Lambda_\Lambda \langle \oplus_B (\Lambda \oplus \cdots \oplus \Lambda)_\Lambda$. Then by Prop. 1.4 [9], $V_\Lambda(V_\Lambda(B)) = B$. Hence Proposition 1.2 can be applied to this case.

Proposition 1.3. *Let Λ be an H-separable extension of Γ such that ${}_\Gamma\Gamma_\Gamma \langle \oplus_\Gamma \Lambda_\Gamma$, $\Delta = V_\Lambda(\Gamma)$, C' the center of Δ and $\Lambda' = \Delta\Gamma$. Then,* 1) *Δ is separable over C and $V_\Lambda(\Delta) = \Gamma$,* 2) *$\Lambda' = V_\Lambda(C')$ and $C' = V_\Lambda(\Lambda')$,* 3) *Λ' is a direct summand of Λ as two sided Λ'-module,* 4) *Λ is H-separable over Λ' and Λ' is H-separable over Γ with ${}_\Gamma\Lambda'_\Gamma \langle \oplus_\Gamma (\Gamma \oplus \cdots \oplus \Gamma)_\Gamma$.*

Proof. 1). See Lemma 1.1 and Prop. 1.2 [9]. 2). $C' = V_\Delta(\Delta) = \Delta \cap V_\Lambda(\Delta)$ $V_\Lambda(\Gamma) \cap V_\Lambda(\Delta) = V_\Lambda(\Delta\Gamma) = V_\Lambda(\Lambda')$. Since Δ is central separable over C', $V_\Lambda(C') = V_\Lambda(\Delta) \otimes_{C'} \Delta = \Delta\Gamma$ by Theorem 1.2 [9]. 3). Since Δ is separable over C, C' is separable over C. Hence ${}_{\Lambda'}\Lambda'_{\Lambda'} \langle \oplus_{\Lambda'}\Lambda_{\Lambda'}$ by 2) and Proposition 1.1. 4). Since Δ is central separable over C', $\Lambda' = \Gamma \otimes_{C'} \Delta$ is H-separable over Γ by Prop. 3.4 [5] (or Prop. 1.7 [9]). It is clear that ${}_\Gamma\Lambda'_\Gamma \langle \oplus_\Gamma (\Gamma \oplus \cdots \oplus \Gamma)_\Gamma$, since $\Lambda' = \Gamma \otimes_{C'} \Delta$. Since Λ' is separable over Γ, Λ is H-separable over Λ' by 3) and Prop. 4.3 [6].

Proposition 1.4. *Let Λ be an H-separable extension of Γ. Then for any separable subextension B of Γ such that ${}_B B_B \langle \oplus_B \Lambda_B$, we have;* 1) *$\Delta$ is a left (resp. right) finitely generated projective D-module,* 2) *D is a separable C-algebra and $V_\Lambda(V_\Lambda(B)) = B$,* 3) *if furthermore $B \subset \Delta\Gamma$, D is a C'-separable algebra.*

Proof. 1) has been proved in Prop. 1.3 [9]. 2). Since B is separable over

Γ, Λ is H-separable over B. Hence D is separable over C, as ${}_BB_B\langle\oplus_B\Lambda_B$. $V_\Lambda(V_\Lambda(B))=B$ is due to Prop. 1.3 [9]. 3). $D=V_\Lambda(B)\supset V_\Lambda(\Delta\Gamma)=C'$. D is separable over C', since D is separable over C.

Proposition 1.5. *Let Λ be a central separable C-algebra. Then for any separable C-subalgebra Γ of Λ, Γ is a direct summand of Λ as two sided Γ-module and $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$.*

Proof. Let C' be the center of Γ. Then, C' is separable over C. Hence by Proposition 1.1, $V_\Lambda(C')$ is a direct summand of Λ as two sided $V_\Lambda(C')$-module. Since Λ is H-separable over Γ by Prop. 4.3 [6], $V_\Lambda(\Gamma)$ is a finitely generated projective C-module. But since C' is separable over C and $V_\Lambda(\Gamma)\supset C'$, $V_\Lambda(\Gamma)$ is finitely generated projective over C'. Then we see C' is a direct summand of $V_\Lambda(\Gamma)$ as C'-module. On the other hand, since Γ is central separable over C', $V_\Lambda(C')\cong V_\Lambda(\Gamma)\otimes_{C'}\Gamma$. Therefore, we have ${}_\Gamma\Gamma_\Gamma\langle\oplus_\Gamma\Lambda_\Gamma$. This implies $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$ by Prop. 1.3 [9].

REMARK. The equality $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$ in Proposition 1.5 has been proved in Theorem 2 [7]. And in more general case, this equality has been proved in Theorem 4.2 [2].

Proposition 1.6. *Let Λ be an H-separable extension of Γ such that ${}_\Gamma\Gamma_\Gamma\langle\oplus$ ${}_\Gamma\Lambda_\Gamma$. Then for any C'-separable subalgebra D of Δ, we have*: 1) *$V_\Lambda(D)$ is separable over Γ,* 2) *$V_\Lambda(V_\Lambda(D))=D$,* 3) *$V_\Lambda(D)$ is a direct summand of Λ as two sided $V_\Lambda(D)$-module and is contained in $\Gamma\Delta$.*

Proof. 1). Since Δ is central separable over C' and D is separable over C', Δ is H-separable over D. Hence $V_\Lambda(\Delta)\otimes_{C'} V_\Delta(D)=V_\Lambda(D)$, while $V_\Delta(D)$ is separable over C', since ${}_DD_D\langle\oplus_D\Delta_D$ by Proposition 1.5. Then $V_\Lambda(D)$ is separable over $V_\Lambda(\Delta)$ which is equal to Γ. 2). $V_\Lambda(V_\Lambda(D))=V_\Lambda(\Gamma V_\Delta(D))$ $=V_\Lambda(\Gamma)\cap V_\Lambda(V_\Delta(D))=\Delta\cap V_\Lambda(D))=V_\Delta(V_\Delta(D))=D$, where $V_\Lambda(D)=\Gamma V_\Delta(D)$ has been shown in 1), and the last equality is due to Proposition 1.5. 3). Since D is separable over C' and C' is separable over C, D is separable over C. Then $V_\Lambda(D)$ is a two sided $V_\Lambda(D)$-direct summand of $V_\Lambda(C)$ which is equal to Λ. $V_\Lambda(D)=\Gamma\otimes_{C'}V_\Delta(D)$ implies $V_\Lambda(D)=\Gamma V_\Delta(D)\subset\Gamma\Delta$.

Theorem 1.2. *Let Λ be an H-separable extension of Γ with ${}_\Gamma\Gamma_\Gamma\langle\oplus_\Gamma\Lambda_\Gamma$. Then the correspondence V: $A \rightsquigarrow V_\Lambda(A)$ provides a one to one correspondence such that V^2=identity between the set of separable subextensions B of Γ in $\Gamma\Delta$ such that ${}_BB_B\langle\oplus_B\Lambda_B$ and the set of separable C'-subalgebras of Δ.*

Corollary 1.4. *Let Λ be an H-separable extension of Γ with ${}_\Gamma\Lambda_\Gamma\langle\oplus$ ${}_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$. Then the above correspondence V provides a one to one correspondence between the set of separable subextensions B in Λ such that ${}_BB_B\langle\oplus$ ${}_B\Lambda_B$ and the set of separable C-subalgebras of Δ.*

Proof. Theorem 1.2 is an immediate consequence of Proposition 1.4 and Proposition 1.6. In case ${}_{\Gamma}\Lambda_{\Gamma}<\oplus_{\Gamma}(\Gamma\oplus\cdots\oplus\Gamma)_{\Gamma}$, $C'=C$ and $\Lambda=\Delta\Gamma$. Hence, Corollary 1.4 is direct from Theorem 1.2.

In Theorem 1.2 we could treat only separable subextensions contained in $\Delta\Gamma$. Denote $\Lambda'=\Delta\Gamma$. Then by Proposition 1.3 we see that Λ is an H-separable extension of Λ' with ${}_{\Lambda'}\Lambda'_{\Lambda'}<\oplus_{\Lambda'}\Lambda_{\Lambda'}$ and $V_{\Lambda}(\Lambda')=C'$, $V_{\Lambda}(C')=\Lambda'$, where C' is the center of Λ'. Hence we consider general ring extension $\Lambda\supset\Gamma$ with the above conditions, i.e., the following conditions which we shall call altogether the condition (*):

(*) 1) Λ is an H-separable extension of Γ with ${}_{\Gamma}\Gamma_{\Gamma}<\oplus_{\Gamma}\Lambda_{\Gamma}$.

2) $V_{\Lambda}(\Gamma)=C'$ and $V_{\Lambda}(C')=\Gamma$, where C' is the center of Γ.

Proposition 1.7. *Let $\Lambda\supset\Gamma$ be a ring extension with the condition (*). Then for any separable subextension B of Γ such that ${}_{B}B_{B}<\oplus_{B}\Lambda_{B}$, $B\subset\Lambda$ satisfies the condition (*).*

Proof. (*) 1) is clear by Prop. 4.3 [6]. (*) 2). Denote $D=V_{\Lambda}(B)$. D is a commutative ring, since $D\subset C'$. Hence $D\subset V_{\Lambda}(D)=V_{\Lambda}(V_{\Lambda}(B))=B$. Then the center of $B=V_{B}(B)=B\cap V_{\Lambda}(B)=B\cap D=D$.

Corollary 1.5. *Let $\Lambda\supset\Gamma$ be as in Proposition 1.7. Then for any H-separable subextension B of Γ in Λ, $\Lambda\supset B$ satisfies the condition (*).*

Proof. The proof is clear by the next proposition.

Proposition 1.8. *Let Λ be any H-separable extension of Γ such that ${}_{\Gamma}\Gamma_{\Gamma}<\oplus_{\Gamma}\Lambda_{\Gamma}$. Then for any H-separable subextension B of Γ in Λ, ${}_{B}B_{B}<\oplus_{B}\Lambda_{B}$.*

Proof. Since B is H-separable over Γ, $\Delta=D\otimes_{C''}V_{B}(\Gamma)$, where $\Delta=V_{\Lambda}(\Gamma)$, $D=V_{\Lambda}(B)$ and C'' is the center of B. Since $V_{B}(\Gamma)$ is a finitely generated projective C''-module, ${}_{D}D_{D}<\oplus_{D}\Delta_{D}$, and ${}_{D}\Delta$ and Δ_{D} are finitely generated projective. Hence $\Delta\otimes_{C}\Delta^{0}$ is left $D\otimes_{C}D^{0}$-projective. Then, Δ is $D\otimes_{C}D^{0}$-projective, since Δ is $\Delta\otimes_{C}\Delta^{0}$-projective. But ${}_{D}D_{D}<\oplus_{D}\Delta_{D}$, hence D is $D\otimes_{C}D^{0}$-projective. This implies that D is separable over C. Then $V_{\Lambda}(D)$ is a direct summand of Λ as two sided $V_{\Lambda}(D)$-module by Proposition 1.1. But $V_{\Lambda}(D)=B$ by Lemma 4.5 [6]. Thus we see ${}_{B}B_{B}<\oplus_{B}\Lambda_{B}$.

Proposition 1.9. *Let $\Lambda\supset\Gamma$ be a ring extension with the condition (*). Furthermore suppose that Λ is right Γ-finitely generated projective. Then for any intermediate ring B between Λ and Γ such that ${}_{B}B_{\Gamma}<\oplus_{B}\Lambda_{\Gamma}$, both $\Lambda\supset B$ and $B\supset\Gamma$ satisfy the condition (*).*

Proof. ${}_{B}B_{\Gamma}<\oplus_{B}\Lambda_{\Gamma}$ implies that C' is a finitely generated projective D-module by Prop. 1.3 [9]. Then B is H-separable over Γ by Corollary 1.3, since C' and D are commutative rings and B is right Γ-finitely generated projective. ${}_\Gamma\Gamma_\Gamma < \oplus {}_\Gamma\Lambda_\Gamma$ implies ${}_\Gamma\Gamma_\Gamma < \oplus {}_\Gamma B_\Gamma$. $V_B(\Gamma)=B\cap V_\Lambda(\Gamma)=B\cap C'=C'=$the center of Γ, and $V_B(C')=V_B(V_B(\Gamma))=\Gamma$, since ${}_\Gamma\Gamma_\Gamma < \oplus {}_\Gamma B_\Gamma$. Thus $B\supset\Gamma$ satisfies $(*)$. ${}_B B_\Gamma < \oplus {}_B\Lambda_\Gamma$ and the separability of B over Γ imply ${}_B B_B < \oplus {}_B\Lambda_B$. Hence $\Lambda\supset B$ also satisfies $(*)$ by Proposition 1.7.

Corollary 1.6. *Let Λ and Γ be as in propolition* 1.9. *Then for any separable C-subalgebra R of C', both $V_\Lambda(R)\supset\Gamma$ and $\Lambda\supset V_\Lambda(R)$ satisfy the condition* $(*)$.

Proof. Since R is separable over C, $V_\Lambda(R)$ is a direct summand of Λ as two sided $V_\Lambda(R)$-module.

Now combining Proposition 1.2, 1.7, 1.8, Corollary 1.5 and 1.6, we obtain the next theorem.

Theorem 1.3. *Let Λ be an H-separable extension of Γ with the condition* $(*)$. *If Λ is left (or right) Γ-finitely generated projective*[1], *there exists a one to one correspondence $V: A \rightsquigarrow V_\Lambda(A)$ such that $V^2=$identity between the set of H-separable subextensions of Γ in Λ and the set of separable C-subalgebra R of C' such that $V_\Lambda(V_\Lambda(R))=R$. In this case V corresponds B an H-separable subextension of Γ to its center.*

REMARK. It is an open question whether or not $V_\Lambda(V_\Lambda(R))=R$ holds for any separable C-subalgebra R of C'.

2. Splitting of $B\otimes_\Gamma A\to A$ and semisimple subalgebras of Δ

In this section we consider ring extensions $\Gamma\subset B\subset\Lambda$ with the condition such that the map $\pi: B\otimes_\Gamma\Lambda\to\Lambda$ such that $\pi(x\otimes y)=xy$ for $x, y\in\Lambda$ splits as B-Λ-map. This condition is weaker than the separability of $\Gamma\subset B$. Throughout this section we shall call briefly that $B\otimes_\Gamma\Lambda\to\Lambda$ splits in this case. If Λ is a separable extension of Γ, then $B\otimes_\Gamma\Lambda\to\Lambda$ splits if and only if π splits as B-Γ-map, because there exists $\sum x_i\otimes y_i$ in $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ such that $\sum x_i y_i=1$. Let φ be a B-Γ-map of Λ to $B\otimes_\Gamma\Lambda$ with $\pi\varphi=1_\Lambda$. Then the map φ^* of Λ to $B\otimes_\Gamma\Lambda$ such that $\varphi^*(\lambda)=\sum\varphi(\lambda x_i)y_i$ is a B-Λ-map with $\pi\varphi^*=1_\Lambda$, since $\pi\varphi^*=(\pi\varphi)^*=(1_\Lambda)^*=1_\Lambda$. From this fact we obtain

Proposition 2.1. *Let Λ be a separable R-algebra and B a left semisimple R-subalgebra of Λ, then $B\otimes_R\Lambda\to\Lambda$ splits.*

Proof. Since B is semisimple over R, the map $\pi: B\otimes_R\Lambda\to\Lambda$ splits as B-R-map. Hence the proof is clear by the above remark.

As a slight generalization of Prop. 4.3. [6] we have

1) In the subsequent paper [11], we will see that Λ is left Γ-finitely generated projective if and only if Λ is right Γ-finitely generated projective in this case.

Proposition 2.2. *Let $\Gamma\subset B\subset\Lambda$ be rings such that $B\otimes_\Gamma\Lambda\to\Lambda$ splits. If Λ is H-separable over Γ, Λ is H-separable over B, too.*

Proof. ${}_B\Lambda_\Gamma<\oplus_B B\otimes_\Gamma\Lambda_\Lambda$ implies ${}_\Lambda\Lambda\otimes_B\Lambda_\Lambda<\oplus_\Lambda\Lambda\otimes_B(B\otimes_\Gamma\Lambda)_\Lambda\cong{}_\Lambda\Lambda\otimes_\Gamma\Lambda_\Lambda <\oplus_\Lambda(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda$. Hence Λ is H-separable over B.

Corollary 2.1. *Let Λ be an H-separable R-algebra and Γ a semisimple R-subalgebra of Λ. Then Λ is H-separable over Γ.*

Proof. This is clear by Proposition 2.1 and Proposition 2.2.

Proposition 2.3. *Let Λ be an H-separable extension of Γ. Then for any intermediate ring B between Λ and Γ such that ${}_B B_\Gamma<\oplus_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits, ${}_D D<\oplus_D\Delta$ and $D\otimes_C\Delta\to\Delta$ splits where $D=V_\Lambda(B)$. Conversely for any C-subalgebra D of Δ such that $D\otimes_C\Delta\to\Delta$ splits, ${}_B B_\Gamma<\oplus_B\Lambda_\Gamma$, and if furthermore B_Γ is finitely generated projective and ${}_D D<\oplus_D\Delta$, $B\otimes_\Gamma\Lambda\to\Lambda$ splits, where $B=V_\Lambda(D)$.*

Proof. Splitting of $B\otimes_\Gamma\Lambda\to\Lambda$ induced a left D-splitting π^*: Hom $({}_B\Lambda_\Lambda,{}_B\Lambda_\Lambda)\to$ Hom $({}_B B\otimes_\Gamma\Lambda_\Lambda,{}_B\Lambda_\Lambda)$, which implies ${}_D D<\oplus_D\Delta$. On the other hand,

$$\begin{aligned}{}_D D\otimes_C\Delta&\cong\mathrm{Hom}\,({}_B\Lambda_\Lambda,{}_B\Lambda_\Lambda)\otimes_C\Delta\cong\mathrm{Hom}\,({}_B\mathrm{Hom}\,({}_C\Delta,{}_C\Lambda)_\Lambda,{}_B\Lambda_\Lambda)\\&\cong\mathrm{Hom}\,({}_B\Lambda\otimes_\Gamma\Lambda_\Gamma,{}_B\Lambda_\Gamma)\cong\mathrm{Hom}\,({}_B\Lambda_\Gamma,{}_B\mathrm{Hom}\,(\Lambda_\Lambda,\Lambda_\Lambda)_\Gamma)\\&\cong\mathrm{Hom}\,({}_B\Lambda_\Lambda,{}_B\Lambda_\Lambda)\end{aligned}$$

as D-Δ-map. The composition η of the above maps is such that $\eta(d\otimes e)(b)=dbe$. Then ${}_B B_\Gamma<\oplus_B\Lambda_\Gamma$ induces the splitting of $D\otimes_C\Delta\to\Delta$ by the following commutative diagram of D-Δ-maps.

$$\begin{array}{ccc}D\otimes_C\Delta & \longrightarrow & \mathrm{Hom}({}_B\Lambda_\Gamma,{}_B\Lambda_\Gamma)\\ \downarrow & & \downarrow\\ \Delta & \longrightarrow & \mathrm{Hom}\,({}_B B_\Gamma,{}_B\Lambda_\Gamma)\end{array}$$

Next suppose that D is a C-subalgebra of Δ such that $D\otimes_C\Delta\to\Delta$ splits. Then $B=V_\Lambda(D)\cong\mathrm{Hom}\,({}_D\Delta_\Delta,{}_D\Lambda_\Delta)<\oplus\mathrm{Hom}\,({}_D D\otimes_C\Delta_\Delta,{}_D\Lambda_\Delta)\cong V_\Lambda(C)=\Lambda$ as B-$V_\Lambda(\Delta)$-map. Hence ${}_B B_\Gamma<\oplus_B\Lambda_\Gamma$. Suppose B_Γ is finitely generated projective and ${}_D D<\oplus_D\Delta$. Then,

$$\begin{aligned}B\otimes_\Gamma\Lambda&\cong B\otimes_\Gamma\mathrm{Hom}\,({}_\Lambda\Lambda,{}_\Lambda\Lambda)\cong\mathrm{Hom}\,({}_\Lambda\mathrm{Hom}\,(B_\Gamma,\Lambda_\Gamma),{}_\Lambda\Lambda)\\&\cong\mathrm{Hom}\,({}_\Lambda\Lambda\otimes_D\Delta,{}_\Lambda\Lambda)\quad({}_B B_\Gamma<\oplus_B\Lambda_\Gamma\text{ implies }{}_B B\otimes_\Gamma\Lambda_\Lambda<\oplus_B(\Lambda\oplus\cdots\oplus\Lambda)_\Lambda)\\&\cong\mathrm{Hom}\,({}_D\Delta,{}_D\mathrm{Hom}\,({}_\Lambda\Lambda,{}_\Lambda\Lambda))\cong\mathrm{Hom}\,({}_D\Delta,{}_D\Lambda)\end{aligned}$$

as B-Λ-map. Then in usual way we see that ${}_D D<\oplus_D\Delta$ implies that $B\otimes_\Gamma\Lambda\to\Lambda$ splits.

Theorem 2.1. *Let Λ be an H-separable extension of Γ with ${}_\Gamma\Gamma_\Gamma \langle \oplus {}_\Gamma\Lambda_\Gamma$. Then the correspondence $V: A \rightsquigarrow V_\Gamma(A)$ provides a one to one correspondence such that V^2=identity between the set of intermediate rings B between $\Gamma\Delta$ and Γ such that ${}_BB_\Gamma \langle \oplus {}_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda \to \Lambda$ splits and the set of C'-subalgebras D of Δ such that ${}_DD\langle \oplus_D\Delta$ and $D\otimes_C\Delta \to \Delta$ splits.*

Proof. Let D be any C'-subalgebra of Δ which satisfies the condition of the theorem. Since C' is C-separable, $D\otimes_C\Delta \to \Delta$ splits if and only if $D\otimes_{C'}\Delta \to \Delta$ splits. Then Δ is H-separable over D by Proposition 2.2. Hence we have $V_\Lambda(D)=V_\Lambda(\Delta)V_\Delta(D)=\Gamma V_\Delta(D)$. Then $V_\Lambda(V_\Lambda(D))=V_\Lambda(\Gamma V_\Delta(D))$ $=\Delta\cap V_\Lambda(V_\Delta(D))=V_\Delta(V_\Delta(D))=D$, since ${}_DD\langle\oplus_D\Delta$. Clearly $V_\Lambda(D)=\Gamma V_\Delta(D)$ $\subset\Gamma\Delta$. Denote B by $V_\Lambda(D)$. Since $\Gamma\Delta$ is Γ-finitely generated projective and ${}_BB_\Gamma\langle\oplus_B\Lambda_\Gamma$, ${}_BB_\Gamma\langle\oplus_B\Gamma\Delta_\Gamma$ and B is right Γ-finitely generated projective. Hence $B\otimes_\Gamma\Lambda\to\Lambda$ splits by Proposition 2.3. The rest of the proof of this theorem is immediate by Proposition 2.3, since $B\subset\Gamma\Delta$ implies $V_\Lambda(B)\supset V_\Lambda(\Gamma)\cap V_\Lambda(\Delta)$ $=\Delta\cap\Gamma=C'$

Corollary 2.2. *Let Λ be an H-separable extension of Γ with ${}_\Gamma\Lambda_\Gamma\langle\oplus_\Gamma(\Gamma\oplus\cdots\oplus\Gamma)_\Gamma$. Then the above correspondence V yields a one to one correspondence between the set of intermediate rings B between Λ and Γ such that ${}_BB_\Gamma\langle\oplus_B\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits and the set of C'-subalgebras D of Δ such that ${}_DD\langle\oplus_D\Delta$ and $D\otimes_C\Delta\to\Delta$ splits.*

Proposition 2.4. *Let Δ be an H-separable extension of Γ with ${}_\Gamma\Gamma_\Gamma\langle\oplus_\Gamma\Lambda_\Gamma$. Then for any C'-semisimple subalgebra D of Δ, $V_\Lambda(V_\Lambda(D))=D$, ${}_BB_\Gamma\langle\oplus\Lambda_\Gamma$ and $B\otimes_\Gamma\Lambda\to\Lambda$ splits where $B=V_\Lambda(D)$.*

Proof. Since D is a semisimple subalgebra of a central separable C'-algebra Δ, ${}_DD\langle\oplus_D\Delta$ by Prop. 4.1 [2] and $D\otimes_C\Delta\to\Delta$ splits. Hence the proof is clear by Theorem 2.1.

REMARK. It is easy to see that for any rings $\Gamma\subset A\subset B\subset\Lambda$ the following statements hold;

1) *If $B\otimes_\Gamma\Lambda\to\Lambda$ splits, then $B\otimes_A\Lambda\to\Lambda$ splits.*

2) *If A is separable over Γ (or more generally $A\otimes_\Gamma\Lambda\to\Lambda$ splits) and $B\otimes_A\Lambda\to\Lambda$ splits, then $B\otimes_\Gamma\Lambda\to\Lambda$ splits.*

The proof of the above remark is almost same as Prop. 1.3 and Prop. 1.5 [3], hence we shall omit it. Thus in Theorem 2.1 and the proof of Proposition 2.4, splitting of $D\otimes_C\Delta\to\Delta$ is equivalent to splitting of $D\otimes_{C'}\Delta\to\Delta$.

3. Relation between quasi-Frobenius extension

In this section we shall obtain a necessary and sufficient condition for a quasi-Frobenius extension to be a separable extension.

The next lemma have been shown in [10].

Lemma 3.1. *Let Λ be a ring and Γ a subring of Λ. Then, Λ is a left quasi-Frobenius extension of Γ if and only if ${}_\Gamma\Lambda$ is finitely generated projective and there exist $\sum x_i^{(k)}\otimes y_i^{(k)}\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ and $\alpha_k\in Hom({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ $k=1, 2, \cdots, n$ such that $\sum_k\sum_i x_i^{(k)}\alpha_k(y_i^{(k)})=1$. (Lemma 1.1 [10]).*

In the same way as Lemma 3.1 we have

Lemma 3.2. *Let Λ be a ring and Γ a subring of Λ. Then Λ is a left quasi-Frobenius extension of Γ if and only if ${}_\Gamma\Lambda$ and Λ_Γ are finitely generated projective and there exist $\sum u_j^{(k)}\otimes v_j^{(k)}\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ and $\beta_k\in Hom({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)$ $k=1, 2, \cdots, n$ such that $\gamma=\sum_{j,k}\beta_k\gamma(u_j^{(k)})v_j^{(k)}$ for all $\gamma\in Hom(\Lambda_\Gamma, \Gamma_\Gamma)$.*

Proof. Λ is a left quasi-Frobenius extension of Γ if and only if ${}_\Gamma\Lambda$ and Λ_Γ are finitely generated projective and there exist

$$\psi_k\in \mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Lambda, {}_\Gamma\Lambda_\Lambda), \quad \beta_k\in\mathrm{Hom}({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Gamma_\Gamma)\ k=1, 2, \cdots, n$$

such that $\gamma=\sum\beta_k\circ\psi_k(\gamma)$ for all $\gamma\in\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)$ by Satz 2 [6].

Since Λ_Γ is finitely generated projective, we have the canonical Λ-Λ-isomorphisms

$$\Lambda\otimes_\Gamma\Lambda \to \Lambda\otimes_\Gamma\mathrm{Hom}({}_\Gamma\Gamma, {}_\Gamma\Lambda) \to \mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma), {}_\Gamma\Lambda)$$

Let θ be the composition of the above maps. Then $\theta(\sum u_i\otimes v_i)(\gamma)=\sum\gamma(u_i)v_i$ for any $\sum u_i\otimes v_i\in\Lambda\otimes_\Gamma\Lambda$ and $\gamma\in\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)$, and $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ is mapped onto $\mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Lambda, {}_\Gamma\Lambda_\Lambda)$ by θ. Hence the proof is immediate by Satz 2 [8].

Let $\Lambda\supset\Gamma$ be a left quasi-Frobenius extension, and ψ_k and β_k be as in the proof of Lemma 2.1, i.e, as in the condition 4) in Satz 2 [6]. Then for any Γ-Λ-map ψ from $\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)$ to Λ, $\psi(\gamma)=\sum\psi(\beta_k)\psi_k(\gamma)$ for any $\gamma\in\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)$. Since β_k is a Γ-Γ-map, $r\beta_k=\beta_k\circ r$ for any r in Γ and any $k=1, 2, \cdots, n$. Then $r\psi(\beta_k)=\psi(r\beta_k)=\psi(\beta_k\circ r)=\psi(\beta_k)r$ for any $r\in\Gamma$. This implies that $\psi(\beta_k)\in\Delta$ for any k. Thus $\psi=\sum\psi(\beta_k)\cdot\psi_k\in\sum\Delta\cdot\psi_k$, and we see $\mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Lambda, {}_\Gamma\Lambda_\Lambda)=\sum\Delta\psi_k$ as left Δ-module. On the other hand, let $\sum u_j^{(k)}\otimes v_j^{(k)}$ be the element of $\Lambda\otimes_\Gamma\Lambda$ which is mapped to ψ_k for $k=1, 2, \cdots, n$. Then for any $d\in\Delta$, $\sum u_i^{(k)}\otimes dv_j^{(k)}$ is mapped to $d\cdot\psi_k$ for any k. Thus we see $(\Lambda\otimes_\Gamma\Lambda)^\Lambda=\sum_k(\sum_j u_j^{(k)}\otimes\Delta v_j^{(k)})$, since $(\Lambda\otimes_\Gamma\Lambda)^\Lambda$ is mapped onto $\mathrm{Hom}({}_\Gamma\mathrm{Hom}(\Lambda_\Gamma, \Gamma_\Gamma)_\Lambda, {}_\Gamma\Lambda_\Lambda)$ by θ. Then we obtain

Proposition 3.1. *Let Λ be a left quasi-Frobenius extension of Γ and $\sum u_j^{(k)}\otimes v_j^{(k)}$ $k=1, 2, \cdots, n$ be as in Lemma 3.2. Then Λ is a separable extension* *of* Γ *if and only if* $\sum_k (\sum_j u_j^{(k)} \Delta v_j^{(k)}) = C$, *where by* $(\sum_j u_j^{(k)} \Delta v_j^{(k)})$ *we mean the set* $\{\sum_j u_j^{(k)} d v_j^{(k)} \mid d \in \Delta\}$

Proof. Let π be the map of $\Lambda \otimes_\Gamma \Lambda$ to Λ with $\pi(x \otimes y) = xy$. Then, Λ is a separable extension of Γ if and only if $\pi((\Lambda \otimes_\Gamma \Lambda)^\Lambda) = C$ by Prop. 2.1 [4]. But by the remark stated above $(\Lambda \otimes_\Gamma \Lambda)^\Lambda = \sum_k (\sum_j u_j^{(k)} \otimes \Delta v_j^{(k)})$, which is mapped onto $\sum_k (\sum_j u_j^{(k)} \Delta v_j^{(k)})$. Therefore, Λ is separable over Γ if and only if $\sum_k (\sum_j u_j^{(k)} \Delta v_j^{(k)}) = C$.

HOKKAIDO UNIVERSITY[2)]

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2) This paper was written while the author was at Kobe University.