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HIGH ORDER DERIVATIONS I

Dedicated to Professor Keizo Asano on his 60th birthday

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Introduction

Let k, A be commutative rings with 1 and assume that A is a k-algebra. A q-th order derivation D of A into an A-module F over k is defined as an element of $\mathrm{Hom}_k(A, F)$ such that for any set of $(q+1)$-elements $(x_0, x_1, \cdots, x_q)$ of A we have the identity

$$D(x_0 x_1 \cdots x_q) = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1<\cdots<i_s} x_{i_1} \cdots x_{i_s} D(x_0 \cdots \check{x}_{i_1} \cdots \check{x}_{i_s} \cdots x_q)\,.$$

The first order derivation is just an ordinary derivation. This interesting notion of high order derivations were introduced by H. Osborn in [5] as far as the author knows.[1] In his paper he developed the theory of high Kähler differentials rather than derivations themselves and furnished algebraic foundation in the theory of high order differentials of C^∞ functions. In this paper we shall give fundamental theories for the calculus of high order derivations and some functorial properties of the module of high order differentials. In a subsequent paper[2] we shall treat an application of the present theory to the Galois theory for purely inseparable field extensions of finite exponent.

One word about higher derivations due to H. Hasse and F.K. Schmidt (cf. [2]). As is supposed spontaneously they have close connections with our high order derivations. In fact if $(D_0, D_1, \cdots, D_m, \cdots)$ is a higher derivation of rank finite (or infinite), then m-th component D_m is an m-th order derivation. But an m-th order derivation cannot necessarily be an m-th component of a higher derivation. It would be an interesting problem to find a condition for an m-th order derivation to be an m-th component of a higher dirivation.

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1) After I completed the work it comes to my attention that the same notion has appeared in R.G. Heyneman and M.E. Sweedler: Affine Hopf Algebras I, J. of Algebra 13 (1969), 192–241.

2) The paper will appear in Journal of Science of the Hiroshima University series A-I, Vol 34 (1970).

Notations and terminologies: Any commutative ring in this paper is assumed to contain 1 and any module is assumed to be unitary. Let k and A be commutative rings. We say that A is a k-algebra if there exists a ring homomorphism f such that $f(1)=1$. f is not necessarily injective but we shall often speak as if f were injective and f is not written explicitly when there is no fear of confusion. Thus if $a\in k$ and x is an element of an A-module we shall write ax instead of $f(a)x$. The set of q-th order derivations of a k-algebra A into an A-module F over k will be denoted by $\mathcal{D}_0^{(q)}(A/k, F)$. When $F=A$ we shall use the notation $\mathcal{D}_0^{(q)}(A/k)$ in place of $\mathcal{D}_0^{(q)}(A/k, A)$ and an element of $\mathcal{D}_0^{(q)}(A/k)$ will be called simply a q-th order derivation of A/k. The numbering of the propositions is renewed in each Chapter. To quote the proposition of different Chapters we shall use the notation such as I–6 (Proposition (or Theorem) 6 of the Chapter I). The proposition of the same Chapter will be referred to without the Chapter number.

Contents

CHAPTER I. CALCULUS OF HIGH ORDER DERIVATIONS

1. Definition and fundamental properties

Let k, A be commutative rings with unit elements and let A be a k-algebra. Let F be an A-module. A q-th order derivation Δ of A/k into F is, by definition, a k-homomorphism of A into F satisfying the following identity:

$$\Delta(x_0x_1, \cdots, x_q) = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1<\cdots<i_s} x_{i_1} \cdots x_{i_s}\Delta(x_0 \cdots \check{x}_{i_1} \cdots \check{x}_{i_s} \cdots x_q)$$

for any set $x_0, x_1, \cdots, x_q$ of $(q+1)$-elements in A.

We have $\Delta(\alpha)=0$ for any $\alpha\in k$. In fact $\Delta(\alpha)=\alpha\Delta(1)$ and we have

$$\Delta(1)=\Delta(1^{q+1})=\Big[[(q+1)-\binom{q+1}{2}+\cdots+(-1)^{r+1}\binom{q+1}{r}+\cdots$$
$$+(-1)^{q+1}(q+1)\Big]\Delta(1)=[1+(-1)^{q+1}]\Delta(1)\,.$$

If $q\equiv 1\ (2)$, then $\Delta(1)=2\Delta(1)$, hence $\Delta(1)=0$. If $q\equiv 0\ (2)$, $\Delta(1)=(1-1)\Delta(1)=0$.

Let a be an element of A. We shall denote by a_L an A-homomorphism of F into F such that

$$a_L(x)=ax \qquad \text{for} \quad x\in F.$$

If m is an element of F, we shall denote by m_R an A-homomorphism of A into F such that

$$m_R(a)=am\,.$$

Let a be an element of A and let Δ be an element of $\mathrm{Hom}_k(A, F)$. We shall set

$$[\Delta, a]=\Delta a_L-a_L\Delta-(\Delta(a))_R$$

We shall sometimes use the notation Δ_a instead of $[\Delta, a]$.

We shall also omit the subscript L or R if there is no fear of confusion. Thus we shall write as

$$[\Delta, a]=\Delta a-a\Delta-\Delta(a)$$

It should be remarked that Δa is a homomorphism while $\Delta(a)$ is the value of Δ at a.

Proposition 1. *Let Δ be an element of $\mathrm{Hom}_k(A, F)$ and $x_1, \cdots, x_q$ be any set of q elements of A. Then we have*

$$[\cdots[[\Delta, x_1], x_2], \cdots, x_q]=\Delta x_1\cdots x_q-\Delta(x_1\cdots x_q)$$
$$-\sum_{i=1}^{q} x_i\{\Delta x_1\cdots \check{x}_i\cdots x_q-\Delta(x_1\cdots\check{x}_i\cdots x_q)\}+\cdots$$
$$+(-1)^s\sum_{i_1<\cdots<i_s} x_{i_1}\cdots x_{i_s}\{\Delta x_1\cdots\check{x}_{i_1}\cdots\check{x}_{i_s}\cdots x_q-\Delta(x_1\cdots\check{x}_{i_1}\cdots$$
$$\check{x}_{i_s}\cdots x_q)\}+\cdots+(-1)^q x_1\cdots x_q\Delta\,.$$

Proof is easy by induction on q and will be omitted. It should be noted that the above expression is symmetric in $x_1, \cdots, x_q$.

Theorem 2. *Let Δ be an element of $\mathrm{Hom}_k(A, F)$. Then Δ is a q-th order derivation if and only if we have*

$$[\cdots[[\Delta, x_1], x_2], \cdots, x_q]=0$$

for any set of elements $x_1, \cdots, x_q$ in A.

Proof. From Proposition 1 we see that

$$[\cdots[[\Delta, x_1], x_2], \cdots, x_q](x_0) = \sum_{s=1}^{q} (-1)^{s-1} \sum_{i_1<\cdots<i_s} x_{i_1}\cdots x_{i_s}\Delta(x_0\cdots\check{x}_{i_1}\cdots\check{x}_{i_s}\cdots x_q)$$

The assertion follows immediately from this identity.
The following Propositions are immediate from Theorem 2.

Proposition 3. *An element Δ of* $\mathrm{Hom}_k(A, F)$ *is a q-th order derivation if and only if $[\Delta, a]$ is a $(q-1)$-th order derivation for any $a\in A$.*

Proposition 4. *If Δ is a q-th order derivation, then Δ is also a q'-th order derivation for any $q'\geq q$.*

Proposition 5. *Let $(D_0, D_1, \cdots, D_m, \cdots)$ be a higher derivation. Then the q-th component D_q is a q-th order derivation for $q\geq 1$.*

Proof. By definition we have for any x, y in A

$$D_q(xy) = \sum_{i=0}^{q} D_i(x)D_{q-i}(y)\,.$$

In other words

$$[D_qx-xD_q-D_q(x)](y) = \sum_{i=1}^{q-1} D_i(x)D_{q-i}(y)\,,$$

i.e.,

$$D_qx-xD_q-D_q(x) = \sum_{i=1}^{q-1} D_i(x)D_{q-i}\,.$$

The induction assumption on q implies that any member of the right hand side is a derivation of order $q-1$. Hence D_q is a q-th order derivation by Proposition 3 and 4.

Proposition 6. *Let D, Δ be derivations of A/k. Then we have*

$$[D\Delta, a] = D[\Delta, a]+[D, a]\Delta+[D, \Delta(a)]+D(a)\Delta+\Delta(a)D\,.$$

Proof.

$$\begin{aligned} &D\Delta a-aD\Delta-D(\Delta(a)) \\ &\quad = D[\Delta a-\Delta a-\Delta(a)]+[Da-aD-D(a)]\Delta+D(a)\Delta+D\Delta(a)-D(\Delta(a)) \\ &\quad = D[\Delta, a]+[D, a]\Delta+[D, \Delta(a)]+\Delta(a)D+D(a)\Delta \end{aligned}$$

q.e.d.

Corollary 6.1. *$D\Delta$ is a derivation of order $r+s$, where r, s are orders of D, Δ respectively.*

This is immediate from Proposition 6 and Proposition 3.

Corollary 6.2.

$$[D, \Delta] = D\Delta-\Delta D$$

is a derivation of order $r+s-1$

Proof. We shall use the induction on $r+s$. When $r=s=1$ this result is well known. From Proposition 6 we have

$$[D\Delta-\Delta D, a] = [D, [\Delta, a]]-[\Delta, [D, a]]+[D, \Delta(a)]-[\Delta, D(a)]$$

The right hand side is derivations of order $r+s-2$ by induction assumptions. Hence by Proposition 3 $D\Delta-\Delta D$ is a derivation of order $r+s-1$.

2. The ring of constants and derivation algebra

Let A be a k-algebra and let F be an A-module. By Proposition 4 we have

$$\mathcal{D}_0^{(1)}(A/k, F)\subset \mathcal{D}_0^{(2)}(A/k, F)\subset \cdots$$

An element of $\mathcal{D}_0^{(q)}(A/k, F)$ not contained in $\mathcal{D}_0^{(q-1)}(A/k, F)$ will be called a proper q-th order derivation. $\mathcal{D}_0^{(q)}(A/k, F)$ is a left A-module and a submodule of $\mathrm{Hom}_k(A, F)$. Let $C_q(A/k, F)$ be the subset of A consisting of elements x such that for any Δ in $\mathcal{D}_0^{(q)}(A/k, F)$ we have $\Delta x=0$. From the definition we see that

$$C_1(A/k, F)\supset C_2(A/k, F)\supset \cdots$$

Proposition 7. *Let* Δ *be a q-th order derivation of* A/k *into* F. *Then for any* $x\in C_{q-1}(A/k, F)$ *we have* $[\Delta, x]=0$. *In particular for any* $y\in A$ *we have*

$$\Delta(xy) = x\Delta y+y\Delta x\,.$$

Proof. Let y be an arbitrary element of A. Then we have $[\Delta, y](x)=0$. That is

$$\Delta(xy) = x\Delta(y)+y\Delta(x)$$

This relation can also be read as $[\Delta, x](y)=0$ for any $y\in A$. Hence we must have $[\Delta, x]=0$.

Corollary 7.1. *Let* Δ_q *be a q-th order derivation and let* p (>0) *be the characteristic of* A. *Then* $A_q=k[A^{p^q}]\subseteq C_q(A, F)$ *for any A-module F.*

Proof. Induction on q. The case $q=1$ is well known. Let $x\in A$. Then by induction assumption $\alpha=x^{p^{q-1}}\in C_{q-1}$. Hence $\Delta(x^{p^q})=\Delta(\alpha^p)=p\alpha^{p-1}\Delta(\alpha)=0$.

Proposition 8. $C_q(A/k, F)$ *is a sub-k-algebra of* A.

Proof. It is clear that if $x, y\in C_q$ and $a, b\in k$, then $ax+by$ is also in C_q. Hence it remains to show that $xy\in C_q$. It is also immediate because for any $\Delta\in \mathcal{D}_k^{(q)}(A, F)$ we have $\Delta(xy)=x\Delta(y)+y\Delta(x)$.

Corollary 7.1 can be sharpened much more. In fact if $q<p^f$ then any q-th order derivation Δ of A/k annihilates any element of $A_f=k[A^{p^f}]$ (cf. Proposition 10). Hence if Δ is a p^f-th order derivation Δ induces on A_f an ordinary derivation. Such an operation Δ is called a semi-derivation of height f by J. Dieudonné in [1].

Let A be a k-algebra. We shall set

$$\mathcal{D}_0(A/k)=\bigcup_{q=1}^{\infty}\mathcal{D}_0^{(q)}(A/k)$$

and

$$\mathcal{D}(A/k)=A\oplus\mathcal{D}_0(A/k)$$

where an element a of A is identified with the elements a_L of $\mathrm{Hom}_k(A, A)$. $\mathcal{D}(A/k)$ is not only a subset of $\mathrm{Hom}_k(A, A)$ but also a subring. To see this fact it suffices to show the following:

(1) A sum of two elements of $\mathcal{D}_0(A/k)$ is an element of $\mathcal{D}_0(A/k)$.

(2) A product of any two elements of $\mathcal{D}_0(A/k)$ is again contained in $\mathcal{D}_0(A/k)$.

(3) For any a in A and $D\in\mathcal{D}_0(A/k)$, aD and Da are again contained in $\mathcal{D}(A/k)$.

(1) and the first of (3) are trivial from the definition and (2) is proved in Corollary 6.1. If D is a high order derivation we have

$$Da=D(a)+[D, a]+aD$$

by Definition of $[D, a]$. Hence Da is an element of $\mathcal{D}(A/k)$. Moreover it is easily seen that k is contained in the center of $\mathcal{D}(A/k)$. Thus $\mathcal{D}(A/k)$ is a k-algebra. We shall call it the *derivation algebra* of A/k. The derivation algebra will play the fundamental roles in a subsequent paper.

3. $D(x^n)$

Proposition 9. *Let D be a q-th order derivation of A/k into an A-module F and let x be an element of A. Then we have*

$$\Phi(n, q)\colon D(x^n)=\sum_{s=0}^{q-1}(-1)^s\binom{n}{q-s}\binom{n-q+s-1}{s}x^{n-q+s}D(x^{q-s})$$

for every natural number n.

Proof. We shall use the double induction on n and q. The case $q=1$ is immediate. The case $n=1$ is also immediate for any q. Now assume the formula for any derivation of order $<q$ and $\Phi(m, q)$ is valid for $m\leq n$. We shall show that $\Phi(n+1, q)$ is also true. By Proposition 1, $\Delta=Dx-xD-D(x)$ is a derivation of order $q-1$. By induction assumption we have

$$D(x^{n+1})-xD(x^n)-x^nD(x)$$
$$=\sum_{s=0}^{q-2}(-1)^s\binom{n}{q-1-s}\binom{n-q+s}{s}x^{n-q+s+1}[D(x^{q-s})-xD(x^{q-s-1})-x^{q-s-1}D(x)]\,.$$

Then we get

$$\begin{aligned}D(x^{n+1}) &= \sum_{s=0}^{q-1}(-1)^s\binom{n}{q-s}\binom{n-q+s-1}{s}x^{n-q+s-1}D(x^{q-s})\\ &+\sum_{s=0}^{q-2}(-1)^s\binom{n}{q-1-s}\binom{n-q+s}{s}x^{n-q+s+1}D(x^{q-s})\\ &+\sum_{s=1}^{q-1}(-1)^s\binom{n}{q-s}\binom{n-q+s-1}{s-1}x^{n-q+s+1}D(x^{q-s})\\ &+x^nD(x)+\sum_{s=0}^{q-2}(-1)^{s+1}\binom{n}{q-1-s}\binom{n-q+s}{s}x^nD(x)\\ &=\sum_{s=0}^{q-2}(-1)^s\binom{n+1}{q-s}\binom{n-q+s}{s}x^{n-q+s+1}D(x^{q-s})\\ &+\left\{1+\sum_{s=0}^{q-1}(-1)^{s+1}\binom{n}{q-s-1}\binom{n-q+s}{s}\right.\\ &\left.+(-1)^{q-1}(n+1)\binom{n-1}{q-1}\right\}x^nD(x)\,.\end{aligned}$$

Hence to prove the assertion it suffices to show the following:

Lemma 1. *For any pair of positive integers n, q we have*

$$f(q)=1+\sum_{s=1}^{q}(-1)^s\binom{n}{q-s}\binom{n-q+s-1}{s-1}=0\,.$$

Proof. Induction on q. $f(1)=0$ and we have

$$\begin{aligned}f(q+1)-f(q) &= -\binom{n}{q}+\sum_{s=1}^{q}(-1)^{s+1}\binom{n}{q-s}\left\{\binom{n-q+s-1}{s-1}+\binom{n-q+s-1}{s}\right\}\\ &= -\binom{n}{q}+\sum_{s=1}^{q}(-1)^{s+1}\binom{n}{q-s}\binom{n-q+s}{s}\\ &= -\binom{n}{q}+\sum_{s=1}^{q}(-1)^{s+1}\binom{n}{q}\binom{q}{s}\\ &= \sum_{s=0}^{q}(-1)^s\binom{q}{s}=0\,.\end{aligned}$$

Alternative proof: Apply the defining formula to the calculus of $D(x^{n+1})=D(x^{n+1-q}x\cdots x)$. Then from the induction assumption for small n we get easily

$$D(x^{n+1})=\sum_{s=0}^{q-1}(-1)^s\Big\{\binom{q}{s}$$
$$+\sum_{t=0}^{q-1}(-1)^t\binom{q}{t+1}\binom{n-t}{q-s}\binom{n-t-q+s-1}{s}\Big\}x^{n+1-q+s}D(x^{q-s})\,.$$

Hence the proof is reduced to the following Lemma on binomial coefficients.

Lemma 2. *For any triplet n, q, s of integers such that $n>0$, and $q>s\geq 0$ we have*

$$\binom{q}{s}=\sum_{t=0}^{q}(-1)^t\binom{q}{t}\binom{n-t+1}{q-s}\binom{n-t-q+s}{s}.$$

Proof. A simple calculation yields that the right hand side is equal to

$$\binom{q}{s}\sum_{t=0}^{q}(-1)^t\binom{q}{t}\binom{n-t}{q}\frac{n-t+1}{n-t+1-q+s}$$
$$=\binom{q}{s}\Big\{\sum_{t=0}^{q}(-1)^t\binom{q}{t}\binom{n-t}{q}+\sum_{t=0}^{q}(-1)^t\binom{q}{t}\binom{n-t}{q}\frac{q-s}{n-t+1-q+s}\Big\}$$

Hence the Lemma is reduced to the proof of the following identities:

$$(1)\quad \sum_{t=0}^{q}(-1)^t\binom{q}{t}\binom{n-t}{q}=1\,,$$

$$(2)\quad \sum_{t=0}^{q}(-1)^t\binom{q}{t}\binom{n-t}{q}\frac{1}{n-t-q+s+1}=0\,.$$

Since

$$x^n\Big(1-\frac{1}{x}\Big)^q=\sum_{t=0}^{q}(-1)^t\binom{q}{t}x^{n-t}\,,$$

the left hand side of (1) times $q!$ is equal to

$$\frac{d^q}{dx^q}\Big\{x^n\Big(1-\frac{1}{x}\Big)^q\Big\}_{x=1}=q!\,.$$

Similary the left hand side of (2) is equal, up to a constant factor, to

$$\frac{d^s}{dx^s}\Big[\frac{1}{x}\,\frac{d^{q-s-1}}{dx^{q-s-1}}\Big\{x^n\Big(1-\frac{1}{x}\Big)^q\Big\}\Big]_{x=1}.$$

This is clearly equal to zero.

Proposition 10. *Let A be a k-algebra of characteristic p and let Δ be a q-th order derivation of A/k into F. Then if $q>p^i$, Δ vanishes on A_i.*

Proof. Assume that $q<p^i$ and let x be an element of A. Then by Proposition 9, we have

$$\Delta(x^{p^i}) = \sum_{s=0}^{q-1} (-1)^s \binom{p^i}{q-s} \binom{p^i-q+s+1}{s} x^{p^i-q+s} D(x^{q-s}) .$$

As is well known

$$\binom{p^i}{r} \equiv 0 \qquad (\mathrm{mod.}\ p)$$

for any $1 \leq r < p^i$. Hence Δ vanishes on A_i.

It is natural to raise the following problem. Let, for instance, D be a p-th order derivation. Then D may induce on A_1 a non-trivial ordinary derivation. In this case we can ask that whether any derivation of A_1 can be obtained in this way or not. In other words, whether an arbitrary derivation of A_1 can be extended to a p-th order derivation of A/k or not. These problems will be treated under a restricted situation in a forthcoming paper.

4. $[D, x_1 x_2 \cdots x_n]$

For notational conventions we shall set

$$[\cdots[[D, x_1], x_2], \cdots, x_q] = [D, x_1 * x_2 * \cdots * x_q]$$

Proposition 11. *Let D be an element of* $\mathrm{Hom}_k(A, F)$. *Then we have the following identity*:

$$[D, x_1 \cdots x_n] = [D, x_1 * \cdots * x_n] + \sum_{i=1}^{n} x_i [D, x_1 * \cdots * \check{x}_i * \cdots * x_n]$$

$$+ \sum_{i<j} x_i x_j [D, x_1 * \cdots * \check{x}_i * \cdots * \check{x}_j * \cdots * x_n] + \cdots\cdots + \sum_{i=1}^{n} x_1 \cdots \check{x}_i \cdots x_n [D, x_i] .$$

Proof. Induction on n. We have for $n=2$.

$$\begin{aligned} [D, x_1 x_2] &= D x_1 x_2 - x_1 x_2 D - D(x_1 x_2) \\ &= [D x_1 - x_1 D - D(x_1)] x_2 + x_1 [D x_2 - x_2 D - D(x_2)] \\ &\quad + x_2 D(x_1) + x_1 D(x_2) - D(x_1 x_2) \\ &= D_{x_1} x_2 + x_2 D_{x_2} - D_{x_1}(x_2) \\ &= (D_{x_1} x_2 - x_2 D_{x_1} - D_{x_1}(x_2)) + x_1 D_{x_2} + x_2 D_{x_1} \\ &= [D_{x_1}, x_2] + x_1 D_{x_2} + x_2 D_{x_1} \\ &= [[D, x_1], x_2] + x_1 [D, x_2] + x_2 [D, x_1] . \end{aligned}$$

Assume the Proposition for $<n$. Then

$$[D, x_1 \cdots x_n] = [D, x_1 * \cdots * x_{n-1} x_n]$$

$$+ \sum_{r=1}^{n-2} \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r} [D, x_1 * \cdots * \check{x}_{i_1} * \cdots * \check{x}_{i_r} * \cdots * x_{n-1} x_n]$$

$$+ \sum_{r=1}^{n-2} \sum_{i_1 < \cdots < i_{r-1}} x_{i_1} \cdots x_{i_{r-1}} x_{n-1} x_n [D, x_1 * \cdots * \check{x}_{i_1} * \cdots * \check{x}_{i_{r-1}} * \cdots * x_{n-2}]$$

From the formula for $n=2$ we see that

$$[D,\ x_1*\cdots*x_{n-1}x_n] = [D,\ x_1*\cdots*x_{n-1}*x_n]+x_{n-1}[D,\ x_1*\cdots*x_{n-2}*x_n]$$
$$+x_n[D,\ x_1*\cdots*x_{n-1}]\,.$$

Substituting this relation in the preceding one we get

$$\begin{aligned}[D,\ x_1\cdots x_n] &= [D,\ x_1*\cdots*x_n]\\ &+\sum_{r=1}^{n-2}\Big\{\sum_{i_1<\cdots<i_r} x_{i_1}\cdots x_{i_r}[D,\ x_1*\cdots*\check{x}_{i_1}*\cdots*\check{x}_{i_r}*\cdots*x_{n-1}*x_n]\\ &+x_{i_1}\cdots x_{i_r}x_{n-1}[D,\ x_1*\cdots*\check{x}_{i_1}*\cdots*\check{x}_{i_r}*\cdots*x_{n-2}*x_n]\\ &+x_{i_1}\cdots x_{i_r}x_n[D,\ x_1*\cdots*\check{x}_{i_1}*\cdots*\check{x}_{i_r}*\cdots*x_{n-1}]\Big\}\\ &+\sum_{r=1}^{n-2}\sum_{i_1<\cdots<i_{s-1}} x_{i_1}\cdots x_{i_{r-1}}x_{n-1}x_n[D,\ x_1*\cdots*\check{x}_{i_1}*\cdots*\check{x}_{i_{r-1}}*\cdots*x_{n-2}]\\ &= [D,\ x_1*\cdots*x_n]+\sum_{r=1}^{n-1}\sum_{i_1<\cdots<i_r} x_{i_1}\cdots x_{i_r}[D,\ x_1*\cdots*\check{x}_{i_1}*\cdots*\check{x}_{i_r}*\cdots*x_n]\,.\end{aligned}$$

Corollary 11.1.

$$[D,\ a^n] = \sum_{r=0}^{n-1}\binom{n}{r}a^r[D,\ a^{*n-r}]$$

Corollary 11.2. *If the characteristic is $p>0$, then we have*

$$[D,\ a^{p^f}] = [D,\ a^{*p^f}] \qquad (f=1,\ 2,\ \cdots).$$

Proposition 12. *Let D be a q-th order derivation of A/K. Then we have the following identity:*

$$\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}(xy)^r[D,\ x^{q+1-r}]y^{q-r} = 0\,.$$

Proof. From Corollary 11.1 we have

$$[D,\ x^{q+1-r}] = \sum_{s=0}^{q-r}\binom{q+1-r}{s}x^s[D,\ x^{*q+1-r-s}]\,.$$

Hence the left hand side is equal to

$$\begin{aligned}&\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}(xy)^r\sum_{s=0}^{q-r}x^s[D,\ x^{*q+1-r-s}]y^{q-r}\\ &=\sum_{r=0}^{q}\sum_{s=0}^{q-r}(-1)^r\binom{r+s}{r}\binom{q+1}{r+s}x^{r+s}y^r[D,\ x^{*q+1-r-s}]y^{q-r}\\ &=\sum_{m=0}^{q}x^m\binom{q+1}{m}\sum_{r=0}^{m}(-1)^r\binom{m}{r}y^r[D,\ x^{*q+1-m}]y^{q-r}\,.\end{aligned}$$

$[D, x^{*q+1-m}]$ is a derivation of order $m-1$. Hence the proof of the proposition is reduced to the following:

Proposition 13. *Let D be a q-th order derivation. Then we have*

$$\sum_{r=0}^{q+1}(-1)^r\binom{q+1}{r}y^rDy^{q+1-r}=0$$

Proof. We shall use the induction on q. For a small integer q it is easy to check. Then we have

$$\begin{aligned}&\sum_{r=0}^{q+1}(-1)^r\binom{q+1}{r}y^rDy^{q+1-r}\\&=\sum_{r=0}^{q+1}(-1)^r\left\{\binom{q}{r}+\binom{q}{r-1}\right\}y^rDy^{q+1-r}\\&=\sum_{r=0}^{q}(-1)^r\binom{q}{r}y[D, y]y^{q-r}=0\,,\end{aligned}$$

since $[D, y]$ is a derivation of order $q-1$.

Theorem 14. *Let A be a k-algebra of characteristic $p>0$, and let D be an n-th order derivation of A/k. Then D induces on $A_1=kA^p$ a $\left[\frac{n}{p}\right]$-th order derivation.*

Proof. Induction on $\left[\frac{n}{p}\right]$. Proposition 10 implies that the Theorem is valid when $\left[\frac{n}{p}\right]=0$. Let us set $\left[\frac{n}{p}\right]=q$, i.e., $qp\leq n<(q+1)p$. Let a be an arbitrary element of A. Then $\Delta=[D, a^p]$ is a derivation of order $n-p$ of A by Corollary 11.2. Hence Δ induces on A_1 a $(q-1)$-th order derivation by induction assumption, i.e., $[\cdots[[\Delta, \alpha_1], \cdots, \alpha_{q-1}]=0$ on A_1 with $\alpha_i\in A_1$. In other words we have $[D, \alpha_1*\cdots*\alpha_{q-1}*a^p]\,|\,A_1=0$. We get immediately the assertion.

Corollary 14.1. *Under the circumstances in Theorem* 14, *D induces on $A_i=kA^{p^i}$ a $\left[\frac{n}{p^i}\right]$-th order derivation.*

5. Localization theorem

Theorem 15. *Let A be a k-algebra and let S a be multiplicatively closed set in A. Let D be a q-th order derivation of A into an A_S-module M. Then D can be extended in a unique way to a k-derivation $\bar{D}$ of A_S into M. Moreover the extension is given by the formula*

$$(*) \qquad \bar{D}\left(\frac{x}{s}\right) = \frac{(-1)^q}{s^{q+1}} \sum_{r=0}^{q} (-1)^r \binom{q+1}{r} s^r D(s^{q-r}x) .$$

Proof. We shall use the induction on q. The case $q=1$ is well known. First we shall show that the extension, if possible, is unique. In fact let s be an arbitrary element of S. Then $\Delta = Ds - sD - D(s)$ is a derivation of order $<q$. Hence there exists a unique extension which will also be denoted by the same letter Δ. Then we must have

$$\bar{D}\left(\frac{x}{s}\right) = \frac{1}{s} D(x) - \frac{x}{s} D(s) - \frac{1}{s} \Delta\left(\frac{x}{s}\right)$$

where $\bar{D}$ is the extension of D to A_S. Thus $\bar{D}$ is determined uniquely by D. We shall show that $\bar{D}\left(\frac{x}{s}\right)$ is given by the formula (*). If D is of order 1, then the right hand side is equal to

$$\frac{-1}{s^2} [D(sx) - 2sD(x)] = \frac{sD(x) - xD(s)}{s^2}$$

and (*) is valid. Assume the formula is true for any derivation of order $<q$, and let us set

$$Ds - sD - D(s) = \Delta .$$

Δ is of order $q-1$ and we have

$$\begin{aligned}
& D(x) - s\bar{D}\left(\frac{x}{s}\right) - \frac{x}{s} D(s) = \Delta\left(\frac{x}{s}\right) \\
&= \frac{(-1)^{q-1}}{s^q} \sum_{r=0}^{q-1} (-1)^r \binom{q}{r} s^r \Delta(s^{q-1-r}x) \\
&= \frac{(-1)^{q-1}}{s^q} \sum_{r=0}^{q-1} (-1)^r \binom{q}{r} s^r \{D(s^{q-r}x) - sD(s^{q-r-1}x) - s^{q-1-r}xD(s)\} \\
&= \frac{(-1)^{q-1}}{s^q} \left[\sum_{r=0}^{q-1} (-1)^r \left\{ \binom{q}{r} + \binom{q}{r-1} \right\} s^r D(s^{q-r}x) + (-1)^q s^q D(x) \right] \\
&\quad + \frac{(-1)^q}{s} \sum_{r=0}^{q} (-1)^r \binom{q}{r} s^{q-1} x D(s) \\
&= \frac{(-1)^{q-1}}{s^q} \left\{ \sum_{r=0}^{q} (-1)^r \binom{q+1}{r} s^r D(s^{q-r}x) \right\} + D(x) - \frac{x}{s} D(s) .
\end{aligned}$$

Hence

$$-s\bar{D}\left(\frac{x}{s}\right) = \frac{(-1)^{q-1}}{s^q} \sum_{r=0}^{q} (-1)^r \binom{q+1}{r} s^r D(s^{q-r}x) .$$

Thus the assertion is proved.

To prove the extensibility of the derivation D to a derivation $\bar{D}$ of A_S, it suffices to show the following. *For an element* $\frac{x}{s}$ *of* A_S *we set* $\bar{D}\left(\frac{x}{s}\right)$ *by the formula* $(*)$. *Then* $\bar{D}$ *is actually a derivation of order* q. First we have to show that $\bar{D}$ is well defined, i.e., if $\frac{x}{s}=\frac{y}{t}$, then we have $\bar{D}\left(\frac{x}{t}\right)=\bar{D}\left(\frac{y}{t}\right)$ i.e.,

$$\frac{(-1)^q}{s^{q+1}}\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}s^rD(s^{q-r}x)$$
$$=\frac{(-1)^q}{t^{q+1}}\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}t^rD(t^{q-r}x)\,.$$

Since $\frac{x}{s}=\frac{y}{t}$, there is an element u of S such that $u(tx-sy)=0$. For the sake of convenience we shall denote the expression $(*)$ by $\delta(x, s)$. Then it suffices to show that

$$(**)\qquad \delta(x, s)=\delta(vx, vs)$$

for any $v\in S$. In fact if $(**)$ is valid, then

$$\delta(x, s)=\delta(utx, uts)=\delta(usy, uts)=\delta(y, t)$$

as required.

The relation $(**)$ is reduced to the following identity:

$$v^{q+1}\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}s^rD(s^{q-r}x)$$
$$=\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}(vs)^rD((vs)^{q-r}vx)\,.$$

In other words:

$$\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}(vs)^r\{D(v^{q-r+1}s^{q-r}x)-v^{q-r+1}D(s^{q-r}x)\}$$
$$=\left[\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}(vs)^r\{Dv^{q-r+1}-v^{q-r+1}D\}s^{q-r}\right](x)$$
$$=\left[\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}(vs)^r\{[D, v^{q-r+1}]+D(v^{q-r+1})\}s^{q-r}\right](x)=0\,.$$

From Proposition 12, we have

$$\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}(vs)^r[D, v^{q-r+1}]s^{q-r}=0$$

and

$$\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}v^rD(v^{q-r+1})=0$$

on account of Proposition 13. Thus we could see that $\bar{D}$ is a well defined mapping. Next we have to show that $\bar{D}$ is a derivation.

First we remark that $\bar{D}$ is actually an element of $\mathrm{Hom}_k(A_S, M)$. In fact if x/s and y/t are in A_S, then

$$\bar{D}\left(\frac{x}{s}+\frac{y}{t}\right)=\bar{D}\left(\frac{tx+sy}{st}\right)=\bar{D}\left(\frac{tx}{st}\right)+\bar{D}\left(\frac{sy}{st}\right)=\bar{D}\left(\frac{x}{s}\right)+\bar{D}\left(\frac{y}{t}\right).$$

If D is a q-th order derivation we denote by $\bar{D}$ the element of $\mathrm{Hom}_k(A_S, m)$ defined by $(*)$. Then we shall show, in the next place, that

$$[\bar{D}, u]=\overline{[D, u]}$$

for any $u\in A$.

Now

$$\begin{aligned}
&[\bar{D}, u]\left(\frac{x}{s}\right)-\overline{[D, u]}\left(\frac{x}{s}\right)\\
&=\frac{(-1)^q}{s^{q+1}}\left[\sum_{r=0}^{q}(-1)^r\binom{q+1}{r}s^r\{D(s^{q-r}ux)-uD(s^{q-r}x)\}\right]-D(u)\frac{x}{s}\\
&\quad+\frac{(-1)^q}{s^q}\sum_{r=0}^{q}(-1)^r\binom{q}{r}s^r\{D(s^{q-r-1}ux)-uD(s^{q-r-1}x)-s^{q-r-1}xD(u)\}\\
&=\frac{(-1)^q}{s^{q+1}}\{D(s^qux)-uD(s^qx)\}-D(u)\frac{x}{s}\\
&\quad+\frac{(-1)^q}{s^{q+1}}\sum_{r=1}^{q}(-1)^r\left\{\binom{q+1}{r}-\binom{q}{r-1}\right\}s^r\{D(s^{q-r}ux)-uD(s^{q-r})\}\\
&\quad-\frac{(-1)^q}{s}\sum_{r=0}^{q-1}(-1)^r\binom{q}{r}xD(u)\\
&=\frac{(-1)^q}{s^{q+1}}\sum_{r=0}^{q}(-1)^r\binom{q}{r}s^r\{D(s^{q-r}ux)-uD(s^{q-r}x)\}\\
&=\frac{(-1)^q}{s}\left\{\sum_{r=0}^{q}(-1)^r\binom{q}{r}s[D, u]s^{q-r}\right\}(x)\\
&=0
\end{aligned}$$

on account of Proposition 13 since $[D, u]$ is of order $q-1$.

We are now well prepared to show that $\bar{D}$ is a derivation of order q of A_S into M. Let us set

$$\Delta=[D, s].$$

Then we can see easily that

$$\begin{aligned}
\left[\bar{D}, \frac{x}{s}\right]&=\frac{1}{s}[\bar{D}, x]-\frac{x}{s^2}[\bar{D}, s]-\frac{1}{s}\left[\bar{\Delta}, \frac{x}{s}\right]\\
&=\frac{1}{s}\overline{[D, x]}-\frac{x}{s^2}\overline{[D, s]}-\frac{1}{s}\left[\bar{\Delta}, \frac{x}{s}\right].
\end{aligned}$$

Any members of right hand side is a derivation of A_S into M by induction assumption on q. Hence $\left[\bar{D}, \frac{x}{s}\right]$ is also a derivation of A_S into M of order $q-1$. Hence $\bar{D}$ is a derivation of order q of A_S into M. Thus the proof of Theorem is complete.

CHAPTER II. MODULES OF HIGH ORDER DIFFERENTIALS

1. Modules of high order differentials

Let k, A be as in Chapter 1 and let C be the category of A-modules. Let F be an A-module and let $\mathcal{D}_0^{(q)}(A/k, F)$ be the set of q-th order derivations of A into F over k. $\mathcal{D}_0^{(q)}(A/k, F)$ is a left A-module and the correspondence

$$F \rightsquigarrow \mathcal{D}_0^{(q)}(A/k, F)$$

is easily seen to be a covariant functor of C into C. This functor is representable (Cf. [5]), and the representing module is given in the following way. Let us consider the exact sequence

$$0 \longrightarrow I \longrightarrow A\otimes_k A \xrightarrow{\varphi} A \longrightarrow 0$$

where

$$\varphi\,(\sum_i a_i\otimes b_i) = \sum a_i b_i\,.$$

We consider $A\otimes_k A$ as an A-module via

$$a(x\otimes y) = ax\otimes y\,.$$

The A-module I/I^{q+1} is the module we are looking for. In fact the mapping of A into I/I^{q+1} given by

$$\delta(a) = \{\text{Class of } (1\otimes a - a\otimes 1) \text{ modulo } I^{q+1}\}$$

is a derivation of order q of A/k, and we can see that any q-th order derivation D of A/k into F can be factored through I/I^{q+1}. We shall denote this A-module by $\Omega_k^{(q)}(A)$ and will be called the module of q-th order (Kähler) differentials. The mapping δ will be called the *canonical q-th order derivation of A/k*. The canonical q-th order derivation of A/k will usually denoted by $\delta_{A/k}^{(q)}$ or $\delta_{A/k}$ if no confusion will occur.

Proposition 1. *Let A be a k-algebra and let δ be the canonical q-th order derivation of A/k. Let x be an element of A such that $\delta(x)=0$. Then for any q-th order derivation D of A/k into an A-module M we have $Dx=xD$.*

Proof. Let y be an arbitrary element of A. The assumption $\delta(x)=0$ implies that $1\otimes x - x\otimes 1$ is contained in I^{q+1}. Hence we have

$$1\otimes xy-xy\otimes 1 = (1\otimes x-x\otimes 1)(1\otimes y-y\otimes 1)+y(1\otimes x-x\otimes 1)$$
$$+x(1\otimes y-y\otimes 1)$$
$$= x(1\otimes y-y\otimes 1) \qquad \text{mod } I^{q+1}$$

i.e.,

$$\delta(xy)=x\delta(y)\,.$$

Now let D be as in Proposition. Then there exists an A-homomorphism h of $\Omega_k^{(q)}(A)$ into M such that $D=h\delta$. Hence we have

$$D(xy) = h\cdot\delta(xy) = h(x\delta(y)) = xh\delta(y) = xD(y)$$

for any y in A. This implies that we have $Dx=xD$.

For q-th order differential module we can develope similar considerations as is done for the module of ordinary differentials (Cf. [4]). These are the subjects of the following paragraphs.

2. Polynomial rings

Let k be a commutative ring with unity and let $A=k[X_\lambda,\ \lambda\in\Lambda]$ be a polynomial ring over k in indeterminates $\{X_\lambda,\ \lambda\in\Lambda\}$. In this case $A\otimes_k A$ is again a polynomial ring in two system of indeterminates $1\otimes X_\lambda$ and $X_\lambda\otimes 1$ with the same indices set Λ. If we set

$$Y_\lambda = 1\otimes X_\lambda - X_\lambda\otimes 1$$

and identify $X_\lambda\otimes 1$ with X_λ, then $A\otimes A$ is a polynomial ring $k[X_\lambda,\ Y_\lambda;\ \lambda\in\Lambda]$. The kernel I of the homomorphism $\varphi\colon A\otimes_k A\to A$ is generated by $\{Y_\lambda\}$. Hence $\Omega_k^{(q)}(A)=I/I^{q+1}$ is a free module over A with generators $\{h_\lambda\}$, $\{h_\lambda h_\mu\}, \cdots, \{h_{\lambda_1}\cdots h_{\lambda_q}\}$ where we set

$$h_\lambda = \delta^{(q)}X_\lambda \quad (\delta^{(q)}=\delta^{(q)}_{A/k})\,.$$

Then we have

$$h_\lambda h_\mu = \delta^{(q)}X_\lambda\delta^{(q)}X_\mu, \cdots, h_{\lambda_1}\cdots h_{\lambda_q} = \delta^{(q)}X_{\lambda_1}\cdots\delta^{(q)}X_{\lambda_q}\,.$$

For any polynomial $f(X)$ in A we have

$$\delta^{(q)}(f) = \sum_\lambda (\Delta_\lambda f)\delta^{(q)}X_\lambda + \sum (\Delta_{\lambda\mu}f)\delta^{(q)}X_\lambda\delta^{(q)}X_\mu + \cdots$$
$$\cdots + \sum (\Delta_{\lambda_1\cdots\lambda_q}f)\delta^{(q)}X_{\lambda_1}\cdots\delta X^{(q)}{}_{\lambda_q}\,.$$

The coefficients $\Delta_\lambda f$, $\Delta_{\lambda\mu}f, \cdots$ are determined by the following formula:

$$f(X_\lambda+T_\lambda)-f(X_\lambda) = \sum(\Delta_\lambda f)T_\lambda + \sum(\Delta_{\lambda\mu}f)T_\lambda T_\mu + \cdots\,.$$

Δ_λ, $\Delta_{\lambda\mu}, \cdots, \Delta_{\lambda_1,\cdots,\lambda_q}$ are derivations of orders $1, 2, \cdots, q$ respectively of $k[X_\lambda]/k$ and T's are indeterminates. From the equations

$$\delta^{(q)}(X_{\lambda_1}\cdots X_{\lambda_r}) = \sum_i X_{\lambda_1}\cdots \hat{X}_{\lambda_i}\cdots X_{\lambda_r}\delta^{(q)}(X_i)$$
$$\cdots+\sum_{i<j} X_{\lambda_1}\cdots \hat{X}_{\lambda_i}\cdots \hat{X}_{\lambda_j}\cdots X_{\lambda_r}\delta^{(q)}(X_{\lambda_i})\delta^{(q)}(X_{\lambda_j})+\cdots$$

we can solve $\delta^{(q)}X_\lambda\delta^{(q)}X_\mu, \cdots, \delta^{(q)}X_{\lambda_1}\cdots\delta^{(q)}X_{\lambda_q}$ in terms of $\delta^{(q)}(X_\lambda X_\mu)$, $\delta^{(q)}(X_{\lambda_1}\cdots X_{\lambda_q})$ and they are represented as linear combinations of $\delta^{(q)}(X_\lambda)$ $\delta^{(q)}(X_\lambda X_\mu), \cdots, \delta^{(q)}(X_{\lambda_1}\cdots X_{\lambda_q})$ with coefficients in A. Hence

$$\delta^{(q)}(X_\lambda),\ \delta^{(q)}(X_\lambda X_\mu), \cdots, \delta^{(q)}(X_{\lambda_1}\cdots X_{\lambda_q})$$

from a free basis of $\Omega_k^{(q)}(k[X_\lambda])$.

Proposition 2. *Let $A=k[X_\lambda, \lambda\in\Lambda]$ be a polynomial ring over k. Then $\Omega_k^{(q)}(A)$ is a free module over A generated by*

$$\delta^{(q)}(X_\lambda),\ \delta^{(q)}(X_\lambda X_\mu), \cdots, \delta^{(q)}(X_{\lambda_1}\cdots X_{\lambda_q}) \qquad (\lambda_i\in\Lambda).$$

For any element $f(X)$ of A, $\delta^{(q)}f$ *can* be *represented uniquely as*

$$\delta^{(q)}(f) = \sum(\partial_\lambda^{(q)}f)\delta^{(q)}(X_\lambda)+\sum(\partial_{\lambda\mu}^{(q)}f)\delta^{(q)}(X_\lambda X_\mu)+\cdots$$
$$\cdots+\sum(\partial_{\lambda_1\cdots\lambda_q}^{(q)}f)\delta^{(q)}(X_{\lambda_1}\cdots X_{\lambda_q}),$$

where $\partial_\lambda^{(q)}(f), \cdots, \partial_{\lambda_1\cdots\lambda_q}^{(q)}(f)$ are elements of A.

A detailed formula for $\partial_\lambda^{(q)}(f), \cdots, \partial_{\lambda_1\cdots\lambda_q}^{(q)}(f)$ will be given in case where A is a polynomial ring of one variable.

In the rest of this paragraph let A be a polynomial ring over k in one variable X. We shall define the operation Δ_m $(m=1, 2, \cdots)$ by

$$f(X+T)-f(X) = \sum_{m=1}^{\infty}\Delta_m(f)T^m .$$

We shall also set

$$\Delta_T f = f(X+T)-f(X) .$$

We have then

$$\Delta_T f = \sum_{m=1}^{\infty}\Delta_m(f)T^m .$$

Proposition 3. *Let $A=k[X]$ be a polynomial ring over k in one variable. Then the following identity holds:*

$$T^m = \sum_{s=1}^{m}(-1)^{m-s}\binom{m}{s}X^{m-s}\Delta_T(X^s) .$$

Proof. We shall prove by induction on m. Assume (3). By definition we have

$$\Delta_T(X^{m+1}) = (m+1)X^mT+\binom{m+1}{2}X^{m-1}T^2+\cdots+(m+1)XT^m+T^{m+1} ,$$

i.e.,

$$\begin{aligned}
T^{m+1} &= -\sum_{r=1}^{m}\binom{m+1}{r}X^{m+1-r}T^{r}+\Delta_T(X^{m+1})\\
&= -\sum_{r=1}^{m}\binom{m+1}{r}X^{m+1-r}(-1)^{r}\sum_{s=1}^{r}(-1)^{s}\binom{r}{s}X^{r-s}\Delta_T(X^{s})+\Delta_T(X^{m+1})\\
&= -\sum_{s=1}^{m}X^{m+1-s}\sum_{r=s}^{m}(-1)^{r-s}\binom{r}{s}\binom{m+1}{r}\Delta_T(X^{s})+\Delta_T(X^{m+1})\\
&= -\sum_{s=1}^{m}X^{m+1-s}\binom{m+1}{s}\sum_{r=s}^{m}(-1)^{r-s}\binom{m+1-s}{r-s}\Delta_T(X^{s})+\Delta_T(X^{m+1})\\
&= -\sum_{s=1}^{m}X^{m+1-s}\binom{m+1}{s}\sum_{t=0}^{m-s}(-1)^{t}\binom{m+1-s}{t}\Delta_T(X^{s})+\Delta_T(X^{m+1})\\
&= \sum_{s=1}^{m}(-1)^{m+1-s}\binom{m+1}{s}X^{m+1-s}\Delta_T(X^{s})+\Delta_T(X^{m+1})\\
&= \sum_{s=1}^{m+1}(-1)^{m+1-s}\binom{m+1}{s}X^{m+1-s}\Delta_T(X^{s}) \qquad \text{q.e.d.}
\end{aligned}$$

We shall set $\delta=\delta^{(q)}_{k[X]/k}$. Then for any $f(X)\in k[X]$ we have

$$\begin{aligned}
\delta(f) &= \sum_{m=1}^{q}\Delta_m(f)(\delta X)^{m}\\
&= \sum_{m=1}^{q}\Delta_m(f)\sum_{s=1}^{m}(-1)^{m-s}\binom{m}{s}X^{m-s}\delta(X^{s})\\
&= \sum_{s=1}^{q}\sum_{m=s}^{q}(-1)^{m-s}\Delta_m(f)\binom{m}{s}X^{m-s}\delta(X^{s}).
\end{aligned}$$

We shall set

$$\partial^{(q)}_{s}(f)=\sum_{m=s}^{q}(-1)^{m-s}\binom{m}{s}\Delta_m(f)X^{m-s}.$$

Then we have

$$\delta(f)=\sum_{s=1}^{q}\partial^{(q)}_{s}(f)\delta(X^{s}).$$

Δ_m is a derivation of order m because of the following

Proposition 4. $[\Delta_m, X]=\Delta_{m-1}$ $(m=1, 2,\cdots)$ *where we set* $\Delta_0=0$.

Proof. By definition we have $[\Delta_m, X]=\Delta_m X-X\Delta_m-\Delta_m(X)$. First we shall treat the case $m=1$. In this case Δ_1 is a derivation, hence $[\Delta_1, X]=0$ as asserted. Assume that $m>1$, and let $f(X)$ be an element of $k[X]$. We set

$$f(X+T)=\sum_{m=0}^{\infty}f_m(X)T^{m}.$$

Then

$$\Delta_m(Xf)-X\Delta_m(f)-\Delta_m(X)f=(Xf_m(X)+f_{m-1}(X))-Xf_m(X)=f_{m-1}(X)\,.$$

Hence by definition we have

$$(\Delta_m X-X\Delta_m-\Delta_m(X))(f)=\Delta_{m-1}(f) \qquad \text{q.e.d.}$$

Proposition 5. $\Delta_m(X^n)=\binom{n}{m}X^{n-m}$.

This is immediate from the definition and the proof will be omitted.

Proposition 6. *Let $f(X)$ be an element of $k[X]$. Then we have*

$\det|\Delta_i(X^j f(X))|=(\Delta_1 f)^m \mod f(X),\ (i=1,\cdots,m;\ j=0,1,\cdots,m-1)$.

Proof. We have by preceding Proposition 4

$$\Delta_m(Xg)=X\Delta_m(g)+g\Delta_m(X)+\Delta_{m-1}(g)\,.$$

Hence

$$\begin{vmatrix} \Delta_1 f & \Delta_2 f & \cdots \Delta_m f \\ \Delta_1(Xf) & \Delta_2(Xf) & \cdots \Delta_m(Xf) \\ \vdots & \vdots & \vdots \\ \Delta_1(X^{m-1}f) & \Delta_2(X^{m-1}f) & \cdots \Delta_m(X^{m-1}f) \end{vmatrix}$$

$$=\begin{vmatrix} \Delta_1 f, & \Delta_2 f, & \cdots \Delta_m f \\ X\Delta_1 f+f, & X\Delta_2 f+\Delta_1 f, & \cdots X\Delta_m(f)+\Delta_{m-1}(f) \\ X\Delta_1(Xf)+Xf, & X\Delta_2(Xf)+\Delta_1(Xf) & \cdots X\Delta_m(Xf)+\Delta_{m-1}(Xf) \\ \vdots & \vdots & \vdots \\ X\Delta_1(X^{m-2})+X^{m-2}f, & X\Delta_2(X^{m-2}f)+\Delta_1(X^{m-2}f) & \cdots X\Delta_m(X^{m-2}f)+\Delta_{m-1}(X^{m-2}f) \end{vmatrix}$$

$$\equiv\begin{vmatrix} \Delta_1 f & \Delta_2 f & \cdots \Delta_m f \\ 0 & \Delta_1 f & \cdots \Delta_{m-1} f \\ 0 & \Delta_1(Xf) & \cdots \Delta_{m-1}(Xf) \\ \vdots & \vdots & \vdots \\ 0 & \Delta_1(X^{m-2}f) & \cdots \Delta_{m-1}(X^{m-2}f) \end{vmatrix} \quad (\mathrm{mod}\ f)$$

Then by induction on m we get immediately

$$\det|\Delta_i(X^j f)|\equiv(\Delta_1 f)^m \qquad (\mathrm{mod}\ f)$$

Proposition 7.

$$\partial_s^{(q)}(X^i)=\delta_{si} \qquad \text{if} \quad i\leq q.$$

$$\partial_s^{(q)}(X^i)=\left[\sum_{t=0}^{q-s}(-1)^t\binom{i-s}{t}\right]\binom{i}{s}X^{i+s} \qquad \text{if} \quad i>q.$$

Proof. By definition we have

$$\partial_s^{(q)}(X^i) = \sum_{m=s}^{q}(-1)^{m-s}\binom{m}{s}X^{m-s}\Delta_m(X^i)$$
$$= \left[\sum_{m=s}^{q}(-1)^{m-s}\binom{m}{s}\binom{i}{m}\right]X^{i-s}$$
$$= \binom{i}{s}\left[\sum_{t=0}^{q-s}(-1)^t\binom{i-s}{t}\right]X^{i-s}.$$

Then if $i<s$ we have $\Delta_m(X^i)=0$ for every $m\geqslant s$. If $q\geqslant i\geqslant s$ we have $\sum_{t=0}^{q-s}(-1)^t\binom{i-s}{t} = 1$ or zero according as $i=s$ or $i>s$.

3. Functorial properties

Let A and B be two k-algebras and let h be a k-algebra homomorphism of A into B. Let $\delta_{A/k}^{(q)}$ and $\delta_{B/k}^{(q)}$ be the canonical q-th order derivations of A/k and B/k respectively. Then the mapping $\Delta=\delta_{B/k}^{(q)}h$ is a q-th order derivation of A/k into $\Omega_k^{(q)}(B)$. Hence there exists an A-homomorphism h^* of $\Omega_k^{(q)}(A)$ into $\Omega_k^{(q)}(B)$ such that we have

$$h^*\delta_{A/k}^{(q)} = \delta_{B/k}^{(q)}h\,.$$

Let $\varphi=\varphi_{B/A/k}^{(q)}$ be a B-homomorphism of $B\otimes_A\Omega_k^{(q)}(A)$ into $\Omega_k^{(q)}(B)$ such that $\varphi(b\otimes\delta_{A/k}^{(q)}(a))=bh^*\delta_{A/k}^{(q)}(a)=b\delta_{B/k}^{(q)}h(a)$. Let $N_{B/A/k}^{(q)}=N$ and $\Omega_k^{(q)}(B/A)$ be the kernel and the cokernel of φ respectively. Then we have the exact sequence

$$(1)\qquad 0\longrightarrow N\longrightarrow B\otimes_A\Omega_k^{(q)}(A)\xrightarrow{\varphi}\Omega_k^{(q)}(B)\xrightarrow{j}\Omega_k^{(q)}(B/A)\longrightarrow 0\,.$$

If we set $\hat{\delta}_{B/A/k}^{(q)}=\hat{\delta}=j\hat{\delta}_{B/k}^{(q)}$, δ is also a q-th order derivation of B/k such that $\hat{\delta}(a)=0$ for any a in A. Moreover it is immediately seen that $\Omega_k^{(q)}(B/A)$ has the universal mapping property with respect to such derivations. Thus we have the homomorphism

$$\psi\colon \Omega_k^{(q)}(B/A)\longrightarrow\Omega_A^{(q)}(B)$$

such that

$$\psi\hat{\delta}(b) = \delta_{B/A}^{(q)}(b)\,,\qquad b\in B.$$

Since $\Omega_A^{(q)}(B)$ is generated over B by elements of the form $\delta_{B/A}^{(q)}(x)$ $(x\in B)$ ψ is a surjective homomorphism. Contrary to the case of ordinary derivation (the case of $q=1$) ψ is not necessarily an isomorphism. But we have the

Proposition 8. *Under the above circumstances we have*
(i) $\varphi_{B/A/k}^{(q)}$ *is left invertible if and only if any* $D\in\mathcal{D}_0^{(q)}(A/k, F)$ *can be extended to an element of* $\mathcal{D}_0^{(q)}(B/k, F)$ *where* F *is an arbitrary* B*-module.*
(ii) $\varphi_{B/A/k}^{(q)}$ *is surjective if* h *is surjective.*

The proof of this proposition is quite similar to that of Theorem 1 of [4] and will be omitted.

Theorem 9[3]**.** *Let A be a k-algebra and let S be multiplicatively closed subset of A. Then we have the isomorphism*

$$\Omega_k^{(q)}(A_S) \simeq A_S \otimes \Omega_k^{(q)}(A).$$

Proof. By I–15 high order derivation of A/k into an A_S module can be extended to that of A_S/k. Hence Proposition 8 implies that $\varphi = \varphi_{A_S/A/k}^{(q)}$ is injective. Moreover the formula of I–15 tells that φ is surjective. Hence φ is an isomorphism.

We shall now determine the kernel of the homomorphism ψ. For this purpose we need the following

Proposition 10. *Let D be an element of $\mathcal{D}_0^{(q)}(B/k, F)$ where F is a B-module. Then for any $a \in A$, $d(a) = [D, a]$ is an element of $\mathcal{D}_0^{(q-1)}(B/k, F)$ and the mapping d is an element of $\mathcal{D}_0^{(q-1)}(A/k, \mathcal{D}_0^{(q-1)}(B/k, F))$.*

Proof. We shall use the induction on the order q of D. The case $q=1$ is trivial because we have $d(a)=0$ for any $a \in A$, and hence $d=0$. Let x be an arbitrary but fixed element of A. We shall show that $[d, x]$ is an element of $\mathcal{D}_0^{(q-2)}(A/k, \mathcal{D}_0^{(q-1)}(B/k, F))$. By I–3, $D_x = [D, x]$ is an element of $\mathcal{D}_0^{(q-1)}(B/k, F)$. Hence $d_0(a) = [D_x, a] = [D_a, x]$ is a $(q-2)$–th order derivation of B/k into F. We can see that $[d, x] = d_0$. In fact we have

$$\begin{aligned}[d, x](a) &= d(xa) - xd(a) - ad(x) \\ &= [D, xa] - x[D, a] - a[D, x] = [[D, x], a] = d_0(a)\end{aligned}$$

(cf. I–11). Moreover by induction assumption d_0 is an element of $\mathcal{D}_0^{(q-2)}(A/k, \mathcal{D}_0^{(q-1)}(B/k, F))$. Hence d is a $(q-1)$-th order derivation of A/k into $\mathcal{D}_0^{(q-1)}(B/k, F)$ as asserted.

Here we shall remark that if D vanishes on A, then $d(a)$ also vanishes on A. Henceforse we shall denote by $\hat{\mathcal{D}}_0^{(q)}(B/A/k, F)$ a set of q-th order derivations of B/k into F vanishing on A. Then we have the

Corollary 10.1. *In Proposition* 10, *assume that D vanishes on A. Then d is an element* of $\mathcal{D}_0^{(q-1)}(A/k, \hat{\mathcal{D}}_0^{(q-1)}(B/A/k, F)$.

We shall apply the above Corollary to the q-th order derivation $\hat{\delta} = \hat{\delta}_{B/A/k}^{(q)}$

3) It is possible to give a direct proof of this Theorem without the knowledge of I–15. T. Kikuchi informed me that he proved the isomorphism $\Omega_k^{(q)}(B) = B \otimes \Omega_k^{(q)}(A)$ under the assumption that (1) B is a flat A-module and (2) the ring homomorphism $A \to B$ is an epimorphism in the category of commutative rings.

of B/k into $\Omega_k^{(q)}(B/A)$. Then the mapping ρ of A defined by $\rho(a)=[\hat{\delta}, a] = \hat{\delta}a - a\hat{\delta}$ is an element of $\mathcal{D}(A/k, \hat{\mathcal{D}}_0^{(q-1)}(B/A/k, \Omega_0^{(q)}(B/A))$. Hence there exists a B-homomorphism $\rho^*(a)$ of $\Omega_k^{(q-1)}(B/A)$ into $\Omega_k^{(q)}(B/A)$ such that

$$(2) \qquad \rho^*(a)\hat{\delta}_{B/A/k}^{(q-1)} = \rho(a)\,.$$

We can see easily that ρ^* is a derivation of A/k of order $q-1$ into $\mathrm{Hom}_B(\Omega_k^{(q-1)}(B/A), \Omega_k^{(q)}(B/A))$. Hence there exists an A-homomorphism f of $\Omega_{A/k}^{(q-1)}(A)$ into $\mathrm{Hom}_B(\Omega_k^{(q-1)}(B/A), \Omega_k^{(q)}(B/A))$. such that

$$(3) \qquad \rho^* = f\delta_{A/k}^{(q-1)}\,.$$

Combining (2) and (3) we have for any $a\in A$ and $b\in B$,

$$(f\delta_{A/k}^{(q-1)}(a))(\hat{\delta}_{B/A/k}^{(q-1)}(b)) = \rho(a)(b) = [\hat{\delta}, a](b) = \hat{\delta}(ab) - a\hat{\delta}(b)\,.$$

By the canonical bijection of

$$\begin{aligned}&\mathrm{Hom}_A(\Omega_k^{(q-1)}(A), \mathrm{Hom}_B(\Omega_k^{(q-1)}(B/A), \Omega_k^{(q)}(B/A))\\ &\quad\simeq \mathrm{Hom}_B(\Omega_k^{(q-1)}(A)\otimes_A\Omega_k^{(q-1)}(B/A), \Omega_k^{(q)}(B/A))\,,\end{aligned}$$

We get an element σ of the latter group such that

$$\sigma(\delta_{A/k}^{(q-1)}(a)\otimes\hat{\delta}_{B/A/k}^{(q-1)}(b)) = (f(\delta_{A/k}^{(q-1)}(a))(\hat{\delta}_{B/A/k}^{(q-1)}(b) = [\hat{\delta}, a](b)\,.$$

Now we shall show that the sequence of B-modules

$$(4) \qquad \Omega_k^{(q-1)}(A)\otimes_A\Omega_k^{(q-1)}(B/A) \xrightarrow{\sigma} \Omega_k^{(q)}(B/A) \xrightarrow{\psi} \Omega_A^{(q)}(B) \longrightarrow 0$$

is exact. It is immediate to see that $\mathrm{Im}(\sigma)\subseteq\mathrm{Ker}(\psi)$. Let us consider the mapping Δ of B into $\mathrm{Coker}(\sigma)$ such that

$$\Delta = \varepsilon\hat{\delta}$$

where ε is the natural homomorphism of $\Omega_k^{(q)}(B/A)$ onto $\mathrm{Coker}(\sigma)$. Then Δ is a q-th order derivation of B/k into $\mathrm{Coker}(\sigma)$. We have $\Delta(a)=0$ for any $a\in A$. We have also $[\Delta, a]=0$. In fact for any element b in B we have

$$\begin{aligned}[\Delta, a](b) &= \Delta(ab) - a\Delta(b) = \varepsilon\hat{\delta}(ab) - a\varepsilon\hat{\delta}(b)\\ &= \varepsilon([\hat{\delta}, a](b)) = \varepsilon\sigma(\delta_{A/k}^{(q-1)}(a)\otimes\delta_{B/k}^{(q-1)}(b)) = 0\,.\end{aligned}$$

Hence Δ must be an A-derivation i.e., Δ is an element of $D_0^{(q)}(B/A, \mathrm{Coker}(\sigma))$. Let h be a B-homomorphism of $\Omega_A^{(q)}(B)$ into $\mathrm{Coker}(\sigma)$ corresponding Δ. Then we have $\Delta = h\delta_{B/A}^{(q)}$. Now let $\sum b\hat{\delta}(c)\in\mathrm{Ker}(\psi)$ where b, c's are elements of B. Then $\psi(\sum b\delta(c))=0$. Hence we have

$$0 = \sum bh\delta(c) = \sum bh\delta_{B/A}^{(q)}(c) = \sum b\Delta(c) = \sum b\varepsilon\hat{\delta}(c) = \varepsilon(\sum b\hat{\delta}(c))\,.$$

Hence $\sum b\hat{\delta}(c)\in\mathrm{Ker}(\varepsilon)=\mathrm{Im}(\sigma)$. Thus the exactness of (4) is proved. For the future reference the above results will be summarized in the following

Theorem 11. *Let k, A and B be as above. Then we have the following exact sequences.*

$$B\otimes_A\Omega_k^{(q)}(A) \xrightarrow{\varphi} \Omega_k^{(q)}(B) \xrightarrow{j} \Omega_k^{(q)}(B/A) \longrightarrow 0$$

$$\Omega_k^{(q-1)}(A)\otimes_A\Omega_k^{(q-1)}(B/A) \xrightarrow{\sigma} \Omega_k^{(q)}(B/A) \xrightarrow{\psi} \Omega_A^{(q)}(B) \longrightarrow 0$$

where $\Omega_k^{(q)}(B/A)=\mathrm{Coker}(\varphi)$ *and* ψ *is defined by the formula* $\psi j\delta_{B/k}^{(q)}(b)=\delta_{B/A}^{(q)}(b)$.

Taking the dual sequence we get the

Corollary 11.1[4]. *Being k, A and B as before, we have the exact sequences*

$$0 \longrightarrow \mathcal{D}_0^{(q)}(B/A) \longrightarrow \hat{\mathcal{D}}_0^{(q)}(B/A/k) \xrightarrow{\tau} \mathcal{D}_0^{(q-1)}(A/k,\ \hat{\mathcal{D}}_0^{(q-1)}(B/A/k)$$

$$0 \longrightarrow \hat{\mathcal{D}}_0^{(q)}(B/A/k) \longrightarrow \mathcal{D}_0^{(q)}(B/k) \xrightarrow{\varphi^*} \mathcal{D}_0^{(q)}(A/k,\ B)$$

where $\hat{\mathcal{D}}_0^{(q)}(B/A/k)$ *is the submodule of* $\mathcal{D}_0^{(q)}(B/k)$ *consisting of elements D such that* $D(a)=0$ *for any element a of A and* τ *is defined by* $\tau(D)\,(a)=[D,\ a](a\in A)$.

Proposition 12. *Let k, A and B be as before. Then we have the followings:*

(i) *Assume that* $\Omega_k^{(q)}(A)=\Omega_A^{(q)}(B)=0$ *and either* $\Omega_k^{(q-1)}(A)$ *or* $\Omega_k^{(q-1)}(B/A)$ *vanishes. Then we have* $\Omega_k^{(q)}(B)=0$

(ii) $\Omega_k^{(q)}(B)=0$ *implies* $\Omega_A^{(q)}(B)=0$.

(iii) *Assume that* $\Omega_k^{(q)}(A)=0$. *Then if either one of* $\Omega_k^{(q-1)}(A)$ *or* $\Omega_k^{(q-1)}(B/A)$ *is zero we have an isomorphism.*

$$\Omega_k^{(q)}(B)\simeq\Omega_A^{(q)}(B)\,.$$

Theorem 13. *Let A and B be k-algebras. Then we have the isomorphism*

$$B\otimes_k\Omega_k^{(q)}(A) = \Omega_B^{(q)}(B\otimes_k A)\,.$$

Proof. Let D be a mapping of $B\otimes_k A$ into $B\otimes_k\Omega_k^{(q)}(A)$ defined by $D(b\otimes a)=b\otimes\delta(a)$ where δ is the canonical q-th order derivation of A/k. Then as is easily seen D is a q-th order derivation of $(B\otimes A)/B$ with values in $B\otimes_k\Omega_k^{(q)}(A)$. Hence there exists a B-homomorphism h of $\Omega_B^{(q)}(B\otimes A)$ into $B\otimes_k\Omega_k^{(q)}(A)$ such that $D=h\delta_{(B\otimes A)/B}^{(q)}$. Let us define a mapping δ_1 of A into $\Omega_B^{(q)}(B\otimes A)$ by $\delta_1(a)=\delta_{(B\otimes A)/B}^{(q)}(1\otimes a)$. This is clearly a q-th order derivation of A/k into $\Omega_B^{(q)}(B\otimes A)$. Hence there exists also an A-homomorphism g of $\Omega_k^{(q)}(A)$ into $\Omega_B^{(q)}(B\otimes A)$ such that $\delta_1=g\delta$. We can extend g to a B-homomor-

4) The author proved originally Corollary 11.1 first. A slight modification of the proof gives us the exact sequence

$$0\to\mathcal{D}_0^{(q)}(B/A,\ F)\to\mathcal{D}_0^{(q)}(B/A/K,\ F)\to\mathcal{D}_0{}^{(q-1)}(A/K,\ \mathcal{D}_0{}^{(q-1)}(B/K,\ F)$$

for any B-module F. Then the representability of the functor $\mathcal{D}_0$ yields at once Theorem 11. The direct proof of Theorem 11 given here is due to my friend T. Kikuchi.

phism $\tilde{g}$ of $B\otimes\Omega_k^{(q)}(A)$ into $\Omega_k^{(q)}(B\otimes A)$ by $\tilde{g}(b\otimes\delta(a))=bg\delta(a)$. It is immediate to verify that $h\tilde{g}$ and $\tilde{g}h$ are identities and we get the assertion.

Let A be a k-algebra and let $\mathfrak{a}$ be an ideal of A. We shall define a mapping ρ of $\mathfrak{a}$ into $A/\mathfrak{a}\otimes\Omega_k^{(q)}(A)$ by

$$\rho(a) = 1\otimes\delta a \qquad a\in\mathfrak{a}.$$

where δ is the canonical q-th order derivation of A/k. ρ is a homomorphism of additive groups and vanishes on $\mathfrak{a}^{q+1}$. Hence ρ induces a k-homomorphism of $\mathfrak{a}/\mathfrak{a}^{q+1}$ into $A/a\otimes\Omega_k^{(q)}(A)$. ρ is not necessarily an A-homomorphism. We denote by N the submodule of $A/\mathfrak{a}\otimes\Omega_k^{(q)}(A)$ generated by $\mathrm{Im}(\rho)$.

On the other hand let us consider a mapping $D\colon A\to\Omega_k^{(q)}(A/\mathfrak{a})$ such that

$$D(a) = \bar{\delta}(\bar{a})$$

where $\bar{\delta}=\delta_{(A/\mathfrak{a})/k}^{(q)}$ and $\bar{a}$ is the class of a modulo $\mathfrak{a}$. D is clearly a q-th order derivation of A into $\Omega_k^{(q)}(A/\mathfrak{a})$. Hence D can be written as $D=h\delta$ where $\delta=\delta_{A/k}^{(q)}$ and h is an A-homomorphism of $\Omega_k^{(q)}(A)$ into $\Omega_k^{(q)}(A/\mathfrak{a})$. Let τ be the natural homomorphism of $A/\mathfrak{a}\otimes\Omega_k^{(q)}(A/\mathfrak{a})$ into $\Omega_k^{(q)}(A/\mathfrak{a})$. We shall set

$$\alpha = \tau(1\otimes h)\,.$$

Theorem 14. *The sequence*

$$0 \longrightarrow N \xrightarrow{\beta} (A/a)\otimes\Omega_k^{(q)}(A) \xrightarrow{\alpha} \Omega_k^{(q)}(A/\mathfrak{a}) \longrightarrow 0$$

is exact.

Proof. It is clear that we have $\alpha\beta=0$. If we identify $(A/\mathfrak{a})\otimes\Omega_k^{(q)}(A)$ with the quotient module $\Omega_k^{(q)}(A)/\mathfrak{a}\Omega_k^{(q)}(A)$, then N is equal to $D^{(q)}(\mathfrak{a})+a\Omega_k^{(q)}(A)/\mathfrak{a}\Omega_k^{(q)}(A)$ where $D^{(q)}(\mathfrak{a})$ is the submodule of $\Omega_k^{(q)}(A)$ generated by elements of the form δx, $x\in\mathfrak{a}$. Hence to prove the exactness it suffices to show that

$$\Omega_k^{(q)}(A/\mathfrak{a})\cong\Omega_k^{(q)}(A)/[D^{(q)}(\mathfrak{a})+\mathfrak{a}\Omega_k^{(q)}(A)]\,.$$

The rest of the proof will be omitted because it is literally the same as the proof of Proposition 9 of [4].

Corollary 14.1.

$$\Omega_k^{(q)}(A/\mathfrak{a})\cong\Omega_k^{(q)}(A)/\mathfrak{a}\Omega_k^{(q)}(A)+D^{(q)}(\mathfrak{a})\,.$$

EXAMPLE. Let L be a purely inseparable extension of a field K of exponent e such that there is a primitive element x, i.e., $L=K(x)$. If we set $x^{p^e}=a$ we have an isomorphism

$$L \cong K[X]/(X^{p^e}-a).$$

We shall denote by δ the canonical q-th order derivation of $K[X]/K$.

First assume $q<p^e$. Then by I–7 and I–10 we have

$$\begin{aligned}\delta(X^i(X^{p^e}-a)) &= X^i\delta(X^{p^e}-a)+(X^{p^e}-a)\delta(X^i)\\ &= (X^{p^e}-a)\delta(X^i).\end{aligned}$$

Hence

$$D^{(q)}(\mathfrak{a})\subset\mathfrak{a}\Omega_K^{(q)}(K[X])$$

where $\mathfrak{a}=(X^{p^e}-a)K[X]$ and we have

$$\Omega_K^{(q)}(L) = L\otimes\Omega_K^{(q)}(K[X])$$

i.e.,

$$\Omega_K^{(q)}(L) = L\delta(X)+L\delta(X^2)+\cdots+L\delta(X^q).$$

If $q=p^e$, then we have also

$$\delta(X^i(X^{p^e}-a)) = (X^{p^e}-a)\delta(X^i)+X^i\delta(X^{p^e}-a).$$

Hence

$$D^{(q)}(\mathfrak{a}) = \mathfrak{a}\Omega_K^{(q)}(K[X])+K[X]\delta(X^{p^e}).$$

From this we immediately see that $\Omega_K^{(p^e)}(L)$ is isomorphic to the free module over L generated by $\delta(X),\cdots,\delta(X^{p^e-1})$. Thus we have

$$\Omega_K^{(p^e)}(L)\cong\Omega_K^{(p^e-1)}(L).$$

This isomorphism implies among others that

$$J^{p^e}=J^{p^e}\cdot J$$

where J is the kernel of the homomorphism $L\otimes L\to L$. Then we have

$$J^{p^e}=J^{p^e+1}=J^{p^e+2}=\cdots$$

i.e.,

$$\Omega_K^{(q)}(L) = \Omega_K^{(p^e-1)}(L) \qquad \text{for} \quad q\geqq p^e.$$

For the future reference the above results will be put in the

Proposition 15[5]. *Let L be a purely inseparable extension primitively generated over a field K of exponent e and let us set $L=K(\alpha)$. Then $\Omega_K^{(q)}(L)$ is isomorphic to a free module of rank q with a basis $\delta(\alpha)$, $\delta(\alpha^2),\cdots,\delta(\alpha^q)$ for $q<p^e$.*

5) This result can also be obtained in a more simple way through the analysis of the kernel $L\otimes_k L\to L$. We took up this method here as an application of Corollary 14.1.

When $q \geqq p^e$, $\Omega_K^{(q)}(L)$ is isomorphic to $\Omega_k^{(p^e-1)}(L)$ and is a free module of rank p^e-1 with a basis $\delta(\alpha), \delta(\alpha^2), \cdots, \delta(\alpha^{p-1})$.

Let A and B be two k-algebras. We shall set $\delta=\delta_{A\otimes B/k}^{(q)}$, $\delta_1=\delta_{A/k}^{(q)}$ and $\delta_2=\delta_{B/k}^{(q)}$. $A\otimes_k B$ is an A-algebra (B-algebra) via the natural homomorphism $f_A(f_B)$ such that $f_A(a)=a\otimes 1(f_B(b)=1\otimes b)$. We have a homomorphism $\psi_A(\psi_B)$ of $\Omega_k^{(q)}(A)\otimes_k B(A\otimes_k\Omega_k^{(q)}(B))$ into $\Omega_k^{(q)}(A\otimes B)$ such that

$$\psi_A(\delta_1(a)\otimes b) = (1\otimes b)\delta(a\otimes 1)$$
$$\psi_B(a\otimes\delta_2(b)) = (a\otimes 1)\delta(1\otimes b)$$

We shall consider the mapping $D_1(D_2)$ of $A\otimes B$ into $\Omega_k^{(q)}(A)\otimes B(A\otimes\Omega_k^{(q)}(B))$ such that

$$D_1(a\otimes b) = \delta_1(a)\otimes b$$
$$D_2(a\otimes b) = a\otimes\delta_2(b)\,.$$

It is not difficult to see that D_i's are also q-th order derivations of $A\otimes B$ into $\Omega_k^{(q)}(A)\otimes B$ and $A\otimes\Omega_k^{(q)}(B)$ respectively. Hence we have an $A\otimes B$-homomorphism $\alpha_A(\alpha_B)$ of $\Omega_k^{(q)}(A\otimes B)$ into $\Omega_k^{(q)}(A)\otimes B(A\otimes\Omega_k^{(q)}(B))$ such that

$$\alpha_A\delta(a\otimes b) = \delta_1(a)\otimes b$$
$$\alpha_B\delta(a\otimes b) = a\otimes\delta_2(b)\,.$$

It is immediately seen that we have

$$\alpha_A\psi_A = 1, \quad \alpha_A\psi_B = 0\,.$$
$$\alpha_B\psi_A = 0, \quad \alpha_B\psi_B = 1\,.$$

These homomorphisms give rise at once the

Proposition 16. *Let A and B be two k-algebras. Then there exists a direct sum decomposition*

$$\Omega_k^{(q)}(A\otimes_k B) \simeq \Omega_k^{(q)}(A)\otimes_k B \oplus A\otimes\Omega_k^{(q)}(B)\oplus U_{A\otimes B/k}^{(q)}\,.$$

The submodule $U_{A\otimes B/k}^{(q)}$ has the universal mapping property with respect to derivations of $A\otimes B$ which vanish on $f_A(A)$ and $f_B(B)$.

Corollary 16.1. *We have*

$$\Omega_k^{(1)}(A\otimes_k B) = \Omega_k^{(1)}(A)\otimes_k B\oplus A\otimes\Omega_k^{(1)}(B)\,.$$

The proof is immediate because any ordinary derivation of $A\otimes B$ vanishing on $A\otimes 1$ and $1\otimes B$ must be a trivial mapping.

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