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AN ELEMENTARY CONSTRUCTION OF THE REPRESENTATIONS OF $SL(2, GF(q))$

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1. Introduction

Let $GF(q)$ be a field containing q elements, q odd. Let $\mathcal{G}$ denote $GL(2, GF(q))$, the group of non-singular two-by-two matrices with entries in $GF(q)$, and let G denote $SL(2, GF(q))$, the subgroup of $\mathcal{G}$ consisting of matrices with determinant one. In this paper, assuming a knowledge of certain of the characters of $\mathcal{G}$, we construct all the irreducible unitary representations of G. Our construction involves essentially no technique beyond the theory of induced representations and the orthogonality relations on a finite group. For a similarly elementary computation of the characters of $\mathcal{G}$ we refer the reader to [3]. In future papers we shall generalize the methods employed in this paper to construct the representations of the $n\times n$ matrix groups $GL(n, GF(q))$ and $SL(n, GF(q))$.

Kloosterman [2] was the first to describe all the irreducible matrix representations of $SL(2, GF(q))$. Weil in [5] generalizes and gives an alternative construction for Kloosterman's representations. In [4] Tanaka uses Weil's theory to construct representations and presents a complete and unified description of the representations of G. We also mention the paper [1] of Gelfand-Graev, which classifies but does not detail the actual construction of all the representations of G.

2. The representations

Let B be the upper unipotent, D the diagonal, and T the upper triangular subgroups of G. Then $T=DB$. G has order $q(q^2-1)$ and contains an abelian subgroup R (unique up to conjugacy) of order $q+1$. Except for plus-or-minus the identity of G elements of R have characteristic roots in $GF(q^2)-GF(q)$. R is isomorphic to the subgroup of $GF(q^2)^\times$ comprised of elements of norm one.

The $q+4$ equivalence classes of irreducible representations of G break up roughly into two main classifications. The $\frac{1}{2}(q+5)$ representations of the

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"principal series" all contain B-invariant vectors. Those $\frac{1}{2}(q+3)$ inequivalent representations which do not contain B-invariant vectors we call discrete series.

More precisely, the principal series include:

(1) The trivial representation of degree 1, $U\equiv 1$;

(2) A q-dimensional representation U_1^1 which occurs with $U\equiv 1$ in the induced representation $\underset{T\uparrow G}{\text{ind}}\, 1$;

(3) $\frac{1}{2}(q-3)$ irreducible induced representations $U^{\alpha}=\underset{T\uparrow G}{\text{ind}}\,\alpha$, where α is a one-dimensional representation of T which is not real-valued. U^{α} has degree $q+1$ and $U^{\alpha'}$ is equivalent to U^{α} if and only if $\alpha'=\alpha$ or α^{-1}.

(4) Let $\alpha=\text{sgn}$, where $\text{sgn}\not\equiv 1$ and $\text{sgn}^2\equiv 1$. Then $\underset{T\uparrow G}{\text{ind}}\,\text{sgn}=U^{\text{sgn}}=U_1^{\text{sgn}}+U_2^{\text{sgn}}$, the direct sum of two inequivalent irreducible representations, each of degree $\frac{1}{2}(q+1)$.

The discrete series are as follows:

(5) If π is a non-trivial character of R, then there is a representation U^{π} of G of degree $q-1$ associated with π. U^{π} is characterized by the fact that it does *not* occur in $\underset{R\uparrow G}{\text{ind}}\,\pi$. U^{π} is irreducible if and only if π is not real-valued. U^{π} is equivalent to $U^{\pi'}$ if and only if $\pi'=\pi$ or π^{-1}, so there are $\frac{1}{2}(q-1)$ inequivalent irreducible representations of degree $q-1$.

(6) If $\pi\not\equiv 1$, $\pi^2\equiv 1$, then $U^{\pi}=U_1^{\pi}+U_2^{\pi}$, the direct sum of inequivalent representations of degree $\frac{1}{2}(q-1)$.

3. The construction of principal series

The construction of the representations of the principal series as induced representations is well-known. For completeness we discuss this problem in detail.

Let α be a one-dimensional representation of T. Since B is the commutator subgroup of T, $\alpha(btb')=\alpha(t)$ for any b and $b'\in B$ and $t\in T$. T/B is canonically D, so α is the extension to T of a character of the abelian group D. The mapping which identifies $d\in D$ with its upper diagonal entry regarded as an element of the multiplicative group $GF(q)^{\times}$ is an isomorphism. In this section, when convenient, we regard α as a function on $GF(q)^{\times}$ via this identification. Let U^{α} denote the representation of G induced from α.

By the definition of U^{α}, G acts by right translation in the space V^{α} which consists of complex-valued functions ψ on G satisfying

$$\psi(tg)=\alpha(t)\psi(g) \tag{3.1}$$

for all $t\in T$ and $g\in G$. Any such function is determined by its restriction to a set of representatives of $T\backslash G$. Since two matrices in G with the same lower entries differ only by a left factor in B, $\psi\in V^{\alpha}$ implies $\psi(g)=\psi(g_{21}, g_{22})$, g_{21} and

g_{22} the lower entries of $g \in G$. Equation (3.1) entails

$$\psi(d^{-1}g_{21}, d^{-1}g_{22}) = \alpha(d)\psi(g_{21}, g_{22}) \tag{3.2}$$

for $d \in GF(q)^\times$, g_{21} and g_{22} as before, so ψ is actually determined by its values, which may be chosen arbitrarily, on a set of representatives for the projective line over $GF(q)$.

Theorem 3.1. *Let α be a one-dimensional representation of T. Let U^α be the representation of G induced from α. U^α is right translation in the space V^α defined by relations* (3.1) *and* (3.2).

(1) *The degree of U^α is $q+1$.*

(2) *U^α is irreducible if and only if $\alpha^2 \not\equiv 1$.*

(3) *$U^{\alpha'}$ is equivalent to U^α if and only if $\alpha' = \alpha$ or α^{-1}.*

(4) *U^1 decomposes into the direct sum of an irreducible representation of degree q and the unique one-dimensional representation of G.*

(5) *U^{sgn}, where* $\mathrm{sgn} \not\equiv 1$ *but* $\mathrm{sgn}^2 \equiv 1$, *decomposes into the direct sum of two inequivalent representations of degree $\frac{1}{2}(q+1)$.*

Proof.

(1) A set of representatives for the projective line over $GF(q)$ (e.g. $\{(0, 1), (-1, z) \mid z \in GF(q)\}$) has cardinality $q+1$. In view of the above remarks this proves that V^α has dimension $q+1$.

(2) The proofs of the remaining parts of this theorem depend upon an analysis of the commuting algebra of U^α.

Let C^α be the convolution algebra of all complex-valued functions f on G satisfying $f(tgt') = \alpha(tt')f(g)$ for any $t, t' \in T$ and $g \in G$. Then $U^\alpha(g_0)(f*\psi) = f*U^\alpha(g_0)\psi$ for any $g_0 \in G$ and $\psi \in V^\alpha$, since $f \in C^\alpha$ acting from the left by convolution keeps V^α stable and commutes with right translation. Frobenius' reciprocity theorem says precisely that C^α is large enough to be the full commuting algebra of U^α.

$f \in C^\alpha$ is determined by its values on a set of representatives for the double cosets $T\backslash G/T$, e.g. $\left\{e = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, w = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}\right\}$. Clearly, $\dim C^\alpha \leq 2$. $\operatorname{Dim} C^\alpha = 2$ if and only if $f(w) \neq 0$ for some $f \in C^\alpha$, if and only if $\alpha(t)f(w) = f(tw) = f(wt^{-1}) = \alpha^{-1}(t)f(w)$ for all $t \in D$. Thus $\dim C^\alpha = 2$ if and only if $\alpha^2(t) \equiv 1$, so (2) is true.

(3) The space of intertwining operators between V^α and $V^{\alpha'}$, $\alpha \neq \alpha'$, is canonically the vector space $T^{\alpha,\alpha'}$ of complex-valued functions on G satisfying $f(tgt') = \alpha(t)f(g)\alpha'(t')$ for all $t, t' \in T$ and $g \in G$. It is spanned by any function f which satisfies $\alpha(t)f(w) = f(w)\alpha'(t^{-1})$ for all $t \in D$. $f(w) \neq 0$ implies $\alpha' = \alpha^{-1}$.

(4) V^1 contains the constant functions on G as a stable subspace. The orthogonal complement of this one dimensional module must be an irreducible q-dimensional representation space for G.

(5) By the analysis in (2) we know that U^{sgn} decomposes into the direct sum of two inequivalent representations, $U_1^{\mathrm{sgn}}+U_2^{\mathrm{sgn}}=U^{\mathrm{sgn}}$. By Frobenius' reciprocity theorem $\operatorname{res}_{G\downarrow T} U_\nu^{\mathrm{sgn}}$, for $\nu=1$ or 2, contains sgn and no other one-dimensional representation of T. Since $G/\{\pm e\}$ is a simple group, G has no non-trivial one-dimensional representations. Therefore, Lemma (4.3) implies that the degree of U_ν^{sgn} is $\frac{1}{2}(q+1)$, $\nu=1$ or 2.

REMARK. To complete our description of the representations of the principal series we need to be more specific about the G-stable subspaces V_1^{sgn} and V_2^{sgn} of V^{sgn}. Set $\phi(-1, z)=\Phi(z)$ for $z\in GF(q)$, where Φ is an additive character of $GF(q)$; let $\phi(0, 1)=0$. Then ϕ extends uniquely to a function in V^{sgn} and $U^{\mathrm{sgn}}(b(u))\phi=\Phi(u)\phi$, where u is the super diagonal entry of $b(u)\in B$. Moreover, $U^{\mathrm{sgn}}(d)\phi(-1, z)=\operatorname{sgn}(d)\phi(-1, d^{-2}z)=\operatorname{sgn}(d)\Phi(d^{-2}z)$ for all $z\in GF(q)$, $d\in D$ (identified with $GF(q)^\times$); $U^{\mathrm{sgn}}(d)\phi(0, 1)=0$. Let $\Phi\not\equiv 1$. Then the $\frac{1}{2}(q-1)$ functions ϕ' which correspond to characters Φ' such that $\Phi'(d^{-2}z)=\Phi(z)$ for some $d\in GF(q)^\times$ belong to V_ν^{sgn}; the other non-trivial additive characters of $GF(q)$ must correspond to elements of $V_{\nu'}^{\mathrm{sgn}}$, $1\leq\nu\neq\nu'\leq 2$. V_ν^{sgn} also contains a vector ψ satisfying $\psi(tgt')=\operatorname{sgn}(tt')\psi(g)$ for all $t, t'\in T$, $g\in G$. In fact ψ may be chosen to be an idempotent in C^{sgn}.

Proposition 3.2. *Set*

$$\psi(g)=\frac{1}{2}\frac{|G|}{|T|}\operatorname{sgn}(t)\,, \qquad \text{if} \quad g=t\in T\,;$$
$$=\frac{1}{2}\frac{|G|}{|T|}(q\operatorname{sgn}(-1))^{-1/2}\operatorname{sgn}(t)\,,$$

if $g=twb$, with $t\in T$, $b\in B$, and $w=\begin{Vmatrix} 0 & 1\\ -1 & 0\end{Vmatrix}$.
Then ψ is an idempotent in the algebra C^{sgn}. There are two choices ψ, ψ' depending on the sign of $[\operatorname{sgn}(-1)]^{1/2}$. Clearly, $\psi+\psi'$ is the identity in C^{sgn}. The function ϕ defined in the preceding remark and corresponding to the non-trivial character Φ of $GF(q)$ belongs to the same G-irreducible subspace of V^{sgn} as ψ if and only if $\sum_{x\neq 0}\operatorname{sgn}(x)\Phi(-x)=[q\operatorname{sgn}(-1)]^{1/2}$ (with the same choice for the sign of the right hand side as in the definition of ψ).

Proof. To show that ψ is an idempotent in C^{sgn} it suffices to show that $\psi*\psi(e)=\psi(e)$ and $\psi*\psi(w)=\psi(w)$. We have

$$\psi*\psi(g)=\frac{1}{|G|}\sum_{x\in G}\psi(x)\psi(x^{-1}g)=\frac{|T|}{|G|}\sum_{x\in G/T}\psi(x)\psi(x^{-1}g)$$
$$=\frac{|T|}{|G|}\{\psi(e)\psi(g)+\sum_{u\in GF(q)}\psi(w)\psi(w^{-1}b^{-1}(u)g)\}\,.$$

Therefore,

$$\psi*\psi(e) = \frac{|T|}{|G|}\cdot\frac{1}{4}\frac{|G|^2}{|T|^2}\{1+[q\,\mathrm{sgn}(-1)]^{-1}\mathrm{sgn}(-1)q\}$$
$$= \psi(e)\,.$$
$$\psi*\psi(w) = \frac{|T|}{|G|}\cdot\frac{1}{4}\frac{|G|^2}{|T|^2}\{[q\,\mathrm{sgn}(-1)]^{-1/2}+[q\,\mathrm{sgn}(-1)]^{-1/2}$$
$$+[q\,\mathrm{sgn}(-1)]^{-1}\sum_{\substack{u\in GF(q)\\ u\neq 0}}\mathrm{sgn}(-u)\}\,.$$

The last term on the right, being a character sum, is zero. It arises from the relation

$$* \qquad w^{-1}b^{-1}(u)w = \begin{Vmatrix}0 & -1\\ 1 & 0\end{Vmatrix}\begin{Vmatrix}1 & -u\\ 0 & 1\end{Vmatrix}\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}$$
$$= \begin{Vmatrix}-u^{-1} & -1\\ 0 & -u\end{Vmatrix}\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}\begin{Vmatrix}1 & u^{-1}\\ 0 & 1\end{Vmatrix}, \quad \text{if} \quad u\neq 0\,.$$

Thus, $\psi*\psi(w)=\psi(w)$.

Finally, since ψ is a minimal idempotent in C^{sgn}, $\psi*\phi=\phi$, if $\psi\in V_\nu^{\mathrm{sgn}}$ and $\phi\in V_\nu^{\mathrm{sgn}}$. If $\phi\notin V_\nu^{\mathrm{sgn}}$, then $\phi\in V_{\nu'}^{\mathrm{sgn}}$, so $\psi*\phi=0$.

$$\psi*\phi(w) = \frac{|T|}{|G|}\sum_{x\in G/T}\psi(x)\phi(x^{-1}w)$$
$$= \frac{|T|}{|G|}\{\psi(e)\phi(w)+\psi(w)\sum_{u\in GF(q)}\phi(w^{-1}b^{-1}(u)w)\}\,.$$

Using relation $*$ as well as the definitions of ψ and ϕ, we obtain

$$\psi*\phi(w) = \phi(w)\left\{\frac{1}{2}+\frac{1}{2}[q\,\mathrm{sgn}(-1)]^{-1/2}\sum_{u\neq 0}\mathrm{sgn}(-u)\Phi(u)\right\},$$

which implies the last part of the proposition.

4. The construction of discrete series for $GL(2, GF(q))$

Let Π be a character of $GF(q^2)^\times$ whose restriction to the elements of norm one is non-trivial. Then Π corresponds to a representation $\mathcal{U}^\Pi$ of the discrete series of $\mathcal{G}=GL(2, GF(q))$ (i.e. $\mathrm{res}_{\mathcal{G}\downarrow B}\,\mathcal{U}^\Pi \not\ni 1$). It turns out that $\mathrm{res}_{\mathcal{G}\downarrow\mathcal{T}}\,\mathcal{U}^\Pi$ is an irreducible representation of $\mathcal{T}$, the triangle subgroup of $\mathcal{G}$. To determine a space of functions which transforms under $\mathcal{G}$ as $\mathcal{U}^\Pi$ we find an irreducible representation m of $\mathcal{T}$ such that $m=\mathrm{res}_{\mathcal{G}\downarrow\mathcal{T}}\,\mathcal{U}^\Pi$. Then, using the trace of $\mathcal{U}^\Pi$ (which we assume known) we extend the matrix coefficients of m to $\mathcal{G}$. To determine the discrete series of G we study $\mathrm{res}_{\mathcal{G}\downarrow G}\,\mathcal{U}^\Pi$.

Let $\mathcal{D}$ be the diagonal subgroup of $\mathcal{G}$ and let α be a character of $\mathcal{D}$. $\mathrm{Ind}_{\mathcal{D}\uparrow\mathcal{T}}\,\alpha = M^\alpha$ is right translation in the space of complex-valued functions on $\mathcal{T}$

which satisfy $\psi(dt)=\alpha(d)\psi(t)$ for all $d\in\mathcal{D}$ and $t\in\mathcal{T}$. Since B represents $\mathcal{D}\backslash\mathcal{T}$, we may consider M^α as acting in a vector space B^α of complex-valued functions on B. We write $\psi\in B^\alpha$ as a function of the super diagonal entries of elements of B. Then

$$M^\alpha(db(u))\psi(x) = \alpha(d)\psi(d_{11}^{-1}d_{22}x+u) \tag{4.1}$$

for any $d\in\mathcal{D}$ and $b(u)$ the element of B with superdiagonal entry $u\in GF(q)$, d_{11} and d_{22} the non-zero entries of d.

To see how M^α decomposes take as an orthonormal basis of B^α the q characters of B. The operators $M^\alpha(b)$ for $b\in B$ obviously diagonalize with respect to this basis. Let Φ_0 be the trivial character of B. Clearly Φ_0 transforms under M^α as the one-dimensional representation α of $\mathcal{T}$. Now let Φ be a fixed non-trivial character of B. For $i\in GF(q)^\times$ set $\Phi_i(x)=\Phi(ix)$ for all $x\in GF(q)$. Then Φ_i is a non-trivial character of B and every non-trivial character of B is of the form Φ_i for some $i\in GF(q)^\times$. (4.1) entails that, except for scalar factors, $\mathcal{D}$ acts transitively on the non-trivial characters of B. Since M^α is completely reducible, we see that the $(q-1)$-dimensional subspace of B^α spanned by the non-trivial characters of B must be irreducible. Call the resulting representation m_α.

Lemma 4.1. *An irreducible representation of $\mathcal{T}$ is either of degree one or $q-1$. An irreducible $(q-1)$-dimensional representation of $\mathcal{T}$ is determined by its restriction to the center of $\mathcal{T}$.*

Proof. If an irreducible representation of $\mathcal{T}$ is not one-dimensional, it is equivalent to a representation m_α for some character α of $\mathcal{D}$. Thus it is $(q-1)$-dimensional. By Frobenius' reciprocity theorem characters α' which occur in $\operatorname{res}_{\mathcal{T}\downarrow\mathcal{D}} m_\alpha$ occur with multiplicity one. Since m_α is irreducible, every α' contained in $\operatorname{res}_{\mathcal{T}\downarrow\mathcal{D}} m_\alpha$ must have the same values on the center of $\mathcal{T}$ (i.e. the scalars). There are $q-1$ distinct characters of $\mathcal{D}$ which agree on the scalars, so they must all occur in $\operatorname{res}_{\mathcal{T}\downarrow\mathcal{D}} m_\alpha$. By Frobenius' theorem, m_α is equivalent to $m_{\alpha'}$, for all such α'.

Lemma 4.2. *Let Φ be a non-trivial character of B. For $i\in GF(q)^x$ set $\Phi_i(x)=\Phi(ix)$ for all $x\in GF(q)$ (considered as super-diagonal entries of elements of B). The matrix coefficients of the representation m_α with respect to the basis for B^α consisting of the $q-1$ non-trivial characters $\{\Phi_i\}_{i\in GF(q)^\times}$ of B are the $(q-1)^2$ functions*

$$\begin{aligned} m_{ij}^\alpha(t) &= \langle m_\alpha(t)\Phi_j, \Phi_i\rangle, \quad i \text{ and } j\in GF(q)^\times, \\ &= \alpha(d)\Phi_j(u), \quad \text{if } \Phi_j(d_{11}^{-1}d_{22}x)=\Phi_i(x) \text{ for all } x\in GF(q); \\ &= 0, \quad \text{otherwise.} \end{aligned} \tag{4.2}$$

In (4.2) $t=db(u)$, *where* $u\in GF(q)$ *is the super-diagonal entry of the matrix* $b(u)\in B$ *and* d_{11} *and* d_{22} *are the diagonal entries of* $d\in\mathcal{D}$.

Proof. Immediate from equation (4.1).

Lemma 4.3. *Let* m_α *be an irreducible representation of* $\mathcal{T}$ *of degree* $q-1$. *Then* $\operatorname{res}_{\mathcal{T}\downarrow T} m_\alpha$ *decomposes into inequivalent representations of degree* $\frac{1}{2}(q-1)$. *Any irreducible representation of* T *is either one-dimensional or* $\frac{1}{2}(q-1)$*-dimensional.*

Proof. $\operatorname{Res}_{\mathcal{T}\downarrow B} m_\alpha$ decomposes simply; if $\operatorname{res}_{\mathcal{T}\downarrow T} m_\alpha$ decomposes, the component representations must be inequivalent. By (4.1) $M^\alpha(d)\Phi(x)=\alpha(d)\Phi(d_{11}^{-2}x)$ for $d\in D$, so two characters Φ and Φ' of B occur in the restriction to B of the same irreducible subrepresentation of $\operatorname{res}_{\mathcal{T}\downarrow T} m_\alpha$ if and only if $\Phi'(x)=\Phi(a^2x)$ for some $a\in GF(q)^\times$ and all $x\in GF(q)$. Since half the characters of B satisfy this relation and half do not, $\operatorname{res}_{\mathcal{T}\downarrow T} m_\alpha$ contains two irreducible representations, each of degree $\frac{1}{2}(q-1)$. The last statement in Lemma (4.3) follows from the fact that any irreducible representation of T occurs in the restriction to T of some irreducible representation of $\mathcal{T}$.

Lemma 4.4. *Let* G *be a finite group and* H *a subgroup of* G. *Let* U *be a unitary representation of* G *whose degree is* d *and character is* X. *Assume* $\operatorname{res}_{G\downarrow H} U$ *is irreducible. Then, for any matrix coefficient* u_{ij} *of* U, $1\le i,j\le d$, *and any* $g\in G$

$$u_{ij}(g)=\frac{d}{|H|}\sum_{h\in H}u_{ij}(h)X(h^{-1}g)\,.$$

Proof.

$$\begin{aligned}\frac{d}{|H|}\sum_{h\in H}u_{ij}(h)X(h^{-1}g)&=\frac{d}{|H|}\sum_{h\in H}u_{ij}(h)\sum_{k=1}^{d}u_{kk}(h^{-1}g)\\&=\frac{d}{|H|}\sum_{h\in H}u_{ij}(h)\sum_{k=1}^{d}\sum_{l=1}^{d}u_{kl}(h^{-1})u_{lk}(g)\\&=\frac{d}{|H|}\sum_{l,k}u_{lk}(g)\sum_{h\in H}u_{ij}(h)\bar{u}_{lk}(h)\\&=u_{ij}(g)\,,\end{aligned}$$

by Schur's orthogonality relations on G.

Lemma (4.1) implies that for any representation $\mathcal{U}^{\text{II}}$ of the discrete series of $\mathcal{G}$, $\operatorname{res}_{\mathcal{G}\downarrow\mathcal{T}}\mathcal{U}^{\text{II}}$ is equivalent to an irreducible representation m_α, where m_α is, up to equivalence, the unique irreducible $(q-1)$-dimensional representation of $\mathcal{T}$ which agrees with $\mathcal{U}^{\text{II}}$ on the scalars. Since $\mathcal{G}=\mathcal{T}\cup\mathcal{T}wB$, $w=\begin{Vmatrix}0&1\\-1&0\end{Vmatrix}$, it suffices to compute the matrix coefficients for $\mathcal{U}^{\Pi}$ at w in order to extend them from $\mathcal{T}$ to all of $\mathcal{G}$. For this purpose we need the character X^{Π} of $\mathcal{U}^{\Pi}$ (To find directions for the easy computation of X^{Π} consult [3], p. 227.). Figure 1 presents X^{Π}.

Conjugacy Classes on G	Values of X^{Π}
$\lambda\begin{Vmatrix}1 & 0\\ 0 & 1\end{Vmatrix}$	$\Pi(\lambda)(q-1)$
$\lambda\begin{Vmatrix}1 & 0\\ 1 & 1\end{Vmatrix}$	$-\Pi(\lambda)$
$\lambda\begin{Vmatrix}t & 0\\ 0 & 1\end{Vmatrix}^*$	0
$\begin{Vmatrix}\varepsilon^q & 0\\ 0 & \varepsilon\end{Vmatrix} \sim \begin{Vmatrix}\alpha & \beta\\ \beta\zeta & \alpha\end{Vmatrix}^*$	$-(\Pi(\varepsilon)+\Pi(\varepsilon^q))$

t, $\lambda\in GF(q)^{x}$, $t\neq 1$; $\varepsilon=\alpha+\beta\sqrt{\zeta}$, $\alpha, \beta, \zeta\in GF(q)$ with ζ not a square and $\beta\neq 0$. * Matrices with the same characteristic roots are conjugate.

Figure 1.

Lemma 4.5. *Let X^{Π} be the character of a representation $\mathcal{U}^{\Pi}$ of the discrete series of $\mathcal{G}$. Let α be a character of $\mathcal{D}$ such that $\alpha(\lambda)=\Pi(\lambda)$ for any scalar matrix $\lambda\in\mathcal{D}$. Then m_α is equivalent to $\operatorname{res}_{\mathcal{S}\downarrow\mathcal{T}}\mathcal{U}^{\Pi}$. Fix a non-trivial character Φ of B. Let $\{m^{\alpha}_{ij}\}_{i,j\in GF(q)^\times}$ be the matrix coefficients of m_α with respect to the basis $\{\Phi_i\}_{i\in GF(q)^x}$ of B^α (see Lemma (4.2) and relation (4.2)). The matrix coefficients m^{α}_{ij} are the restrictions to $\mathcal{T}$ of matrix coefficients $u^{\Pi;\alpha}_{ij}$ of $\mathcal{U}^{\Pi}$. For $w=\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix}$ and $\delta(a, b)$ the diagonal matrix with diagonal entries $\delta_{11}=a$ and $\delta_{22}=b$,*

$$u^{\Pi;\alpha}_{ij}(w) = -\alpha^{-1}(\delta(i, j))\,q^{-1}\sum_{\varepsilon:\varepsilon\varepsilon^q=ij}\Pi(\varepsilon)\Phi(\varepsilon+\varepsilon^q)\,. \tag{4.3}$$

Proof. By Lemma (4.4) and relation (4.2)

$$\begin{aligned} u^{\Pi;\alpha}_{ij}(w) &= \frac{(q-1)}{|\mathcal{T}|}\sum_{t\in\mathcal{T}} m^{\alpha}_{ij}(t)X^{\Pi}(t^{-1}w)\\ &= q^{-1}\sum_{u\in GF(q)}\alpha(\delta)\Phi_j(u)X^{\Pi}(b^{-1}(u)\delta^{-1}w)\,, \end{aligned}$$

where $\delta=\delta(j, i)$ and $b(u)\in B$ has super diagonal entry u. Use of the explicit formula for X^{Π} easily yields (4.3).

Theorem 4.6. *Let Π be a character of $GF(q^2)^\times$ whose restriction to the elements of norm one is not trivial. Let X^{Π} be the character of the irreducible representation of $\mathcal{G}$ associated with Π. Let α be any character of $\mathcal{D}$ which agrees* *with* Π *on the scalar matrices. Then* m_α *is* $\operatorname{res}_{\mathcal{S}\downarrow\mathcal{T}} \mathcal{U}^{\Pi}$ *and* B^α, *the representation space of* m_α, *is a representation space for* $\mathcal{U}^{\Pi}$. *Fix a non-trivial character* Φ *of* B *and write it as a function of the super-diagonal entries of elements of* B. *Take as a basis for* B^α *the* $q-1$ *non-trivial characters* $\{\Phi_i\}_{i\in GF(q)^\times}$, *where* $\Phi_i(x)=\Phi(ix)$ *for all* $x\in GF(q)$. *Matrix coefficients for* $\mathcal{U}^{\Pi}$ *acting in* B^α *are as follows. For* $i, j\in GF(q)^\times$ *set* $u_{ij}^{\Pi;\alpha}=\langle \mathcal{U}^{\Pi}(g)\Phi_j, \Phi_i\rangle$. *If* $g=db(u)$, *where* $d\in\mathcal{D}$ *has diagonal entries* d_{11} *and* d_{22} *and* $b(u)\in B$ *has super-diagonal entry* $u\in GF(q)$, *then*

$$(4.4)\qquad u_{ij}^{\Pi;\alpha}(g) = \alpha(d)\Phi_j(u)\,, \quad \textit{provided } d_{11}^{-1}d_{22}=j^{-1}i; \qquad = 0\,, \quad \textit{otherwise.}$$

If $g=b(v)wdb(u)$, *where* $d\in\mathcal{D}$ *has diagonal entries* d_{11} *and* d_{22}, $w=\begin{Vmatrix} 0 & 1 \\ -1 & 0\end{Vmatrix}$, *and* $b(u)$ *and* $b(v)\in B$ *have superdiagonal entries* u *and* v *respectively then*

$$(4.5)\qquad u_{ij}^{\Pi;\alpha}(g) = \Phi(iv+ju)[-\Pi(d_{11})\alpha^{-1}(\delta(i,j))q^{-1}\sum_{\varepsilon:\varepsilon\varepsilon^q=l}\Pi(\varepsilon)\Phi(\varepsilon+\varepsilon^q)]$$

where $\delta(i,j)$ *is the diagonal matrix with upper entry* i *and lower entry* j *and* $l=ijd_{11}^{-1}d_{22}$.

Proof. Relation (4.4) is the same as (4.2), so no proof is needed. To prove (4.5) note first that $u_{ij}^{\Pi;\alpha}(b(v)gb(u))=\Phi_i(v)u_{ij}^{\Pi;\alpha}(g)\Phi_j(u)$. Moreover, $u_{ij}^{\Pi;\alpha}(wd)=\alpha(d)u_{i,jd_{11}^{-1}d_{22}}^{\Pi,\alpha}(w)$. Use of (4.3) to express $u_{i,jd_{11}^{-1}d_{22}}^{\Pi,\alpha}(w)$ as an exponential sum leads to a proof of (4.5).

5. Discrete series of G

Let Π be a character of $GF(q^2)^\times$ whose restriction to N^1, the elements of norm one in $GF(q^2)^\times$, is not trivial. Let π be Π restricted to N^1. Let $\mathcal{U}^{\Pi}$ be the representation of the discrete series of $\mathcal{G}$ associated with Π. Set $U^\pi=\operatorname{res}_{\mathcal{G}\downarrow G}\mathcal{U}^{\Pi}$. The trace X^π of U^π is the restriction to G of X^{Π}, so, up to equivalence, U^π depends only on the values of Π restricted to N^1. Furthermore, U^π and $U^{\pi'}$ are equivalent if and only if $\pi'=\pi$ or π^{-1}, since, if π' is the restriction to N^1 of a character Π' of $GF(q^2)^\times$, $X^\pi=X^{\pi'}$ if and only if $\pi'=\pi$ or π^{-1}.

We may take as representatives for the conjugacy classes in G those representatives for conjugacy classes in $\mathcal{G}$ which lie in G (see Figure 1.). However, $\begin{Vmatrix}1 & 0\\ 1 & 1\end{Vmatrix}$ and $\begin{Vmatrix}1 & 0\\ \zeta & 1\end{Vmatrix}$, ζ a non-square, are not conjugate in G; similarly $-\begin{Vmatrix}1 & 0\\ 1 & 1\end{Vmatrix}$ and $-\begin{Vmatrix}1 & 0\\ \zeta & 1\end{Vmatrix}$.

Theorem 5.1. *Let* π *be a non-trivial character of* N^1 *and let* U^π *be the corresponding representation of* G *defined above.* U^π *is irreducible if and only if* $\pi^2\not\equiv 1$. *If* $\pi^2\equiv 1$, $U^\pi=U_1^\pi+U_2^\pi$, *the direct sum of inequivalent* $\frac{1}{2}(q-1)$-*dimensional representations.*

Proof. It suffices to show that $|G|^{-1}\sum_{g\in G}|X^{\pi}(g)|^2=1$, if $\pi^2\not\equiv 1$, and 2, otherwise. The computation is easy and we omit it. In the case that U^{π} is reducible, the components are $\frac{1}{2}(q-1)$-dimensional and inequivalent, since, according to Lemma (4.3), this statement holds already for $\underset{G\downarrow T}{\text{res}}\, U^{\pi}$. We may use Lemma (4.3) to obtain representation spaces for U_1^{π} and U_2^{π}.

There are $q+4$ conjugacy classes in G and we have accounted for this many equivalence classes of irreducible representations, so our description of the irreducible representations of $SL(2, GF(q))$ is complete.

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