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ON MULTIPLY TRANSITIVE GROUPS VIII

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1. Introduction

Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$, and let P be a Sylow 2-subgroup of a stabilizer of four points in G. By a theorem of M. Hall [2, Theorem 5.8.1] and a lemma of E. Witt [7, Theorem 9.4], we have that P fixes exactly four, five, six, seven or eleven points and the normalizer of P in G restricted on the set of the fixed points of P is S_4, S_5, A_6, A_7 or M_{11}. (cf. H. Nagao and T. Oyama [5], Lemma 1).

The purpose of this paper is to prove the following

Theorem. *Let G be a 4-fold transitive group. If a Sylow 2-subgroup of a stabilizer of four points in G fixes exactly six points, then G must be A_6.*

The above theorem of M. Hall is that if a stabilizer of four points in G is of odd order then G must be one of the following groups: S_4, S_5, A_6, A_7 or M_{11}. Therefore to prove our theorem we may assume that a Sylow 2-subgroup of a stabilizer of four points in G is not identity.

2. Definitions and notations

A permutation x is called semi-regular if there is no point fixed by x. A permutation group G is called semi-regular if every non-identity element of G is semi-regular on the points actually moved by G.

For a permutation group G on Ω the subgroup of G consisting of all the elements fixing the points $i, j, \cdots, k$ of Ω will be denoted by $G_{ij\cdots k}$, which we shall call the stabilizer of the points $i, j, \cdots, k$. The totality of points left fixed by a subset X of G will be denoted by $I(X)$, and if a subset Δ of Ω is a fixed block of X, then the restriction of X on Δ will be denoted by X^{Δ}. A G-orbit of minimal length is called a minimal orbit of G.

For subsets X, Y of a group G, $\langle X, Y\rangle$ is the subgroup of G generated by the elements of X and Y, and $N_G(X)$ is the normalizer of X in G.

3. Proof of the theorem

In the following two lemmas we assume that G is a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$, and P is a Sylow 2-subgroup of G_{1234}. For a point t of a minimal orbit of P in $\Omega\text{-}I(P)$ let $P_t=Q$, $N_G(Q)=N$ and $I(Q)=\Delta$.

Lemma 1. *Let R be a Sylow 2-subgroup of N_{ijkl} for $\{i, j, k, l\}\subseteqq\Delta$. Then R^Δ, which is a Sylow 2-subgroup of $(N^\Delta)_{ijkl}$, is semi-regular and $|I(R)|=|I(P)|$.*

Proof. Let φ be a natural homomorphism of N_{ijkl} onto $(N^\Delta)_{ijkl}$. Then $\varphi(R)=R^\Delta$. Since R is a Sylow 2-subgroup of N_{ijkl}, R^Δ is a Sylow 2-subgroup of $(N^\Delta)_{ijkl}$.

Let P' be a Sylow 2-subgroup of G_{ijkl} containing R. Since Q is a normal 2-subgroup of N_{ijkl} and R is a Sylow 2-subgroup of N_{ijkl}, $P'\geqq R\geqq Q$ and $I(P')\subseteqq I(R)\subseteqq\Delta$. Since G is 4-fold transitive, P and P' are conjugate, and $|I(P)|=|I(P')|$. Therefore $I(P')$ is a proper subset of Δ. For any point r of $\Delta\text{-}I(P')$, $P'_r\geqq Q=P_t$. From the assumption that t belongs to a minimal P-orbit we have

$$|P': P'_r|=|r^{P'}|\geqq|t^P|=|P: P_t|=|P': Q|.$$

Therefore $P'_r=Q$ and $R_r=Q$. Thus R^Δ is identity or semi-regular and $I(R^\Delta)=I(P')$. Since $\Delta=I(Q)\supsetneqq I(P')$ and $P'_\Delta=Q$, $N_{P'}(Q)^\Delta$ is a non-identity 2-group of $(N^\Delta)_{ijkl}$. Therefore R^Δ, which is a Sylow 2-subgroup of $(N^\Delta)_{ijkl}$, is semi-regular and $|I(R)|=|I(R^\Delta)|=|I(P')|=|I(P)|$.

Lemma 2. *In Lemma 1 if $|I(P)|=6$, that is $N_G(P)^{I(P)}=A_6$, then R^Δ is an elementary abelian group, $N_N(R)^{I(R)}\leqq A_6$ and $|\Delta|\geqq 10$.*

Proof. If R has an element $x=(i'\ j'\ k'\ l')\cdots$, where $\{i', j', k', l'\}\subset\Delta$, then x^Δ has no 2-cycle by Lemma 1. On the other hand, x normalizes $G_{i'j'k'l'}$ and Q. Therefore x normalizes some Sylow 2-subgroup P' of $G_{i'j'k'l'}$ containing Q. Form the assumption $x^{I(P')}\in A_6$. Hence $x^{I(P')}$ has a 2-cycle. Since $I(P')\subset\Delta$, x^Δ has a 2-cycle, which is a contradiction. Thus R^Δ is elementary abelian.

From Lemma 1 $|I(R)|=6$. If $N_N(R)^{I(R)}\nleqq A_6$, then $N_N(R)$ has a 2-element x such that $x^{I(R)}$ is an odd permutation. On the other hand x normalizes some Sylow 2-subgroup P'' of $G_{I(R)}$. Since $G_{I(R)}$ contains a Sylow 2-subgroup of G_{ijkl}, P'' is a Sylow 2-subgroup of some stabilizer of four points in G, and $I(P'')=I(R)$. Then $x^{I(R)}=x^{I(P'')}\in N_G(P'')^{I(P'')}=A_6$, which is a contradiction. Thus $N_N(R)^{I(R)}\leqq A_6$.

Since $|I(R)|=6$ and $\Delta\supsetneqq I(R)$, $|\Delta|\geqq 8$. Suppose that $|\Delta|=8$. Let $\Delta=\{i, j, k, l, r, s, u, v\}$ and $I(R)=\{i, j, k, l, r, s\}$. Then R has the following 2-element

$$a=(i)\ (j)\ (k)\ (l)\ (r)\ (s)\ (u\ v)\ \cdots.$$

Since a normalizes G_{ijuv} and Q, a normalizes a Sylow 2-subgroup P''' of G_{ijuv} containing Q. It follows from $I(P''')\subset\Delta$ that $a^{I(P''')}$ is a transposition. This is a contradiction since $N_G(P'')^{I(P'')}=A_6$. Therefore $|\Delta|\geqq 10$.

From now on we consider a permutation group G on $\Omega=\{1, 2, \cdots, n\}$, which is not necessarily 4-fold transitive. In the following two lemmas we assume that G satisfies the following condition.

(*) *Let P be a Sylow 2-subgroup of any stabilizer of four points in G. Then*

(i) *P is semi-regular and elementary abelian,*

(ii) $|\Omega|\geqq 10$ *and* $|I(P)|=6$,

(iii) $N_G(P)^{I(P)}\leqq A_6$.

By Lemma 1 and Lemma 2, N^{Δ} satisfies the assumption (*). In Lemma 4 we shall show that there is no group satisfying (*). Thus if G is a 4-fold transitive group as in Theorem, then G_{1234} is of odd order, and hence G must be A_6.

Lemma 3. *G is a doubly transitive group and $N_G(P)^{I(P)}$ is A_6 or A_6^*, where A_6^* is a doubly transitive group of degree 6 isomorphic to A_5.*

Proof. For any two points i_1 and i_2 let P' be a Sylow 2-subgroup of $G_{i_1i_2i_3i_4}$, where $\{i_3, i_4\}\subset\Omega-\{i_1, i_2\}$. Then we have an involution a of P', which has the following form

$$a=(i_1)\,(i_2)\,(i_3)\,(i_4)\,(i_5)\,(i_6)\,(j_1\,j_2)\cdots.$$

where $I(P')=\{i_1, i_2, \cdots, i_6\}$. Since a normalizes $G_{i_1i_2j_1j_2}$, a normalizes a Sylow 2-subgroup P'' of $G_{i_1i_2j_1j_2}$. From (*iii*)

$$a^{I(P'')}=(i_1)\,(i_2)\,(j_1\,j_2)\,(j_3\,j_4)\,,$$

where $I(P'')=\{i_1, i_2, j_1, j_2, j_3, j_4\}$. Hence $\langle a, P''\rangle$ is a 2-group and fixes exactly two point i_1 and i_2. From a lemma of D. Livingstone and A. Wagner [4, Lemma 6] G is a doubly transitive group on Ω.

Since G is doubly transitive, for any two points u and v of $I(P)$ there is a Sylow 2-subgroup S of G_{uv}, which contains P and fixes only two points u and v. Let $T=N_S(P)$. Then $T\gneqq P$ and $I(S)\subseteqq I(T)\subseteqq I(P)$. Suppose that $I(S)\subsetneqq I(T)$, then $|I(T)|=4$ or 6. By (*iii*) $|I(T)|=6$. Therefore $I(T)=I(P)$. Since P is a Sylow 2-subgroup of $G_{I(P)}$, $T=P$, which is a contradiction. Therefore $I(T)=I(S)=\{u, v\}$. This shows that for any two points u, v of $I(P)$ $N_G(P)^{I(P)}$ contains a 2-group, which fixes only two points u and v. Using also Lemma 6 of [3] we have that $N_G(P)^{I(P)}$ is doubly transitive. Therefore, by assumption (*iii*), $N_G(P)^{I(P)}$ is either A_6 or A_6^*, where A_6^* is a doubly transitive group of degree 6 isomorphic to A_5 (see B. Huppert [3] II, 4.7 Satz).

To prove the next lemma we need the following result in [6]: Let G be a 4-fold transitive group. If a Sylow 2-subgroup P of a stabilizer of four points in G is semi-regular and not identity, then $|I(P)|\neq 6$.

Lemma 4. *There is no group satisfying* (*).

Proof. Assume that G satisfies (*). We may assume that P is a Sylow

2–subgroup of G_{1234} and $I(P)=\{1, 2, \cdots, 6\}$. In the case $N_G(P)^{I(P)}=A_6^*$, we may assume that A_6^* is generated by $\{(3\ 4)\ (5\ 6), (1\ 2\ 3)\ (4\ 5\ 6), (1\ 3\ 5\ 6\ 4)\}$ (see W. Burnside [1] §166).

For any point i of Ω-$\{1, 2, 3\}$ let P' be a Sylow 2–subgroup of G_{123i}. By (*ii*) P' fixes six points, say 1, 2, 3, i, j and k. Let Q be a Sylow 2–subgroup of G_{123} containing P'. Since $|\Omega|$ is even, $|I(Q)|$ is also even. Therefore Q fixes at least four points, and hence Q fixes six points by (*ii*), which are the points of $I(P')$. Thus $Q=P'$ is a Sylow 2–subgroup of G_{123}, and any point in $\Omega-\{1, 2, 3\}$ belongs to a G_{123}-orbit of odd length. On the other hand P and P' are conjugate in G_{123}, hence G_{123} has an element taking $\{4, 5, 6\}$ into $\{i, j, k\}$. Thus G_{123} has exactly one or three orbits in Ω–$\{1, 2, 3\}$. If G_{123} has three orbits in Ω–$\{1, 2, 3\}$, then three points 4, 5 and 6 belong to different G_{123}-orbits, say T_4, T_5 and T_6 respectively.

Suppose that G_{123} is transitive on Ω–$\{1, 2, 3\}$. Since a Sylow 2–subgroup of G_{12} fixes only two points 1 and 2, G_{12} has an element taking 3 into some point of Ω–$\{1, 2, 3\}$. Therefore G_{12} is transitive on Ω–$\{1, 2\}$. It follows from Lemma 3 that G is 4–fold transitive on Ω. But this contradicts the theorem in [6].

From now on we assume that G_{123} has three orbits T_4, T_5 and T_6 in Ω–$\{1, 2, 3\}$. Suppose that $N_G(P)^{I(P)}=A_6$. Then $N_G(P)$ contains an element x of the form

$$x = (1)\ (2)\ (3)\ (4\ 5\ 6) \cdots .$$

Since $x\in G_{123}$, 4, 5 and 6 belong to the same G_{123}-orbit, which is a contradiction.

Thus we have that $N_G(P')^{I(P')}=A_6^*$, for any Sylow 2–subgroup P' of an arbitrary stabilizer of four points in G.

Suppose that P has two involutions x and y. Since P is elementary abelian by (*i*), we may assume by (*ii*) that x and y are of the following forms

$$x = (1)\ (2) \cdots (6)\ (i\ j)\ (k\ l) \cdots ,$$
$$y = (1)\ (2) \cdots (6)\ (i\ k)\ (j\ l) \cdots .$$

$\langle x, y\rangle$ normalizes some Sylow 2–subgroup of G_{ijkl}. Hence the restriction of $\langle x, y\rangle$ on the set of the points fixed by this Sylow 2–subgroup is a four group and fixes two points. But a stabilizer of two points in A_6^* is of order 2, which is a contradiction. Therefore P is of order 2, and any Sylow 2–subgroup of a stabilizer of four points in G is also of order 2.

Let a be an involution of P. We may assume by (*ii*) that a is of the form

$$a = (1)\ (2) \cdots (6)\ (i\ j) \cdots .$$

Then a normalizes a Sylow 2–subgroup of G_{12ij}, and hence a commutes with some involution b in G_{12ij}. Since b fixes only six points, $b^{I(a)}$ is not identity. Since $N_G(P)^{I(P)}=A_6^*$, b must be of the form

$$b=(1)(2)(3\ 4)(5\ 6)(i)(j)(k)(l)\cdots,$$

and then a have a 2–cycle $(k\ l)$. If a and b have two 2–cycles $(i'\ j')\ (k'\ l')$ and $(i'\ k')\ (j'\ l')$ respectively, then $\langle a, b\rangle$ normalizes some Sylow 2–subgroup of $G_{i'j'k'l'}$. Using the same argument as above we have a contradiction. Therefore if a has 2–cycles in $\Omega-\{1, 2, \cdots, 6, i, j, k, l\}$, then b has the same 2–cycles. Since a commutes with b, ab is also an involution, and fixes two points 1 and 2. Therefore $|I(ab)|=2$ or 6 by (*ii*). If $|I(ab)|=2$, then

$$a=(1)(2)\cdots(6)(i\,j)(k\ l),$$
$$b=(1)(2)(3\ 4)(5\ 6)(i)(j)(k)(l),$$

and hence $|\Omega|=10$. If $|I(ab)|=6$, then

$$a=(1)(2)\cdots(6)(i\,j)(k\ l)(i'\ j')(k'\ l'),$$
$$b=(1)(2)(3\ 4)(5\ 6)(i)(j)(k)(l)(i'\ j')(k'\ l'),$$

and hence $|\Omega|=14$. On the other hand from the assumption that A_6^* has an element (1 2 3) (4 5 6), there is an element

$$c=(1\ 2\ 3)(4\ 5\ 6)\cdots$$

in $N_G(P)$. Then c normalizes G_{123}, and since the G_{123}-orbits in $\Omega-\{1, 2, 3\}$ are T_4, T_5 and T_6, c takes T_4 into T_5, T_5 into T_6 and T_6 into T_4. Therefore T_4, T_5 and T_6 are of the same length and $|\Omega-\{1, 2, 3\}|$ is divisible by 3. But $|\Omega|-3=7$ or 11, which is not divisible by 3. This contradiction arises from the first assumption that there is a group which satisfies the conditions (*i*), (*ii*) and (*iii*).

Now by Lemma 2 and 4 we have that, for a 4–fold transitive group G as in the theorem, the stabilizer of four points in G is of odd order. Therefore G must be A_6.

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