Takeuchi, M.
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NICE FUNCTIONS ON SYMMETRIC SPACES

Dedicated to Professor Atuo Komatu for his 60th birthday

MASARU TAKEUCHI

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Introduction

A smooth function f on a compact smooth manifold M is called a *Morse function* on M if the critical points of f are all non-degenerate. A Morse function f on M is called a *nice function* on M if

$$\textit{Index of } f \textit{ at } p = f(p) \quad \textit{for any cirtical point } p \textit{ of } f.$$

The existence of a nice function was proved by S. Smale and successfully used by him in solving the Poincaré conjecture (Smale [4]). For any Morse function f on M, the Morse inequality:

$$\textit{Number of critical points of } f \geqslant \dim H_*(M, \boldsymbol{K})$$

holds for any coefficient field $\boldsymbol{K}$. A Morse function f is called *economical* for $\boldsymbol{K}$ if the equality holds in the above Morse inequality for $\boldsymbol{K}$.

The purpose of the present note is to show that for a symmetric R-space M (For the definition of an R-space, see Section 1.) we have a nice function on M, which is also economical for $\boldsymbol{Z}_2$, by choosing a suitable spherical function on M.

Recently A. Hattori constructed as follows a nice function on the Grassmann manifold of m-subspaces of $(m+n)$-space $\boldsymbol{F}^{m+n}$ over $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$ or the algebra $\boldsymbol{H}$ of real quaternions: Let

$$x_j = \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{m+n\, j} \end{pmatrix} \in \boldsymbol{F}^{m+n} \qquad (1 \leqslant j \leqslant m)$$

be an orthonormal basis of an m-subspace x of $\boldsymbol{F}^{m+n}$ with respect to the standard metric $\sum \alpha_i \bar{\alpha}_i$ of $\boldsymbol{F}^{m+n}$, where $\alpha \mapsto \bar{\alpha}$ is the canonical involution of $\boldsymbol{F}$. We put

$$l_i = \sum_{j=1}^{m} x_{ij}\bar{x}_{ij} \qquad (1 \leqslant i \leqslant m+n)\,.$$

Then Hattori's nice function f is given by

$$f(x) = d\left\{\sum_{i=1}^{m+n} i\,l_i - \frac{m(m+1)}{2}\right\}, \quad d=\dim_{\boldsymbol{R}} \boldsymbol{F}.$$

The class of symmetric R-spaces includes the Grassmann manifolds and we can confirm that our spherical functions for them are nothing but Hattori's nice functions.

In addition we shall show that for an R-space M we have another economical Morse function on M for $\boldsymbol{Z}_2$ by choosing a suitable length function on M defined by means of an imbedding of M into a Euclidean space, which generalizes length functions on classical groups constructed by S. Ramanujam [3] and is essentially the same as our spherical function.

1. Spherical functions on $\boldsymbol{R}$-spaces

We recall here the notion of R-spaces and some properties of them. Let G be a connected semi-simple Lie group with finite center and $\mathfrak{g}$ the Lie algebra of G. An element Z of $\mathfrak{g}$ is called *real semi-simgle* if adZ is a semi-simple endomorphism of $\mathfrak{g}$ whose eigenvalues are all real. For a real semi-simple element Z of $\mathfrak{g}$, the sum $\mathfrak{n}^+(Z)$ of positive eigenspaces of adZ is a nilpotent subalgebra of $\mathfrak{g}$. A subgroup U of G is called *parabolic* if there exists a real semi-simple element Z of $\mathfrak{g}$ such that U is the normalizer in G of $\mathfrak{n}^+(Z)$. The quotient space $M=G/U$ of a connected semi-simple Lie group G with finite center modulo a parabolic subgroup U of G is called an *R-space*.

Let $M=G/U$ be an R-space and Z a real semi-simple element of the Lie algebra $\mathfrak{g}$ of G such that U is the normalizer in G of $\mathfrak{n}^+(Z)$. Let $\mathfrak{k}$ be a maximal compact subalgebra of $\mathfrak{g}$, which is perpendicular to Z with respect to the Killing form $(\ ,\ )$ of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$. Then the maximal compact subgroup K of G generated by $\mathfrak{k}$ is transitive on $M=G/U$ (Takeuchi [5]). It follows that if we put $K^*=K\cap U$, we have $M=K/K^*$. Moreover we have (Takeuchi [5])

$$(*) \qquad K^* = \{x\in K\ ;\ AdxZ = Z\}\,.$$

The smooth function f_X on $M=K/K^*$ for $X\in\mathfrak{p}$ defined by

$$f_X(xo) = (AdxZ, X) \qquad for \quad x\in K$$

where o is the origin of M, is a spherical function on $M=K/K^*$ associated with the representation $(Ad, \mathfrak{p})$ of K.

Now we take a maximal abelian subalgebra $\mathfrak{h}^-$ of $\mathfrak{p}$ containing Z and extend it to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let $\mathfrak{g}_C$ be the complexification of $\mathfrak{g}$ and σ the complex conjugation of $\mathfrak{g}_C$ with respect to the real form $\mathfrak{g}$ of $\mathfrak{g}_C$. The real part $\mathfrak{h}_0$ of the complexification $\mathfrak{h}_C$ of $\mathfrak{h}$ is equal to $\sqrt{-1}\,\mathfrak{h}^+ + \mathfrak{h}^-$, where $\mathfrak{h}^+=\mathfrak{h}\cap\mathfrak{k}$. The root system $\tilde{\mathfrak{r}}$ of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$ is identified with a subset of $\mathfrak{h}_0$ by means of the duality defined by the Killing form (,) of $\mathfrak{g}_C$. We introduce a linear order $>$ on $\mathfrak{h}_0$ in such a way that for any $\alpha \in \tilde{\mathfrak{r}}$ we have

$$\sigma\alpha \neq -\alpha \quad \text{and} \quad \alpha > 0 \Rightarrow \sigma\alpha > 0\,,$$
$$\alpha > 0 \quad \Rightarrow \quad (\alpha, Z) \geqslant 0\,.$$

The Weyl group $\tilde{W}$ of $\mathfrak{g}_C$ on $\mathfrak{h}_C$ is a subgroup of the orthogonal group on $\mathfrak{h}_0$ with respect to the Killing form of $\mathfrak{g}_C$. For an element s of $\tilde{W}$ we put

$$\Phi_s = \{\alpha \in \tilde{\mathfrak{r}};\ \alpha > 0,\ s^{-1}\alpha < 0\}\,.$$

We denote the cardinality $\sharp\Phi_{s^{-1}}$ of $\Phi_{s^{-1}}$ by $n(s)$ and call it the *index* of s. We put $\tilde{\mathfrak{r}}_1 = \{\alpha \in \tilde{\mathfrak{r}};\ (\alpha, Z) = 0\}$ and define

$$W^1 = \{s \in \tilde{W};\ s\sigma = \sigma s,\ \Phi_s \cap \tilde{\mathfrak{r}}_1 = \phi\}\,.$$

Then for any element s of W^1 we can find an element $a(s)$ of the normalizer in K of $\mathfrak{h}_0$ such that $Ad\, a(s) = s$ on $\mathfrak{h}_0$. The element $a(s)^{-1}o$ of M does not depend on the choice of $a(s)$ so that we shall denote the element $a(s)^{-1}o$ by $s^{-1}o$.

Now we take an element H of $\mathfrak{h}^-$ such that $(\alpha, H) \neq 0$ for any root α with $\sigma\alpha \neq -\alpha$. Then (Takeuchi-Kobayashi [6], Takeuchi [5]) $s \mapsto s^{-1}o$ gives a bijective correspondence of W^1 to the set of critical points of f_H. Moreover (Takeuchi [5]) $s^{-1}o$ is the "origin" of the $n(s)$-dimensional cell V_s of the standard cellular decomposition $M = \bigcup_{s \in W^1} V_s$ of M, which is economical for $\boldsymbol{Z}_2$ in the sense that $\{V_s;\ s \in W^1\}$ gives a basis of $H_*(M, \boldsymbol{Z}_2)$.

Theorem 1. *Let H_0 be an element of the negative Weyl chamber of $\mathfrak{h}^-$, that is,*

$(\alpha, H_0) < 0$ *for any positive root α with $\sigma\alpha \neq -\alpha$.*

Then the spherical function f_{H_0} on M is a Morse function and for any element s of W^1 we have

$$\text{Index of } f_{H_0} \text{ at } s^{-1}o = \text{Index } n(s) \text{ of } s.$$

Corollary. *f_{H_0} is an economical Morse function on M for $\boldsymbol{Z}_2$.*

Proof. For $X \in \mathfrak{k}$ and $s \in W^1$ we have

$$\begin{aligned}(Xf_{H_0})(s^{-1}o) &= \frac{d}{dt}(Ad(\exp tX a(s)^{-1})Z, H_0)|_{t=0} \\ &= \left(\frac{d}{dt}(Ad \exp tX)(s^{-1}Z)|_{t=0}, H_0\right) = ([X, s^{-1}Z], H_0) \\ &= -(s^{-1}Z, [X, H_0])\,.\end{aligned}$$

It follows that the Hessian $\mathcal{H}$ of f_{H_0} at $s^{-1}o$ is given by

$$\mathcal{H}(Xs^{-1}o, Ys^{-1}o) = (s^{-1}Z, [X, [Y, H_0]]) \qquad for \quad X, Y \in \mathfrak{k}\,.$$

Now we want to find a basis of the tangent space $M_{s^{-1}o}$ of M at $s^{-1}o$, convenient for the computation of the quantity $(s^{-1}Z, [X, [Y, H_0]])$.

Let τ be the anti-linear automorphism of $\mathfrak{g}_C$ such that $\tau|\mathfrak{k}=1$ and $\tau|\mathfrak{p}=-1$. Then there exist root vectors $\{X_\alpha\}$ of $\mathfrak{g}_C$ with respect to $\mathfrak{h}_C$ with $[X_\alpha, X_{-\alpha}]=-(2/(\alpha, \alpha))\alpha$ and $\tau X_\alpha = X_{-\alpha}$. For a positive root α with $\sigma\alpha \neq -\alpha$ we define $S_\alpha \in \mathfrak{k}$ and $T_\alpha \in \mathfrak{p}$ as follows. If $\sigma\alpha=\alpha$, $S_\alpha=(1+\tau)X$, $T_\alpha=(1-\tau)X$. If $\sigma\alpha<\alpha$ and $\alpha+\sigma\alpha$ is not a root, $S_\alpha=(1+\tau)(1+\sigma)X_\alpha$, $S_{\sigma\alpha}=(1+\tau)\sqrt{-1}(1-\sigma)X_\alpha$, $T_\alpha=(1-\tau)(1+\sigma)X_\alpha$, $T_{\sigma\alpha}=(1-\tau)\sqrt{-1}(1-\sigma)X_\alpha$. If $\sigma\alpha<\alpha$ and $\alpha+\sigma\alpha$ is a root, $S_\alpha=\sqrt{2}(1+\tau)(1+\sigma)X_\alpha$, $S_{\sigma\alpha}=\sqrt{2}(1+\tau)\sqrt{-1}(1-\sigma)X_\alpha$, $T_\alpha=\sqrt{2}(1-\tau)(1+\sigma)X_\alpha$, $T_{\sigma\alpha}=\sqrt{2}(1-\tau)\sqrt{-1}(1-\sigma)X_\alpha$. Let $\bar{\lambda}$ denote the orthogonal projection to $\mathfrak{h}^-$ of an element λ of $\mathfrak{h}_0$. Then we have (Takeuchi [5])

1) $[H, S_\alpha]=(\alpha, H)T_\alpha$, $[H, T_\alpha]=(\alpha, H)S_\alpha$ *for* $H \in \mathfrak{h}^-$,
2) $[S_\alpha, T_\alpha]=(4/(\bar{\alpha}, \bar{\alpha}))\bar{\alpha}$,
3) $\alpha \neq \beta \Rightarrow ([S_\alpha, T_\beta], \mathfrak{h})=\{0\}$.

On the other hand, $\mathfrak{k}$ is spanned over $\boldsymbol{R}$ by the centralizer $\mathfrak{k}_0$ in $\mathfrak{k}$ of $\mathfrak{h}^-$ and $\{S_\alpha\}$. But $\mathfrak{k}_0 s^{-1}o = Ad\, a(s)^{-1}\mathfrak{k}_0\, o=\{o\}$ since $Ad\, a(s)^{-1}\mathfrak{k}_0=\mathfrak{k}_0$ because of $s\mathfrak{h}^-=\mathfrak{h}^-$ and since $\mathfrak{k}_0$ is contained in the Lie algebra of K^*. It follows that the tangent space $M_{s^{-1}o}$ of M at $s^{-1}o$ is spanned over $\boldsymbol{R}$ by $\{S_\alpha s^{-1}o\}$. We have from 1), 2), and 3)

$$\begin{aligned} \mathcal{H}(S_\alpha s^{-1}o, S_\beta s^{-1}o) &= (s^{-1}Z, [S_\alpha, [S_\beta, H_0]]) \\ &= -(\beta, H_0)(s^{-1}Z, [S_\alpha, T_\beta]) \\ &= \begin{cases} 0 & if \quad \alpha \neq \beta \\ \dfrac{-4(\alpha, H_0)}{(\bar{\alpha}, \bar{\alpha})}(s^{-1}Z, \alpha) & if \quad \alpha=\beta\,. \end{cases} \end{aligned}$$

We note here that $-4(\alpha, H_0)/(\bar{\alpha}, \bar{\alpha})>0$. Now we need the following lemma giving the signature of $(s^{-1}Z, \alpha)$.

Lemma 1. *For a positive root α we have*

1) $\sigma\alpha \neq -\alpha$ *and* $(s^{-1}Z, \alpha)<0$
$\Leftrightarrow$ $(s^{-1}Z, \alpha)<0$
$\Leftrightarrow$ $\alpha \in \Phi_{s^{-1}}$
2) $\sigma\alpha \neq -\alpha$ *and* $(s^{-1}Z, \alpha)>0$
$\Leftrightarrow$ $(s^{-1}Z, \alpha)>0$

Proof of Lemma 1. Assume that $\sigma\alpha=-\alpha$. Then $(s^{-1}Z, \alpha)=(Z, s\alpha)=(\sigma Z, \sigma s\alpha)=(Z, s\sigma\alpha)=-(Z, s\alpha)=-(s^{-1}Z, \alpha)$ so that $(s^{-1}Z, \alpha)=0$. Therefore it suffices to show that

$$(s^{-1}Z, \alpha) < 0 \Leftrightarrow s\alpha < 0 .$$

If $(s^{-1}Z, \alpha)<0$, then $(Z, s\alpha)<0$. It follows from the choice of our linear order on $\mathfrak{h}_0$ that $s\alpha<0$. Conversely if $s\alpha<0$, then $-s\alpha>0$ and $s^{-1}(-s\alpha)=-\alpha<0$ so that $-s\alpha\in\Phi_{s^{-1}}$. But since $\Phi_s\cap\tilde{\mathfrak{r}}_1=\phi$ because s is an element of W^1, we have $(s^{-1}Z, \alpha)=(s\alpha, Z)\neq 0$. On the other hand we have $(s\alpha, Z)\leqslant 0$ from the choice of the order again. Thus we have $(s^{-1}Z, \alpha)<0$.

From the above lemma we see that the negative space $M^-{}_{s^{-1}o}$ of $\mathcal{H}$ is spanned by $\{S_\alpha s^{-1}o; \alpha\in\Phi_{s^{-1}}\}$ and the positive space $M^+{}_{s^{-1}o}$ of $\mathcal{H}$ is spanned by $\{S_\alpha s^{-1}o; \alpha>0, (s^{-1}Z, \alpha)>0\}$. But $\dim M=\#\{\alpha\in\tilde{\mathfrak{r}}; (\alpha, Z)<0\}=\#\{\alpha\in\tilde{\mathfrak{r}}; (s^{-1}Z, \alpha)<0\}$ since the Lie algebra of U is the sum of non-negative eigenspaces of adZ on $\mathfrak{g}$ (Takeuchi [5]). It follows from Lemma 1 that $\dim M=\#\{\alpha>0; (s^{-1}Z, \alpha)<0\}+\#\{\alpha<0; (s^{-1}Z, \alpha)<0\}=\#\Phi_{s^{-1}}+\#\{\alpha>0; (s^{-1}Z, \alpha)>0\}$. Therefore the Hessian $\mathcal{H}$ is non-degenerate and the index of f_{H_0} at $s^{-1}o=\dim M^-{}_{s^{-1}o}=\#\Phi_{s^{-1}}=$the index $n(s)$ of s. q.e.d.

REMARK. If X is a *regular* element of $\mathfrak{p}$, that is, there exists an element k of K such that $H_0=AdkX$ is an element of the negative Weyl chamber of $\mathfrak{h}^-$. then f_X is always an economical Morse function on M for $\boldsymbol{Z}_2$, since then $f_X(xo)=f_{H_0}(kxo)$ for $x\in K$. If M is the Grassmann manifold over $\boldsymbol{C}$ or $\boldsymbol{H}$, the dimensional consideration of cells yields that f_X for regular X is an economical Morse function on M for any coefficient field.

2. Nice functions on symmetric $\boldsymbol{R}$-spaces

Throughout this section we assume that the eigenvalues of adZ are 0, 1 and -1. Then the inner automorphism $\exp ad\,\pi\sqrt{-1}\,Z$ of $\mathfrak{g}_C$ is involutive, leaves $\mathfrak{k}$ invariant and is extended to the automorphism θ of K. Let $K_\theta=\{k\in K; \theta k=k\}$. Then K^* lies between K_θ and the connected component of K_θ. It follows that $M=K/K^*$ is symmetric. Conversely, if $M=G/U$ is an R-space such that $M=K/K^*$ is symmetric, then U is determined by an element Z of $\mathfrak{g}$ such that eigenvalues of adZ are 0, 1 and -1 (Nagano [2]).

Lemma 2. (Takeuchi [5]) *Let $\{\alpha_1, \cdots, \alpha_l\}$ be the fundamental root system with respect to the linear order on $\mathfrak{h}_0$ we have chosen in Section 1. Then for any element s of W^1 there exist fundamental roots $\alpha_{i_1}, \cdots, \alpha_{i_{n(s)}}$ such that*

$$Z-s^{-1}Z=\sum_{k=1}^{n(s)} p_{i_k}\alpha_{i_k}, \quad p_{i_k}=\frac{2(Z, s\alpha_{i_k})}{(\alpha_{i_k}, \alpha_{i_k})}=\frac{2}{(\alpha_{i_k}, \alpha_{i_k})}.$$

Let $\delta=\frac{1}{2}\sum_{\alpha>0}\alpha$ and $\delta_0=\bar{\delta}$. It is known that $2(\delta, \alpha_i)/(\alpha_i, \alpha_i)=1$ for any i, thus we have $(\delta, \alpha)>0$ for any positive root α. It follows that for any positive root α with $\sigma\alpha\neq-\alpha$ we have $(-\delta_0, \alpha)=-\left(\frac{1}{2}(\delta+\sigma\delta), \alpha\right)=-\frac{1}{2}((\delta, \alpha)$

$+(\delta, \sigma\alpha))<0$ since $\sigma\alpha>0$. Therefore $-\delta_0$ is an element of the negative Weyl chamber of $\mathfrak{h}^-$.

Theorem 2. *Let $M=G/U=K/K^*$ be a symmetric R-space. Then*

$$f=f_{-\delta_0}+\frac{1}{2}\dim M$$

is a nice function on M.

Proof. Recalling that $\dim M=\#\{\alpha\in\tilde{\mathfrak{r}};\ (\alpha, Z)<0\}=\#\{\alpha\in\tilde{\mathfrak{r}};\ (\alpha, Z)>0\}$ and considering that $(\alpha, Z)=0$ or 1 for any positive root α, we have $(Z, \delta_0)=(Z, \delta)=\frac{1}{2}\sum_{\alpha>0}(Z, \alpha)=\frac{1}{2}\dim M$. For an element s of W^1 we take an expression of $Z-s^{-1}Z$ as in Lemma 2. Then we have

$$\begin{aligned} f(s^{-1}o) &= (s^{-1}Z, -\delta_0)+\frac{1}{2}\dim M \\ &= -(s^{-1}Z, \delta_0)+(Z, \delta_0) = (Z-s^{-1}Z, \delta_0) \\ &= (Z-s^{-1}Z, \delta) \\ &= \sum_{k=1}^{n(s)}\frac{2(\alpha_{i_k}, \delta)}{(\alpha_{i_k}, \alpha_{i_k})} = n(s)\,. \end{aligned}$$

It follows from Theorem 1 that $f(s^{-1}o)$ equals the index of f at $s^{-1}o$. q.e.d.

3. Length functions on R-spaces

Now we come back to a general R-space $M=G/U=K/K^*$. The group K acts on $\mathfrak{p}$ under the adjoint action as isometries of Euclidean space $\mathfrak{p}$ with respect to the Killing form (,) of $\mathfrak{g}$. Owing to the equality (*) in Section 1, we may identify $M=K/K^*$ with the K-orbit through Z in $\mathfrak{p}$. Then the spherical function f_X on M is nothing but the height function on M with respect to the direction $X\in\mathfrak{p}$, that is,

$$f_X(Y)=(Y, X) \qquad \textit{for} \quad Y\in M\subset\mathfrak{p}\,.$$

Now we consider the length function L_X on M from the point X of $\mathfrak{p}$ defined by

$$L_X(Y)=(Y-X, Y-X) \qquad \textit{for} \quad Y\in M\subset\mathfrak{p}\,.$$

Then we have

$$L_X(Y)=-2(X, Y)+(Y, Y)+(X, X)=-2f_X(Y)+(Z, Z)+(X, X)\,.$$

It follows from Section 1 that if H is an element of $\mathfrak{h}^-$ such that $(\alpha, H)\neq 0$ for any root α with $\sigma\alpha\neq-\alpha$, then $s\mapsto s^{-1}Z$ gives a bijective correspondence of W^1 to the set of critical points of L_H.

Theorem 3. *If X is a regular element of $\mathfrak{p}$, then the length function L_X on M is an economical Morse function for $\boldsymbol{Z}_2$. In particular if H_0 is an element of the positive Weyl chamber of $\mathfrak{h}^-$, that is,*

$$(\alpha, H_0) > 0 \qquad \text{for any positive root } \alpha \text{ with } \sigma\alpha \neq -\alpha\,,$$

then for any element s of W^1 we have

$$\text{Index of } L_{H_0} \text{ at } s^{-1}Z = \text{Index } n(s) \text{ of } s\,.$$

Proof. It follows from Remark in Section 1 and the above equality that L_X is an economical Morse function for $\boldsymbol{Z}_2$. Moreover Theorem 1 implies the second statement since $-H_0$ is an element of the negative Weyl chamber of $\mathfrak{h}^-$.

The second statement may be derived as follows by means of the diagram of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$, which will give another proof of Theorem 1. The classical Morse theory for geodesics yields that if $s^{-1}Z$ is a non-degenerate critical point of L_{H_0}

$$\text{Index of } L_{H_0} \quad \text{at} \quad s^{-1}Z = \sum_{0<t<1} \delta(t)$$

where $\delta(t)$ is the multiplicity of the point $tH_0+(1-t)s^{-1}Z$ if this is a focal point relative to M along the transversal geodesic segment $\{\tau H_0+(1-\tau)s^{-1}Z;\, 0\leqslant\tau\leqslant 1\}$ to M and $\delta(t)=0$ otherwise. On the other hand it is known (Bott-Samelson [1]) that L_{H_0} is a Morse function on M and

$$\delta(t) = \#\{\alpha>0;\, \sigma\alpha\neq-\alpha,\, (\alpha,\, tH_0+(1-t)s^{-1}Z) = 0\}\,.$$

But for a positive root α with $\sigma\alpha\neq-\alpha$, the equation $(\alpha, tH_0+(1-t)s^{-1}Z)=0$ has a solution t such that $0<t<1$ if and only if $(\alpha, s^{-1}Z)<0$. It follows from Lemma 1 that $\sum_{0<t<1}\delta(t)$ equals the cardinality of $\Phi_{s^{-1}}$, that is, the index $n(s)$ of s.
q.e.d.

OSAKA UNIVERSITY

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