Siu, Y.-T.
Osaka J. Math.
5 (1968), 273-277

LOCAL COHOMOLOGY AND CONNECTEDNESS OF ANALYTIC SUBVARIETIES

YUM-TONG SIU

(Received April 10, 1968)
(Revised August 17, 1968)

Suppose X is an analytic subvariety in some open neighborhood G of the origin 0 in $\boldsymbol{C}^n$ with $\operatorname{codim}_{G;0}(X)=r$, where $\operatorname{codim}_{G;0}(X)$ denotes the codimension at 0 of X as a subvariety of G. Let ${}_n\mathfrak{O}$ be the structure sheaf of $\boldsymbol{C}^n$. Let $H^p_{X;0}({}_n\mathfrak{O})$, or simply $H^p_{X;0}$, denote the direct limit of $\{H^{p-1}(U-X, {}_n\mathfrak{O})\mid U$ is an open neighborhood of 0 in $G\}$ for $p\geqq 1$. ($H^p_{X;0}$ agrees with the stalk at 0 of the sheaf defined by the p-th local cohomology groups at X with coefficients in ${}_n\mathfrak{O}$, [1], p. 79). We say that X is *locally a complete intersection* at 0 if X can be defined locally at 0 by r holomorphic functions. If X is locally a complete intersection, obviously we have

$$H^p_{X;0}=0 \qquad \text{for} \quad p>r\,. \tag{1}$$

The question naturally arises: to what extent does (1) characterize a local complete intersection? Not much is known about the characterization of local complete intersections. In [3] Hartshorne introduces a concept of connectedness which in our case is equivalent to the following: X is *locally connected in codimension k* at 0 if the germ of X at 0 cannot be decomposed as the union of two subvariety-germs which are both different from the germ of X at 0 and whose intersection is a subvariety-germ Y with $\operatorname{codim}_{X;0}(Y)>k$. He shows that, if X is locally a complete intersection, then X is locally connected in codimension 1 at 0 (and also locally connected in codimension 1 at 0 in some properly defined formal sense). In this note we prove that (1) is a stronger necessary condition for local complete intersections than the connectedness condition. The following is our main theorem:

Theorem 1. *Suppose $q\geqq 0$. If $H^p_{X;0}=0$ for $p>q+r$, then X is locally connected in codimension $q+1$ at* 0.

For the proof of Theorem 1 we need the following:

Lemma 1. *Suppose Y is a 1-dimensional subvariety in some open neighborhood H of 0 in $\boldsymbol{C}^n$. Suppose 0 is the only singular point of Y and Y is locally irreducible at 0. Then $H^n_{Y;0}=0$.*

Proof. Suppose D is an arbitrary open neighborhood of 0 in H. By changing linearly the coordinates system of $\boldsymbol{C}^n$, we can find $U=\{(z_1, \cdots, z_n)\in\boldsymbol{C}^n \mid |z_i|<\delta_i,\ 1\leqq i\leqq n\}\subset D$ for some $\delta_i>0$, $1\leqq i\leqq n$, such that the projection $\pi\colon \boldsymbol{C}^n\to\boldsymbol{C}$ defined by $\pi(z_1, \cdots, z_n)=z_1$ makes $Y\cap U$ an irreducible analytic cover of s sheets over $U_1=\{z_1\in\boldsymbol{C} \mid |z_1|<\delta_1\}$ with $\{0\}$ as the critical set in U_1 (III, B. 3, [2]) and $\pi^{-1}(0)\cap Y\cap U=\{0\}$. Let $\tilde{U}_1=\{t\in\boldsymbol{C} \mid |t|<\sqrt[s]{\delta_1}\}$. We are going to define holomorphic functions g_k on $\tilde{U}_1$, $2\leqq k\leqq n$, such that

$$Y\cap U=\{(t^s, g_2(t), \cdots, g_n(t)) \mid t\in\tilde{U}_1\}\,. \tag{2}$$

Fix $z^*=(z_1^*, \cdots, z_n^*)\in Y\cap U$ with $z_1^*\neq 0$ and fix t^* with $(t^*)^s=z_1^*$. Take $t\in\tilde{U}_1-\{0\}$. Let γ be a continuous map from $[0, 1]$ to $\tilde{U}_1-\{0\}$ such that $\gamma(0)=t^*$ and $\gamma(1)=t$. Let $\hat{\gamma}$ be the continuous map from $[0, 1]$ to $U_1-\{0\}$ defined by $\hat{\gamma}(c)=(\gamma(c))^s$ for $c\in[0, 1]$. Then $\hat{\gamma}(0)=z_1^*$. Since $Y\cap U-\{0\}$ is a topological covering over $U_1-\{0\}$, there is a continuous map $\tilde{\gamma}\colon [0, 1]\to Y\cap U-\{0\}$ such that $\pi\tilde{\gamma}=\hat{\gamma}$ and $\tilde{\gamma}(0)=z^*$. Let $\tilde{\gamma}(1)=(z_1, \cdots, z_n)$. Define $g_k(t)=z_k$, $2\leqq k\leqq n$. Set $g_k(0)=0$, $2\leqq k\leqq n$. It is readily verified that g_k, $2\leqq k\leqq n$, are well-defined and holomorphic. (2) is satisfied, because $Y\cap U$ is irreducible. Define $F\colon \boldsymbol{C}^n\to\boldsymbol{C}^n$ by $F(w_1, \cdots, w_n)=((w_1)^s, w_2, \cdots, w_n)$. Let $\tilde{Y}=F^{-1}(Y\cap U)$ and let $\tilde{U}=F^{-1}(U)$. Let $e_1, \cdots, e_s$ be all the distinct s-th roots of unity. Let $Y_p=\{(w_1, \cdots, w_n)\in\boldsymbol{C}^n \mid w_1\in\tilde{U}_1,\ w_k=g_k(e_p w_1),\ 2\leqq k\leqq n\}$, $1\leqq p\leqq s$. $F(w_1, \cdots, w_n)\in Y\cap U$ if and only if for some $t\in\tilde{U}_1$ $(w_1)^s=t^s$ and $w_k=g_k(t)$, $2\leqq k\leqq n$. Hence $\bigcup_{p=1}^s Y_p=\tilde{Y}$. Since Y_p is defined by $n-1$ holomorphic functions, $H^q(\tilde{U}-Y_p, {}_n\mathfrak{O})=0$ for $q\geqq n-1$ and $1\leqq p\leqq s$. The following portion of the Mayor-Vietoris sequence is exact: $H^q(\tilde{U}-Y_{p+1}, {}_n\mathfrak{O})\oplus H^q(\tilde{U}-\bigcup_{i=1}^p Y_i, {}_n\mathfrak{O})\to H^q(\tilde{U}-\bigcup_{i=1}^{p+1} Y_i, {}_n\mathfrak{O})\to H^{q+1}(\tilde{U}-(Y_{p+1}\cap(\bigcup_{i=1}^p Y_i)), {}_n\mathfrak{O})$, $q\geqq 0$, $1\leqq p<s$. Since $H^{q+1}(\tilde{U}-(Y_{p+1}\cap(\bigcup_{i=1}^p Y_i)), {}_n\mathfrak{O})=0$ for $q\geqq n-1$ (see Problème 1, [4] or Th., [5]), by induction on p we conclude that $H^q(\tilde{U}-\bigcup_{i=1}^p Y_i, {}_n\mathfrak{O})=0$ for $1\leqq p\leqq s$ and $q\geqq n-1$. In particular, $H^{n-1}(\tilde{U}-\tilde{Y}, {}_n\mathfrak{O})=0$. Let $\mathfrak{F}$ be the zeroth direct image of ${}_n\mathfrak{O}$ under F. Then, since $H^{n-1}(\tilde{U}-\tilde{Y}, {}_n\mathfrak{O})=0$,

$$H^{n-1}(U-Y, \mathfrak{F})=0\,. \tag{3}$$

We claim that

$$\mathfrak{F}\approx {}_n\mathfrak{O}^s\,. \tag{4}$$

Consider the subvariety $Z=\{(z_0, z_1, \cdots, z_n) \mid z_1=(z_0)^s\}$ in $\boldsymbol{C}^{n+1}$. Let ${}_Z\mathfrak{O}$ be the structure sheaf of Z. Let $\theta\colon \boldsymbol{C}^{n+1}\to\boldsymbol{C}^n$ be defined by $\theta(z_0, z_1, \cdots, z_n)=(z_1, \cdots, z_n)$. Let $T\colon \boldsymbol{C}^n\to Z$ be defined by $T(w_1, \cdots, w_n)=(w_1, (w_1)^s, w_2, \cdots, w_n)$. T is biholomorphic and $\theta T=F$. Let $\mathfrak{G}$ be the zeroth direct image of ${}_Z\mathfrak{O}$ under θ. To prove (4), we need only prove that $\mathfrak{G}\approx {}_n\mathfrak{O}^s$. Suppose Q is a bounded nonempty Stein open subset in $\boldsymbol{C}^n$ and $f\in\Gamma(\theta^{-1}(Q)\cap Z, {}_Z\mathfrak{O})$. Then $f=\tilde{f}\,|\,\theta^{-1}(Q)\cap Z$ for some $\tilde{f}\in\Gamma(\theta^{-1}(Q), {}_{n+1}\mathfrak{O})$. By methods analogous to the usual proof of the

Weierstrass division theorem, we obtain $\tilde{f}=u((z_0)^s-z_1)+\sum_{i=0}^{s-1}(v_i\circ\theta)(z_0)^i$, where u is a holomorphic function on $\theta^{-1}(Q)$ and v_i, $0\leqq i\leqq s-1$, are holomorphic functions on Q. It is easily seen that v_i, $0\leqq i\leqq s-1$, are uniquely determined by f. $f\mapsto(v_0, \cdots, v_{s-1})$ defines a map h_Q from $\Gamma(\theta^{-1}(Q)\cap Z, {}_Z\mathfrak{O})$ to $\Gamma(Q, {}_n\mathfrak{O}^s)$. $\{h_Q\,|\,Q$ is a bounded Stein open subset of $\boldsymbol{C}^n\}$ induces a sheaf-isomorphism from $\mathfrak{G}$ to ${}_n\mathfrak{O}^s$. (4) is proved. (3) and (4) imply that $H^{n-1}(U-Y, {}_n\mathfrak{O})=0$. Hence $H^n_{Y;0}=0$. q.e.d.

Proof of Theorem 1.

Suppose X is not locally connected in codimension $q+1$ at 0. We are going to prove that $H^p_{X;0}\neq 0$ for some $p>q+r$. For some open neighborhood U of 0 in G we have $X\cap U=X_1\cup X_2$ and $X_1\cap X_2=Z$, where (i) for $i=1, 2$ X_i is a subvariety of $X\cap U$ whose germ at 0 is different from the germ of X at 0 and (ii) $\operatorname{codim}_{X;0}(Z)>q+1$. We can assume w.l.o.g. that no branch-germ X_1 at 0 contains a branch-germ of X_2 at 0 and vice versa. We have $n>q+r+1$.

(a) First we prove the case where $Z=\{0\}$. By shrinking U, we can find for $i=1, 2$ a 1-dimensional subvariety Y_i in X_i such that 0 is the only singular point of Y_i and Y_i is locally irreducible at 0. For any open neighborhood W of 0 in U we have the following portion of the Mayor-Vietoris sequence:

$$H^{n-2}(W-X, {}_n\mathfrak{O})\rightarrow H^{n-1}(W-\{0\}, {}_n\mathfrak{O})\xrightarrow{\alpha_W} H^{n-1}(W-X_1, {}_n\mathfrak{O})\oplus H^{n-1}(W-X_2, {}_n\mathfrak{O}),$$

where $\alpha_W=\alpha_W^{(1)}\oplus(-\alpha_W^{(2)})$ and $\alpha_W^{(i)}: H^{n-1}(W-\{0\}, {}_n\mathfrak{O})\rightarrow H^{n-1}(W-X_i, {}_n\mathfrak{O})$, $i=1, 2$, are the restriction maps. Moreover, we have the following commutative diagram:

$$\begin{array}{ccc} H^{n-1}(W-\{0\}, {}_n\mathfrak{O}) & \xrightarrow{\alpha_W} & H^{n-1}(W-X_1, {}_n\mathfrak{O})\oplus H^{n-1}(W-X_2, {}_n\mathfrak{O}) \\ \beta_W\downarrow & & \| \\ H^{n-1}(W-Y_1, {}_n\mathfrak{O})\oplus H^{n-1}(W-Y_2, {}_n\mathfrak{O}) & \xrightarrow{\gamma_W} & H^{n-1}(W-X_1, {}_n\mathfrak{O})\oplus H^{n-1}(W-X_2, {}_n\mathfrak{O}), \end{array}$$

where $\beta_W=\beta_W^{(1)}\oplus(-\beta_W^{(2)})$, $\gamma_W=\gamma_W^{(1)}\oplus\gamma_W^{(2)}$, and $\beta_W^{(i)}: H^{n-1}(W-\{0\}, {}_n\mathfrak{O})\rightarrow H^{n-1}(W-Y_i, {}_n\mathfrak{O})$ and $\gamma_W^{(i)}: H^{n-1}(W-Y_i, {}_n\mathfrak{O})\rightarrow H^{n-1}(W-X_i, {}_n\mathfrak{O})$, $i=1, 2$, are the restriction maps. Passing to direct limits, we have the following commutative exact diagram:

$$(5)\qquad \begin{array}{ccccc} H^{n-1}_{X;0} & \rightarrow & H^n_{\{0\};0} & \longrightarrow & H^n_{X_1;0}\oplus H^n_{X_2;0} \\ & & \downarrow & & \| \\ & & H^n_{Y_1;0}\oplus H^n_{Y_2;0} & \rightarrow & H^n_{X_1;0}\oplus H^n_{X_2;0} \end{array}$$

The cocycle in $Z^{n-1}(\mathfrak{A}, {}_n\mathfrak{O})$, where $\mathfrak{A}=\{A_i\}_{i=1}^n$ and $A_i=\{(z_1, \cdots, z_n)\in\boldsymbol{C}^n\,|\,z_i\neq 0\}$, defined by $(z_1\cdots z_n)^{-1}\in\Gamma(\cap_{i=1}^n A_i, {}_n\mathfrak{O})$ is not mapped to 0 under any restriction map $H^{n-1}(\boldsymbol{C}^n-\{0\}, {}_n\mathfrak{O})\rightarrow H^{n-1}(D-\{0\}, {}_n\mathfrak{O})$ for any polydisc neighborhood D of 0 in $\boldsymbol{C}^n$. Hence $H^n_{\{0\};0}\neq 0$. Since $H^n_{Y_i;0}=0$ for $i=1, 2$ by Lemma 1, the exact diagram in (5) implies that $H^{n-1}_{X;0}\neq 0$. Since $n-1>q+r$, $H^{p}_{X;0}\neq 0$ for some $p>q+r$.

(b) In the general case, suppose $H^{p}_{X;0}=0$ for $p>q+r$. We are going to derive a contradiction. In view of (a) we can assume that the germ of Z at 0 has positive dimension. Let $h=\operatorname{codim}_{U;0}(Z)$. Then $r+q+2\leqq h<n$. After a linear transformation of the coordinates system of $\boldsymbol{C}^n$ and after a shrinking of U, we can assume that $Z\cap \boldsymbol{C}^h=\{0\}$, where $\boldsymbol{C}^h$ is regarded as a linear subspace of $\boldsymbol{C}^n$ through the embedding sending $(z_1, \cdots, z_h)\in \boldsymbol{C}^h$ to $(z_1, \cdots, z_h, 0, \cdots, 0)\in \boldsymbol{C}^n$. Suppose W is an arbitrary open neighborhood of 0 in U. Consider the exact sequences $0\rightarrow {}_n\mathfrak{O}/\sum_{i=k+1}^n z_i\,{}_n\mathfrak{O}\xrightarrow{f_k}{}_n\mathfrak{O}/\sum_{i=k+1}^n z_i\,{}_n\mathfrak{O}\rightarrow {}_n\mathfrak{O}/\sum_{i=k}^n z_i\,{}_n\mathfrak{O}\rightarrow 0$, $h+1\leqq k\leqq n$, where f_k is defined by multiplication by z_k and $\sum_{i=n+1}^n z_i\,{}_n\mathfrak{O}=0$. These give us exact sequences $H^p(W-X, {}_n\mathfrak{O}/\sum_{i=k+1}^n z_i\,{}_n\mathfrak{O})\rightarrow H^p(W-X, {}_n\mathfrak{O}/\sum_{i=k}^n z_i\,{}_n\mathfrak{O})\rightarrow H^{p+1}(W-X, {}_n\mathfrak{O}/\sum_{i=k+1}^n z_i\,{}_n\mathfrak{O})$, $p\geqq 0$, $h+1\leqq k\leqq n$. Passing to direct limits, we have the following exact sequences:

$$(6)\qquad \begin{aligned}&\text{dir. lim.}_W\, H^p(W-X, {}_n\mathfrak{O}/\textstyle\sum_{i=k+1}^n z_i\,{}_n\mathfrak{O})\\ &\text{dir. lim.}_W\, H^p(W-X, {}_n\mathfrak{O}/\textstyle\sum_{i=k}^n z_i\,{}_n\mathfrak{O})\\ &\text{dir. lim.}_W\, H^{p+1}(W-X, {}_n\mathfrak{O}/\textstyle\sum_{i=k+1}^n z_i\,{}_n\mathfrak{O})\,,\\ &p\geqq 0,\ \ h+1\leqq k\leqq n\,.\end{aligned}$$

Since $\text{dir. lim.}_W\, H^p(W-X, {}_n\mathfrak{O}/\sum_{i=n+1}^n z_i\,{}_n\mathfrak{O})=H^{p+1}_{X;0}=0$ for $p\geqq q+r$, by (6) and by backward induction on k we conclude that $\text{dir. lim.}_W\, H^p(W-X, {}_n\mathfrak{O}/\sum_{i=k}^n z_i\,{}_n\mathfrak{O})=0$ for $p\geqq q+r$ and $h+1\leqq k\leqq n+1$. Since for $p\geqq 0$ $H^{p+1}_{X\cap C^h;0}({}_h\mathfrak{O})\approx$ $\text{dir. lim.}_W\, H^p(W-X, {}_n\mathfrak{O}/\sum_{i=h+1}^n z_i\,{}_n\mathfrak{O})$, we have

$$(7)\qquad H^{p+1}_{X\cap C^h;0}({}_h\mathfrak{O})=0 \qquad \text{for}\quad p\geqq q+r\,.$$

Since no branch-germ of X_1 at 0 contains a branch-germ of X_2 at 0 and vice versa, $\operatorname{codim}_{U;0}(X_i)<\operatorname{codim}_{U;0}(Z)=h$ for $i=1, 2$. Hence the germ of $X_i\cap \boldsymbol{C}^h$ at 0 is positive dimensional for $i=1, 2$. We are in the situation of Part (a). $H^{h-1}_{X\cap C^h;0}({}_h\mathfrak{O})\neq 0$. Since $h\geqq q+r+2$, this contradicts (7). q.e.d.

REMARK. The converse of Theorem 1 is not true as is shown in the following example: In $\boldsymbol{C}^6$ let $X_1=(\{z_1=z_2=0\}\cup\{z_2=z_3=0\}\cup\{z_3=z_4=0\})\cap\{z_5=0\}$ and $X_2=(\{z_2=z_1=0\}\cup\{z_1=z_4=0\}\cup\{z_4=z_3=0\})\cap\{z_6=0\}$. Let $X=X_1\cup X_2$. For $i=1, 2$, X_i is of codimension 3 and can be defined by 3 global holomorphic functions, because $X_1=\{z_1z_3+z_2z_4=0, z_2z_3=0, z_5=0\}$ and $X_2=\{z_1z_3+z_2z_4=0, z_1z_4=0, z_6=0\}$. Hence $H^{p}_{X_i;0}=0$ for $p>3$ and $i=1, 2$. $X_1\cap X_2=(\{z_1=z_2=0\}\cup\{z_3=z_4=0\})\cap\{z_5=z_6=0\}$ is of codimension 4 and is not locally connected in codimension 1 at 0, because $X_1\cap X_2=Y_1\cup Y_2$ and $Y_1\cap Y_2=\{0\}$, where $Y_1=\{z_1=z_2=z_5=z_6=0\}$ and $Y_2=\{z_3=z_4=z_5=z_6=0\}$. Hence $H^{p}_{X_1\cap X_2;0}\neq 0$ for some $p>4$. By taking direct limits of Mayor-Vietoris sequences, we obtain exact sequences $H^p_{X;0} \to H^{p+1}_{X_1 \cap X_2;0} \to H^{p+1}_{X_1;0} \oplus H^{p+1}_{X_2;0}$, $p>0$. Hence $H^p_{X;0} \neq 0$ for some $p>3$. On the other hand, the 6 branch-germs of X are given by $Z_1=\{z_1=z_2=z_5=0\}$, $Z_2=\{z_2=z_3=z_5=0\}$, $Z_3=\{z_3=z_4=z_5=0\}$, $Z_4=\{z_1=z_2=z_6=0\}$, $Z_5=\{z_1=z_4=z_6=0\}$, and $Z_6=\{z_3=z_4=z_6=0\}$. It can be easily verified that we cannot divide these 6 branch-germs into two groups so that the intersection of the union of one group with the union of another group is of dimension <2. X serves also as an example of a non local complete intersection which is locally connected in codimension 1.

UNIVERSITY OF NOTRE DAME

References

[1] H. Cartan: *Faisceaux analytiques cohérents*, C.I.M.E. (Varenna), 1963, Inst. Math. d. Unvi., Roma, 1–88.

[2] R.C. Gunning and H. Rossi: Analytic Functions of Several Complex Variables, Prentice-Hall, Englewood Cliffs, N. J., 1965.

[3] R. Hartshorne: *Complete intersections and connectedness*, Amer. J. Math. **84** (1962), 496–508.

[4] B. Malgrange: *Faisceaux sur des variétés analytiques-réelles*, Bull. Soc. Math. France **87** (1957), 231–237.

[5] Y.-T. Siu: *Analytic sheaf cohomology groups of dimension n of n-dimensional non-compact complex manifolds*, to appear in Pacific J. Math.