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AFFINE STRUCTURES ON COMPLEX MANIFOLDS

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Let M be a complex manifold of complex dimension n and let $C=\{U_i, \varphi_i\}_{i\in I}$ be the maximal atlas defining the complex structure on M. A subset $A=\{U_j, \varphi_j\}_{j\in J}$, $J\subset I$, of C is called an affine atlas of M, if $\varphi_{jk}=\varphi_j\circ\varphi_k^{-1}$ is a complex affine transformation of $\boldsymbol{C}^n$ whenever $U_j\cap U_k\neq\phi$. We can define the notion of a maximal affine atlas of M and we say that each maximal affine atlas of M defines a complex affine structure of the complex manifold M. We shall denote by $\boldsymbol{A}(M)$ the totality of complex affine structures on M.

The aim of this note is to study the structure of $\boldsymbol{A}(M)$ in the case where M is compact and the complex structure of M is homogeneous and we shall prove the following theorems.

Theorem 1. *Let M be a complex torus of complex dimension n. Then there exists a natural one-to-one correspondence between the set $\boldsymbol{A}(M)$ and the set of all commutative associative algebra structure over $\boldsymbol{C}$ in the complex vector space $\boldsymbol{C}^n$. In particular $\boldsymbol{A}(M)$ is a complex affine variety.*

More generally:

Theorem 2. *Let M be a connected compact complex manifold and let $Aut(M)$ be the group of all holomorphic transformations of M. Assume $Aut(M)$ is transitive on M. Then $\boldsymbol{A}(M)$ is a complex affine algebraic variety.*

1. A) Let M be complex manifold and let I be the tensor of the almost complex structure associated with M. For each point $p\in M$, the value I_p of I at p is an endomorphism of the tangent space $T_p(M)$ such that $I_p^2=-1$. Let $T_p^+(M)$ (resp. $T_x^-(M)$) be the subspace of the complexified tangent space $T_p{}^C(M)$ consisting of all u such that $I_p\, u=iu$ (resp. $I_p u=-iu$) with $i=\sqrt{-1}$. Then we have

$$T_p{}^C(M)=T_p^+(M)\oplus T_p^-(M)\,.$$

If $\{z^1, z^2, \cdots, z^n\}$ is a system of complex local coordinates on an open set U, then $\{(\partial/\partial z^i)_p\}_{i=1,\cdots,n}$ and $\{(\partial/\partial \bar{z}^i)_p\}_{i=1,\cdots,n}$ are bases of $T_p^+(M)$ and $T_p^-(M)$ respectively at each point $p\in U$. The totality of complex tangent vectors belonging to $T_p^+(M)$ $(p\in M)$ form a holomorphic vector bundle $T^+(M)$ over M.

Let X be a smooth vector field on M. Then we can write X uniquely in the form

$$X = X^+ + X^-,$$

where $X^+(p) \in T_p^+(M)$ and $X^-(p) \in T_p^-(M)$ at each point $p \in M$ and $X^-(p) = \overline{X^+(p)}$, where—denotes the conjugation of $T_p{}^C(M)$.

A complex vector field W on M is, by definition, a smooth section of the vector bundle $T^+(M) \oplus T^-(M)$. Then we can write W uniquely in the form $W = X + iY$, where X and Y are smooth real vector fields on M. A complex vector field W is called holomorphic if W is a holomoprhic section of $T^+(M)$. A smooth real vector field X is called holomorphic if X^+ is holomorphic.

Let $\mathfrak{g} = \mathfrak{g}(M)$ be the vector space of all holomorphic real vector fields on M. Then $\mathfrak{g}$ is a complex Lie algebra and if M is compact, $\mathfrak{g}$ is identified with the Lie algebra of the group $\mathrm{Aut}(M)$ of holomorphic transformations of M.

In the following we denote by $\mathfrak{X}(M)$ the real vector space of all smooth vector fields on M. Then a complex vector field on M is identified with an element of $\mathfrak{X}(M)^C$.

B) A linear connection ∇ on M is defined by a bilinear mapping $(X, Y) \to \nabla_X Y$ of $\mathfrak{X}(M) \times \mathfrak{X}(M)$ into $\mathfrak{X}(M)$ satisfying the following conditions:

1) $\nabla_{fY} X = f(\nabla_Y X)$;

2) $\nabla_Y fX = f(\nabla_Y X) + Yf \cdot X$.

A linear connection ∇ on M is called a *holomorphic* linear connection if the following two conditions are satisfied:

a) $\nabla_Y IX = I(\nabla_Y X)$ for all $X, Y \in \mathfrak{X}(M)$;

b) if X and Y are holomorphic vector fields defined on an open set O of M, then $\nabla_Y X$ is also holomorphic on O.

If ∇ is a linear connection, we can extend ∇ to a complex bilinear mapping of $\mathfrak{X}(M)^C \times \mathfrak{X}(M)^C$ into $\mathfrak{X}(M)^C$. Then the conditions a) and b) are equivalent to the following two conditions a′) and b′).

a′) $(\nabla_Y X)^+ = \nabla_Y X^+$;

b′) if U and W are complex holomorphic vector fields defined on an open set O, then $\nabla_W U$ is also holomorphic.

C) Let us consider a complex affine structure on M defined by a maximal affine atlas $\{(O, \varphi)\}$. Let $\{z^1, \cdots, z^n\}$ be the local coordinates on O defined by the chart (O, φ). On each of these open sets O we can define uniquely an linear connection ∇^0 on O by the conditions: $\nabla^0_{Z^i} Z^j = \nabla^0_{\bar{Z}^i} Z^j = 0$ $(i, j = 1, 2, \cdots, n)$, where $Z^i = \partial/\partial z^i$ and $\bar{Z}^i = \partial/\partial \bar{z}^i$. Then there exists a unique linear connection ∇ on M such that the restriction of ∇ on each O coincides with ∇^0. This affine connection ∇ is holomorphic and locally flat, i.e. the torsion and the curvature of ∇ are 0. This means that

$$\nabla_X Y - \nabla_Y X = [X, Y];$$
$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) = \nabla_{[X,Y]} Z$$

for any $X, Y, Z \in \mathfrak{X}(M)$.

Thus to each affine structure on M there corresponds a locally flat, holomorphic linear connection on M and to the distinct affine structures there correspond distinct linear connections.

D) Let, conversely, ∇ be any locally flat, holomorphic linear connection on M and let $\tilde{M}$ be the universal covering manifold of M. Then there is defined uniquely a connection $\tilde{\nabla}$ on $\tilde{M}$ such that, for smooth vector fields X and Y on M, we have $\tilde{\nabla}_{Y^*} X^* = (\nabla_Y X)^*$, where, for each vector field X on M, X^* denotes the lift of X. $\tilde{\nabla}$ is also locally flat and holomorphic. Let $\mathfrak{P} = \{P\}$ be the complex vector space of all parallel complex holomorphic vector fields on $\tilde{M}$. Then the map $P \to P(\tilde{a})$ $(\tilde{a} \in \tilde{M})$ is a bijection of $\mathfrak{P}$ onto $T^+_{\tilde{a}}(M)$. In particular $\dim_C \mathfrak{P} = n = \dim_C M$. $\mathfrak{P}$ form an abelian Lie algebra, because $[P, P'] = \tilde{\nabla}_P P' - \tilde{\nabla}_{P'} P = 0$.

Let Γ be the fundamental group of M. Then Γ acts from the right on $\tilde{M}$ and the action of each element of Γ is holomorphic and affine.

Fix a point $a \in M$ and let $\tilde{a}$ be a point of $\tilde{M}$ such that $\pi(\tilde{a}) = a$, π denoting the projection of $\tilde{M}$ onto. M. For each vector $y \in T^+_a(M)$, there exists one and only one $P \in \mathfrak{P}$ such that $d\pi \cdot P(\tilde{a}) = y$. We denote this vector field P by P_y.

Let $\gamma \in \Gamma$. Then γ is a holomorphic affine transformation and hence $d\gamma \cdot \mathfrak{P} \subset \mathfrak{P}$. Put

$$d\gamma \cdot P_y = P_{f(\gamma)y} \,. \tag{1}$$

Then $\gamma \to f(\gamma)$ is a representation of the group Γ in the vector space $T^+_a(M)$.

Now let $\{P_1, \cdots, P_n\}$ $(n = \dim_C M)$ be a basis of the complex vector space $\mathfrak{P}$. Then we can define n holomorphic 1-forms $\{\omega^1, \cdots, \omega^n\}$ on $\tilde{M}$ by the condition

$$\omega^i(P_j) = \delta^i_j \qquad (i, j = 1, \cdots, n)\,.$$

These 1-forms are closed. There exists a basis $\{y_1, \cdots, y_n\}$ of $T^+_a(M)$ such that $P_i = P_{y_i}$ $(i = 1, 2, \cdots,)$. We can define a $T^+_a(M)$-valued holomorphic 1-form θ on $\tilde{M}$ by

$$\theta = \sum_{i=1}^n \omega^i y_i \,.$$

Then we have:

$$\theta(P) = d\pi \cdot P(\tilde{a})\,, \qquad P \in \mathfrak{P}\,, \tag{2}$$

and

$$d\theta = 0\,, \tag{3}$$

Moreover,

$$(4)\qquad (d\gamma)^*\theta = f(\gamma)\cdot\theta\,, \qquad \gamma\in\Gamma\,.$$

In fact, let $P\in\mathfrak{P}$. Then there exists $y\in T_a^+(M)$ such that $P=P_y$. Then

$$((d\gamma)^*\theta)(P) = \theta(d\gamma\ P_y) = \theta(P_{f(\gamma)y}) = f(\gamma)\cdot y \qquad \text{by (2).}$$

On the other hand, by (2) $y=\theta(P)$ and hence $((d\gamma)^*\cdot\theta)(P)=f(\gamma)\cdot\theta(P)$ and this proves the equality (4). For any $\tilde{x}\in\tilde{M}$, let

$$(5)\qquad \varphi(\tilde{x}) = \int_{\tilde{a}}^{\tilde{x}}\theta\,.$$

Then φ is a holomorphic map of $\tilde{M}$ into $T_a^+(M)$. Put

$$(6)\qquad q(\gamma) = \varphi(\gamma\tilde{a})$$

for all $\gamma\in\Gamma$. We have then

$$(7)\qquad \varphi(\gamma\tilde{x}) = f(\gamma)\varphi(\tilde{x})+q(\gamma)\,.$$

In particular for $\tilde{x}=\sigma\tilde{a}$ $(\sigma\in\Gamma)$, we have

$$q(\gamma\sigma) = f(\gamma)q(\sigma)+q(\gamma)\,.$$

This shows that q is a 1–cocycle of the group Γ and that, if we denote $a(\gamma)$ the complex affine transformation $x\to f(\gamma)x+q(\gamma)$ of $T_a^+(M)$, then $\gamma\to a(\gamma)$ is a homomorphism of Γ into the group of complex affine transformations of $T_a^+(M)$. Moreover (7) shows that

$$(7')\qquad \varphi(\gamma\tilde{x}) = a(\gamma)\varphi(\tilde{x})\,.$$

By the definition of φ, we have

$$(d\varphi)(\tilde{x}) = \theta(\tilde{x})$$

and $\theta(\tilde{x})$: $T_{\tilde{x}}^+(M)\to T_a^+(M)$ is bijective. Therefore φ is a holomorphic immersion of $\tilde{M}$ into the n dimensional complex vector space $T_a^+(M)$ which satisfies $(7')$*. Let U be an open subset of M evenly covered by π such that each connected component of $\pi^{-1}(U)$ is mapped bijectively by φ onto an open set in $\boldsymbol{C}^n=T_a^+(M)$. Let $\tilde{U}$ be any one of the connected components of $\pi^{-1}(U)$ and let ψ be the holomorphic bijective map of U onto $\varphi(\tilde{U})$ defined by $\psi=\varphi\circ(\pi\,|\,\tilde{U})^{-1}$. Then it is easy to check that $\{(U,\psi)\}$ defines a complex affine structure on M and that the locally flat holomorphic linear connection associated with this complex affine structure coincides with ∇.

* The mapping φ is the "development" of $\tilde{M}$ in $\boldsymbol{C}^n$, The present way of defining the development φ is due to J.L. Koszul,

Thus there is a one-to-one correspondence between the set $\boldsymbol{A}(M)$ of all complex affine structures on a complex manifold M and the set of all locally flat, holomorphic linear connections on M.

2. In the following we shall denote by $\boldsymbol{A}(M)$ the set of all locally flat, holomorphic linear connections on M. We denote by $\mathfrak{g}$ the complex Lie algebra of all holomorphic vector fields on M. From now on we assume that M is compact. Then $\mathfrak{g}$ is identified with the Lie algebra of the group $\mathrm{Aut}(M)$.

Now let $\nabla\in\boldsymbol{A}(M)$. Then the map $(X, Y)\to-\nabla_Y X$ is a bilinear map of $\mathfrak{g}\times\mathfrak{g}$ into $\mathfrak{g}$. Let

$$X\cdot Y=-\nabla_Y X\,. \tag{8}$$

Then this multiplication in $\mathfrak{g}$ defines an algebra structure on the complex vector space $\mathfrak{g}$ and we denote this algebra by $\mathfrak{g}(\nabla)$.

DEFINITION. Let A be an algebra over a field k and set $[x, y, z]=x(yz)-(xy)z$ and call it the associator of x, y and z. We call A a pre-Lie algebra, if the relation

$$[x, y, z]=[x, z, y]$$

holds for any x, y and z in A.

For example, an associative algebra A is a pre-Lie algebra. Let A be a pre-Lie algebra and set

$$[x, y]=xy-yx$$

for $x, y\in A$. Then we can show and that the bracket product $[x, y]$ defines a Lie algebra. We call this Lie algebra the Lie algebra associated with A.

REMARK. The notion of pre-Lie algebras has been introduced by M. Gerstenhaber in connection with the deformation of algebras. See [1] and [3].

DEFINITION. Let $\mathfrak{g}$ be a Lie algebra over a field k and A a pre-Lie algebra over k. We call A pre-Lie algebra over $\mathfrak{g}$ if the associated Lie algebra of A is $\mathfrak{g}$.

Lemma 1. *Let $\mathfrak{g}$ be the algebra of all holomorphic vector fields on a compact complex manifold M and let ∇ be a locally flat holomorphic linear connection on M. Then the algebra $\mathfrak{g}(\nabla)$ is a pre-Lie algebra over $\mathfrak{g}$.*

This lemma follows easily from the definition of the multiplication in $\mathfrak{g}(\nabla)$ and from the fact that the torsion and the curvature of ∇ are 0.

Let us denote by $\boldsymbol{A}(\mathfrak{g})$ the set of all pre-Lie algebra structures over $\mathfrak{g}$. Then the map $\nabla\to\mathfrak{g}(\nabla)$ defines a map of $\boldsymbol{A}(M)$ into $\boldsymbol{A}(\mathfrak{g})$.

Assume now that M is homogeneous. Then for any $p\in M$, the tangent vectors $X(p)$ $(X\in\mathfrak{g})$ span the tangent space $T_p(M)$. Then we see easily that the map of $\boldsymbol{A}(M)$ into $\boldsymbol{A}(\mathfrak{g})$ is injective.

Now let

$$\mathfrak{g}_p = \{Y \in \mathfrak{g} \mid Y(p) = 0\}\,, \qquad p \in M\,.$$

If $Y \in \mathfrak{g}_p$, then $(\nabla_Y X)(p) = \nabla_{Y(p)} X = 0$ and hence $\mathfrak{g}_p$ is a left ideal of $\mathfrak{g}(\nabla)$.

Conversely we have the following lemma.

Lemma 2. *Let M be a compact homogeneous complex manifold and let $\mathfrak{g}$ be the Lie algebra of all holomorphic vector fields on M. Let A be a pre-Lie algebra over $\mathfrak{g}$ such that $\mathfrak{g}_p$ is a left ideal of A for every $p \in M$. Then there exists a locally flat holomorphic linear connection ∇ on M such that $A = \mathfrak{g}(\nabla)$.*

Proof. Let $p \in M$ and $u \in T_p(M)$. Then there exists a $Y \in \mathfrak{g}$ such that $Y(p) = u$. For any $X \in \mathfrak{g}$ define $\nabla_u X \in T_p(M)$ by putting

$$\nabla_u X = -(X \cdot Y)(p)\,.$$

This definition does not depend on the choice of $Y \in \mathfrak{g}$ such that $Y(p) = u$, because $\mathfrak{g}_p$ is a left ideal of A. For any vector field Y and any $X \in \mathfrak{g}$, $\nabla_Y X$ will denote the vector field on M such that

$$(\nabla_Y X)(p) = \nabla_{Y(p)} X\,.$$

Then the following equalities hold:

1) $\nabla_{fY} X = f \nabla_Y X$, where f is a smooth function on M;

2) $\nabla_{Y+Y'} X = \nabla_Y X + \nabla_{Y'} X$; $\nabla_Y (X + X') = \nabla_Y X + \nabla_Y X'$,

where Y and Y' are smooth vector fields on M and $X, X' \in \mathfrak{g}$;

3) $\nabla_Y X$ is a smooth vector field; in fact, let $p \in M$. Then in a neighborhood U of p, Y is written uniquely in the form $Y = f^1 Y_1 + \cdots + f^n Y_n$, where $Y_1, \cdots, Y_n$ are in $\mathfrak{g}$ and $f^1, \cdots, f^n$ are smooth functions on U. At each point $q \in U$, we have $(\nabla_Y X)(q) = \sum_{i=1}^n f^i(q)(\nabla_{Y_i} X)(q) = -\sum_{i=1}^n f^i(q)(X \cdot Y_i)(q)$ and hence $\nabla_Y X$ is smooth on U.

Next let Y be a smooth vector field and $u \in T_p(M)$. Define $\nabla_u Y \in T_p(M)$ by

$$\nabla_u Y = \nabla_{Y(p)} X + [X, Y](p)\,,$$

where X is a vector field in $\mathfrak{g}$ such that $u = X(p)$. We have to show that this definition is consistent. It suffices to show that

$$\nabla_{Y(p)} X + [X, Y](p) = 0$$

whenever $X(p) = 0$ and $X \in \mathfrak{g}$. To see this let $Y = f^1 Y_1 + \cdots + f^n Y_n$ in a neighborhood U of p, where $Y_1, \cdots, Y_n \in \mathfrak{g}$. Then

$$\nabla_{Y(p)} X = \sum_{i=1}^n f^i(p) \nabla_{Y_i(p)} X \quad \text{and} \quad [X, Y] = \sum_{i=1}^n f^i [X, Y_i] + \sum_{i=1}^n X f^i \cdot Y_i$$

on U. Since $X(p)=0$, we have $(Xf^i)(p)=0$ and, since X and Y_i are in $\mathfrak{g}$, $[X, Y_i]=X\cdot Y_i-Y_i\cdot X$. Therefore $[X, Y_i](p)=(X\cdot Y_i)(p)-(Y_i\cdot X)(p)=-\nabla_{Y_i(p)}X+\nabla_{X(p)}Y_i=-\nabla_{Y_i(p)}X$ and hence $\nabla_{Y(p)}X+[X, Y](p)=\sum_{i=1}^n f^i(p)\nabla_{Y_i(p)}X-\sum_{i=1}^n f^i(p)\nabla_{Y_i(p)}X=0$. Thus we have defined the tangent vector $\nabla_u Y$ for any $u\in T_p(M)$ and any smooth vector field Y. The following conditions hold:

i) $\nabla_u(Y+Y')=\nabla_u Y+\nabla_u Y'$;

ii) $\nabla_{u+v}Y=\nabla_u Y+\nabla_v Y$; $\nabla_{\lambda u}Y=\lambda\nabla_u Y$ $(\lambda\in\boldsymbol{R})$;

iii) $\nabla_u(fY)=f(p)\nabla_u Y+uf\cdot Y$, where f is a smooth function.

Thus we have defined a linear connection ∇ on M and it is easily seen that the torsion of ∇ is 0 and that $\nabla_Y X=-X\cdot Y$ for $X, Y\in\mathfrak{g}$. Then $\nabla_X(\nabla_Y Z)-\nabla_Y(\nabla_X Z)-\nabla_{[X,Y]}Z=[Z, X, Y]-[Z, Y, X]=0$ for $X, Y, Z\in\mathfrak{g}$. It follows then that the curvature of ∇ is 0. Moreover it is easily seen that ∇ is holomorphic and $\mathfrak{g}(\nabla)=A$ and the lemma is proved.

Assume now that $\mathfrak{g}_p=\{0\}$ for all $p\in M$. This is the case if and only if M is of the form $M=G/D$, where G is a complex Lie group and D is a discrete subgroup of G. In this case, the map $\nabla\rightarrow\mathfrak{g}(\nabla)$ establishes a one-to-one correspondence between the set of all complex affine structures on M and the set $\boldsymbol{A}(\mathfrak{g})$ of all pre-Lie algebra structures over $\mathfrak{g}$. In particular, this holds for a complex torus M. In this case the Lie algebra $\mathfrak{g}$ is abelian and Theorem 1 follows from the following lemma and from what we have proved so far.

Lemma 3. *Let A be a pre-Lie algebra over a Lie algebra $\mathfrak{g}$. Assume $\mathfrak{g}$ is abelian. Then A is a commutative associative algebra.*

In fact, $xy-yx=[x, y]=0$ and hence $xy=yx$ for $x, y\in A$. Moreover, $x(yz)-(xy)z=x(zy)-(xz)y$ and $yz=zy$ and hence $(xy)z=(xz)y$. But $(xy)z=z(xy)$ and $(xz)y=(zx)y$ and hence $z(xy)=(zx)y$ and this proves that A is associative.

Now each pre-Lie algebra structure over $\mathfrak{g}$ is identified with an element of the vector space $\mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathfrak{g}$ and $\boldsymbol{A}(\mathfrak{g})$ is identified with an algebraic subset of $\mathfrak{g}^*\otimes\mathfrak{g}^*\otimes\mathfrak{g}$. For each $p\in M$, Let $\boldsymbol{A}_p$ be the subset of $\boldsymbol{A}(\mathfrak{g})$ consisting of all pre-Lie algebra over $\mathfrak{g}$ such that $\mathfrak{g}_p$ is a left ideal. Then $\boldsymbol{A}_p$ is an algebraic subset of $\boldsymbol{A}(\mathfrak{g})$ and by Lemma 2, $\boldsymbol{A}(M)$ is identified with $\bigcap_{p\in M}\boldsymbol{A}_p$: $\boldsymbol{A}(M)=\bigcap_{p\in M}\boldsymbol{A}_p$. Then there exists a finite number of points $p_1, \cdots, p_r$ in M such that $\boldsymbol{A}(M)=\bigcap_{i=1}^r\boldsymbol{A}_{p_i}$ and hence $\boldsymbol{A}(M)$ is a complex affine variety. This proves Theorem 2.

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