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Osaka J. Math.
5 (1968), 189-198

ON SEMI-PRIMARY ABELIAN CATEGORIES

Dedicated to Professor Atuo Komatu for his 60th birthday

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(Received June 7, 1968)

Let $\boldsymbol{C}$ be an abelian category with exact direct limits, namely cocomplete C_3-category ([5], p. 83).

In this note we always assume that $\boldsymbol{C}$ contains a generator U, and hence $\boldsymbol{C}$ is locally small by [5], p. 71. In [2], Gabriel and Popesco have given a characterization of U being projective and small by using the concept of localization in [1]. We shall give another proof without localization in the section 1.

In the section 2, we shall define a function φ of $\boldsymbol{C}$ into itself, which is analogous to the radical of semi-primary ring.

We shall show that $\boldsymbol{C}$ has such a function when the endomorphism ring $[U, U]$ is a semi-primary ring, and we shall give some criteria by means of φ that U is small and projective.

In the section 3, we shall add some remarks in the previous author's work on category of tri-angular matrices, [4].

In this note we shall freely make use of concepts in categories from [5].

The author would like to express his thanks to professor O.E. Villamayor for inviting him to Universidad de Buenos Aires.

1. Preliminary results

In this section we shall summarize all results which we need in the following sections.

Almost all results in this section have been proved in [2] and [6] by using the concepts of localization in [1]. However, we shall give here another approach to them by means of rather homological method.

Let $\boldsymbol{C}$ be an abelian cocomplete C_3-category ([5], p. 81) and U an object of $\boldsymbol{C}$. Let $A=[U, U]$. By mod A we mean the category of A-right modules. Let $T:\boldsymbol{C}\rightarrow \text{mod } A$; $T(V)=[U, V]$ for any $V\in\boldsymbol{C}$ be the functor of $\boldsymbol{C}$ into mod A. In this case we can define a coadjoint S of T such that $S(M)=M\otimes_A U$ by [5], p. 143, namely η: $[M, T(V)]\approx[S(M), V]_{\boldsymbol{C}}$. Furthermore, we have natural transformation ψ_V: $ST(V)\rightarrow V$ and φ_M: $M\rightarrow TS(M)$, (see[5], pp 118–119).

Theorem 0 (Gabriel and Popesco [2]). *Let $\boldsymbol{C}$, U and A be as above. Then the following statements are equivalent;*

1) *U is a generator,*

2) *T is a completely faithful (namely, full and faithful).*
3) ψ_V *is isomorphic for all* $V \in C$ *and* S *is an exact functor.*

Proof. 1)$\leftrightarrow$2). See [2] or [5] in which we do not need the concept of localization.
3)$\rightarrow$2). $[ST(V), V'] \underset{\eta}{\approx} [T(V), T(V')]$ and $[ST(V), V'] \approx [V, V']$ for $V, V' \in \boldsymbol{C}$.
2)$\rightarrow$3). $[ST(V), V'] \underset{\eta}{\approx} [T(V), T(V')] \underset{\alpha}{\approx} [V, V']$. Hence, $[ST(V), \ ]$ and $[V, \ ]$ give the equivalent functors. Therefore, $\psi_V = \eta^{-1}\alpha^{-1} I_V$ is isomorphic.
Thus, it remains to show that S is exact. First, we show that if $M \in \operatorname{mod} A$ is contained in a free module F, then $0 \to S(M) \to S(F)$ is exact. In order that, we assume first that M is finitely generated, say $M = (m_1, m_2, \cdots, m_n)$ and hence we may assume that F is also finitely generated. Then we have a commutative diagram

1)

$$\begin{array}{ccccccc} 0 & \longleftarrow & M & \overset{f}{\longleftarrow} & \sum_{i=1}^{n} \oplus A v_{\beta_i} & \longleftarrow & {}^{K}A = \sum_{k \in K} \oplus A w_k \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \alpha} & & \\ & F = & \sum_{i=1}^{m} \oplus A u_{\alpha_i} & = & \sum_{i=1}^{m} \oplus A u_{\alpha_i}, & & \end{array}$$

where u_{α_i}, v_{β_i} and w_k are free bases and i is the inclusion map, f is a natural mapping such that $f(v_{\beta_i}) = m_i$, $\alpha = if$, and $K = \ker f$.

Operating S on the above 1) we obtain commutative exact diagram:

2)

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ & & & & K' & \overset{i_1}{\nwarrow} & \\ & & & & \downarrow & & \\ 0 & \longleftarrow & S(M) & \overset{S(f)}{\longleftarrow} & \sum_{i=1}^{n} \oplus U & \overset{i_2}{\longleftarrow} & V \\ & & \downarrow & & \downarrow{\scriptstyle S(\alpha)} & & \\ & & S(F) & = & S(F) & & \end{array}$$

where $V = \operatorname{im}({}^{K}U \xrightarrow{\beta} \sum_{1}^{n} U)$ and $K' = \ker S(\alpha)$.
It is clear that there exists the inclusion map i_1 of V into K'. Operating again T on 2) we have

3)

$$\begin{array}{ccccccc} 0 & & & & & & \\ \downarrow & & & & & & \\ T(K') & \overset{T(i_1)}{\nwarrow} & & & & & \\ \downarrow{\scriptstyle i_3} & & & & & & \\ \sum \oplus A v_{\beta_i} & \underset{T(i_2)}{\longleftarrow} & T(V) & \underset{T(\beta)}{\longleftarrow} & T({}^{K}U) & \underset{\varphi_{K_A}}{\longleftarrow} & {}^{K}A \\ \downarrow{\scriptstyle \alpha} & & & & & & \\ \sum \oplus A u_{\alpha_i} & & & & & & \end{array}$$

where the vertical line is exact and $T(i_1)$, $T(i_2)$ are inclusions. Since K is also $\ker\alpha$, there exists a unique isomorphism θ such that

$$\begin{array}{ccc} T(K') & \xrightarrow{\theta} & K \\ \downarrow i_3 & \swarrow i_4 & \\ \sum \oplus A_{\beta_i} & & \end{array}$$

is commutative. Let $a \in T(K')$ and put $k = \theta a$. Then $T(i_2)T(\beta)\varphi_{KA}w_k = i_3 a = i_4\theta a = i_4 k = i_3 T(i_1)T(\beta)\varphi_{KA}w_k$ by the naturality of φ. Put $b = T(\beta)\varphi w_k \in T(V)$, $i_3 a = i_3 T(i_1) b$. Since i_3 is injective, $a = T(i_1)b$. Hence, $T(i_1)$ is isomorphic. Since T is faithful, i_1 is isomorphic by [5], p. 56. Therefore, $0 \to S(M) \to S(F)$ is exact from 2). Next, let M be any submodule of free A-module F: $0 \to M \to F$. Then M is a direct limit of the family of finitely generated A-submodules M_{α_i}; $M = \varinjlim M_{\alpha_i}$. Since S is colimit and exact preserving by [5], p. 85 and p. 55, $0 \to S(M) = \varinjlim S(M_{\alpha_i}) \to S(F)$ is exact from the first argument. Hence, $\mathrm{Tor}^1(M, U) = 0$ for all $M \in \mathrm{mod}\, A$, ([5], p. 112, § 8), which implies that S is exact.

From now on we fix a generator U in $\boldsymbol{C}$ and $A = [U, U]$. Then for any subobject U' in U it is clear that $[U, U']$ is identified to a right ideal in A, and we shall denote it by $\boldsymbol{r}_{U'}$ or $\boldsymbol{r}$. By $\boldsymbol{K}U$ we mean the image of f: $\sum_{k \in \boldsymbol{K}} U_k \to U$ defined by $f(U_k) = kU$ for any subset $\boldsymbol{K}$ in A. We note from the definitions that $\boldsymbol{r}_{U'}U = ST(U')$. Then we have from [5], p. 71.

Lemma 1. *For any subobject U' in U we have $U' = \boldsymbol{r}_{U'}U$.*

Lemma 2. *Let U be a generator in $\boldsymbol{C}$ and $\boldsymbol{r}_1$, $\boldsymbol{r}_2$ right ideals in A. Then we have*

1) $(\boldsymbol{r}_1 + \boldsymbol{r}_2)U = \boldsymbol{r}_1 U \cup \boldsymbol{r}_2 U$.
2) $(\boldsymbol{r}_1 \cap \boldsymbol{r}_2)U = \boldsymbol{r}_1 U \cap \boldsymbol{r}_2 U$.

Proof. 1) is trivial from the definition.
2) We have the following row exact and commutative diagrams:

4)
$$\begin{array}{ccccccccc} 0 \to & \boldsymbol{r}_2 U & \to & \boldsymbol{r}_1 U \cup \boldsymbol{r}_2 U & \to & (\boldsymbol{r}_1 U \cup \boldsymbol{r}_2 U)/\boldsymbol{r}_2 U & \to 0 \\ & \uparrow & & \uparrow & & \uparrow \approx & \\ 0 \to & \boldsymbol{r}_1 U \cap \boldsymbol{r}_2 U & \to & \boldsymbol{r}_1 U & \to & \boldsymbol{r}_1 U/\boldsymbol{r}_1 U \cap \boldsymbol{r}_2 U & \to 0 \\ & \uparrow & & \uparrow \approx & & \uparrow & \\ 0 \to & (\boldsymbol{r}_1 \cap \boldsymbol{r}_2)U & \to & \boldsymbol{r}_1 U & \to & \boldsymbol{r}_1 U/(\boldsymbol{r}_1 \cap \boldsymbol{r}_2)U, & \end{array}$$
and

5)
$$\begin{array}{ccccccc} 0 \to & \boldsymbol{r}_2 & \to & \boldsymbol{r}_2 \cup \boldsymbol{r}_1 & \to & (\boldsymbol{r}_2 \cup \boldsymbol{r}_1)/\boldsymbol{r}_2 & \to 0 \\ & \uparrow & & \uparrow & & \uparrow \approx & \\ 0 \to & \boldsymbol{r}_1 \cap \boldsymbol{r}_2 & \to & \boldsymbol{r}_1 & \to & \boldsymbol{r}_1/\boldsymbol{r}_1 \cap \boldsymbol{r}_2 & \to 0 \end{array}$$

Since S is an exact functor, we obtain $(\boldsymbol{r}_1 \cap \boldsymbol{r}_2)U = \boldsymbol{r}_1 U \cap \boldsymbol{r}_2 U$ from 4) by operating S on 5).

The following proposition is an immediate consequence of [6], Prop. 1.1 and [5], p. 104. However, we shall prove it without localization.

Proposition 3. *Let* $\boldsymbol{C}$, *U and A be as above and U a generator. Then the following statements are equivalent.*

1) $S(\ ) = \ \otimes U$ *is an equivalent functor.*

2) $T(\ ) = [U,\ ]$ *and* $S(\ )$ *give a one-to-one correspondence between right ideals and subobjects in U.*

3) *For any maximal right ideal* $\boldsymbol{r}$ *in A* $S(A/\boldsymbol{r}) \neq 0$.

4) *U is projective and small in* $\boldsymbol{C}$.

Proof. 1)→2)→3) are trivial.

4)→1) is proved in [5], p. 104.

3)→4) It is clear from 3) that for any non-zero A-module M, $S(M) = M \otimes U \neq 0$, since S is exact by Theorem 0. Let $V_1 \overset{\alpha}{\to} V_2 \to 0$ be exact in $\boldsymbol{C}$ and $T(V_1) \to T(V_2) \to K \to 0$ be exact in mod A. Since S is exact, $ST(V_1) = V_1 \overset{\alpha}{\to} ST(V_2) = V_2 \to S(K) \to 0$ is exact. Hence, $S(K) = 0$, which means $K = 0$ from the above. Therefore, T is exact and hence, U is projective. Finally we shall show that U is small. Let $f: U \to \sum_{i \in I} V_i$ be a morphism in $\boldsymbol{C}$, where V's are any objects in $\boldsymbol{C}$. Put $U_J = f^{-1}(\sum_{k \in J} V_k)$, where J is a finite set of I. Since $\boldsymbol{C}$ is C_3-category, $U = \cup U_J$ by [5] p. 83. Then $A = \bigcup_{J}^{t} \boldsymbol{r}_J$ by Lemma 2 and 3), where $\boldsymbol{r}_J = [U, U_J]$. Put $1 = \sum_{i=1}^{t} f_i$, $f_i \in \boldsymbol{r}_{J_i}$. Then $U = \bigcup_{i=1}^{t} U_{J_i}$, which implies $\operatorname{im} f \subset \sum_{i=1}^{t} \sum_{l \in J_i} V_l$.

An object V in $\boldsymbol{C}$ is called minimal if there exist no proper subobjects in V. If V' is a directsum of minimal sub-objects, then V' is called semi-simple. We note that some properties of semi-simple modules are valid in $\boldsymbol{C}$.

Lemma 4. *For any artinian and noetherian object V,* $[V, V]$ *is a semi-primary ring.*

It is well known in mod A, and its proof is valid in $\boldsymbol{C}$.

2. Semi-primary category $\boldsymbol{C}$

Let $\boldsymbol{C}$ be an abelian category mentioned in the section 1. We shall consider a function φ of object in $\boldsymbol{C}$ into itself which is similar to the radical of a ring.

I. $\varphi(C)$ is a subobject in C for any C in $\boldsymbol{C}$ such that $C/\varphi(C)$ is semi-simple.

II. $C = \varphi(C)$ if and only if $C = 0$.

III. If C/C' is semi-simple for some subobject C' in C, then $C' \supset \varphi(C)$.

Let φ, φ_1 be functions in $\boldsymbol{C}$ satisfying I and II. We note in this case that every non-zero object contains a maximal subobject. If $\varphi_1(C) \supseteqq \varphi_2(C)$ for all

$C \in \boldsymbol{C}$, then we shall say φ_2 is smaller than φ_1. Furthermore, if φ_2 satisfies III and $\boldsymbol{C}$ is locally small, then φ_2 is a unique minimal function among those satisfying I and II, since $\varphi_2(C) = \cap D$, where D runs all maximal subobjects in C. In this case φ_2 is a functor which satisfies the following commutative diagram

$$6) \qquad \begin{array}{ccc} C & \xrightarrow{f} & C' \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i'} \\ \varphi(C) & \xrightarrow{\varphi(f)} & \varphi(C') \end{array},$$

where $f \in \boldsymbol{C}$ and i, i' are inclusions and $\varphi(f)$ is defined as follows: Let V be a maximal subobject in C' then $f^{-1}(V) = C$ or $C/f^{-1}(V) \approx C'/V$ ([5], pp. 22–24), and hence $f(\varphi(C)) \subset V$, which implies $\operatorname{im}(f \mid \varphi(C)) \subset \varphi(C')$. Conversely, if φ satisfying I, II induces a functor in $\boldsymbol{C}$ satisfying 6), then φ satisfies III. In fact, let $V \neq 0$ in $\boldsymbol{C}$, then V contains a maximal subobject V_0. The commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & V/V_0 \\ \uparrow & & \uparrow \\ \varphi(V) & \longrightarrow & \varphi(V/V_0) = 0 \end{array}$$

shows $\varphi(V) \subseteq V_0$.

We put $\varphi^1(U) = \varphi(U)$, $\varphi^i(U) = \varphi(\varphi^{i-1}(U))$.

Lemma 5. *Let U be a generator of $\boldsymbol{C}$. If φ^i is defined in U such that $\varphi^n(U) = 0$ for some n and satisfies I, II (resp. I, II and III), then φ induces a function $\tilde{\varphi}$ in $\boldsymbol{C}$ such that $\tilde{\varphi}$ satisfies I, II (reps. I, II and III).*

Proof. First, we define $\tilde{\varphi}(\varphi^i(U)) = \varphi^{i+1}(U)$ for all i. Let V be any object in $\boldsymbol{C}$ which is different from any $\varphi^i(U)$, and $g: \sum_{[U,V] \ni f} \oplus U_f \to V$ the canonical morphism defined by $f: U_f \to V$. We assume that $\operatorname{im}(g \mid \sum \oplus \varphi^i(U)) = V$ and $\operatorname{im}(g \mid \sum \oplus \varphi^{i+1}(U)) \neq V$. Then define $\tilde{\varphi}(V) = \operatorname{im}(g \mid \sum \oplus \varphi^{i+1}(U))$. It is clear that $V/\tilde{\varphi}(V)$ is semi-simple and that $V/\tilde{\varphi}(V) \neq 0$ if $V \neq 0$. Next, we assume φ satisfies III for U. Let V_0 be a maximal subobject in V, then $f^{-1}(V_0) \supset \varphi(U)$. Therefore, $\tilde{\varphi}(V) \subseteq V_0$.

DEFINITION. Let V be an object in $\boldsymbol{C}$. If $[V, V]$ is a semi-primary ring, V is called a *semi-primary* object.

From Lemma 4, every artinian and noetherian object is semi-primary.

Proposition 6. *Let U be a projective, small generator in an abelian C_3-category. Then U is semi-primary if and only if a function φ in U satisfying I, II and III is defined and $U/\varphi(U)$ is a directsum of finite many of simple objects and $\varphi^n(U) = 0$ for some n.*

Proof. It is clear from Theorem 0 and Proposition 3. We note here that $\varphi^i(U)=S(\mathfrak{n}^i U)$, where $\mathfrak{n}$ is the radical of $[U, U]$.

The main purpose of this section is to study some structure of C_3-category with semi-primary generator.

Theorem 7. *Let* $\boldsymbol{C}$ *be an abelian* C_3*-category with semi-primary generator* U. *Then we can define a function* φ *in* $\boldsymbol{C}$ *which satisfies I and II and* $U/\varphi(U)$ *is a finite directsum of simple subobjects and* $\varphi^n(U)=0$ *for some.* n.

Proof. Let $A=[U, U]$ and $\mathfrak{n}$ the radical of A. Put $U_i=\mathfrak{n}^i U$ for all i. It is clear that $U_i\supset U_{i+1}$. Put $\mathfrak{r}_i=[U, U_i]$. Then $\mathfrak{r}_i\supset\mathfrak{n}^i$. Put $\bar{\mathfrak{r}}_{i+1}=\mathfrak{n}^i\cap\mathfrak{r}_{i+1}$. Then $\bar{\mathfrak{r}}_{i+1}U=U_i\cap U_{i+1}=U_{i+1}$ by Lemma 2. Since $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ is semi-simple, so is $\mathfrak{n}^i/\bar{\mathfrak{r}}_{i+1}$, say $\mathfrak{n}^i/\bar{\mathfrak{r}}_{i+1}=\sum_{i=I}\oplus\tilde{\mathfrak{r}}_{\alpha_i}$; $\mathfrak{n}^i\supset\mathfrak{r}_{\alpha_i}\supset\bar{\mathfrak{r}}_{i+1}$, and $\tilde{\mathfrak{r}}_{\alpha_i}$ is simple. Put $U_{\alpha_i}=\mathfrak{r}_{\alpha_i}U$. If $U_{\alpha_i}=U_{i+1}$, $\mathfrak{r}_{\alpha_i}\subset\mathfrak{n}^i\cap\mathfrak{r}_{i+1}$, which is a contradiction. Hence, $U_i\supset U_{\alpha_i}\supsetneqq U_{i+1}$. We shall show that U_{α_i}/U_{i+1} is simple. Let V be a subobject such that $U_{\alpha_i}\supset V\supsetneqq U_{i+1}$. Then $\mathfrak{r}_V\supset\mathfrak{r}_{\alpha_i}$, in fact if $\mathfrak{r}_V\not\supset\mathfrak{r}_{\alpha_i}$, $\mathfrak{r}_V\cap\mathfrak{r}_{\alpha_i}=\bar{\mathfrak{r}}_{i+1}$, and hence, $U_{i+1}=(\mathfrak{r}_V\cap\mathfrak{r}_{\alpha_i})U=V\cap U_{\alpha_i}=V$. Therefore, $V=\mathfrak{r}_V U\supset\mathfrak{r}_{\alpha_i}U=U_{\alpha_i}$. Since $\mathfrak{n}^i=\cup\,\mathfrak{r}_{\alpha_i}$, $U_i=\cup\,U_{\alpha_i}$. On the other hand, $\mathfrak{r}_{\alpha_i}\cap\bigcup_{i\neq j}\mathfrak{r}_{\alpha_j}=\bar{\mathfrak{r}}_{i+1}$. Hence, $U_{\alpha_i}\cap\cup\,U_{\alpha_j}=U_{i+1}$. Since $\boldsymbol{C}$ is C_3-category, $U_i/U_{i+1}\approx\sum\oplus U_{\alpha_i}/U_{i+1}$ is semi-simple. We define $\varphi^i(U)=U_i$. Then $U/\varphi(U)$ is a finite directsum of simple subobjects from the above, and $\varphi^n(U)=0$ if $\mathfrak{n}^n=0$. Then we can define a function $\tilde{\varphi}$ in $\boldsymbol{C}$ from Lemma 5.

Let V_0 be a subobject in V such that $V_0+V'=V$ implies $V=V'$ for any subobject V' in V. V_0 is called *negligible*. By $[U\colon U_1]$ we mean the number of simple components in U/U_1.

Theorem 8. *Let* $\boldsymbol{C}$ *be an abelian* C_3*-category with semi-primary generator* U, *Then the following conditions are equivalent.*

1) *U is projective and small.*
2) $[A\colon\mathfrak{n}]=[U\colon\varphi(U)]$, *where* $\varphi(U)=\mathfrak{n}U$, $A=[U, U]$ *and* $\mathfrak{n}$ *is the radical of* A.
3) $\varphi(U)$ *is negligible in* U.
4) φ *satisfies the condition III.*
5) $T\colon C\to$ *mod* A *is preserving minimal objects.*

Proof. If U is projective and small, then $\boldsymbol{C}$ is equivalent to mod A by Proposition 3. Hence, 2) 3) 4) and 5) are trivial. We assume 2). We put $\mathfrak{a}=[U, \mathfrak{n}U]$. If we restrict the argument in the proof of Theorem 7 to the case of $i=1$, we get $[A\colon\mathfrak{a}]=[U\colon\mathfrak{n}U]$, $=n$. Hence, $\mathfrak{a}=\mathfrak{n}$. For every maximal right ideal $\mathfrak{r}$, $\mathfrak{r}/\mathfrak{n}=\sum_{i=1}^{n-1}\oplus\mathfrak{r}_{\alpha_i}/\mathfrak{n}$, which implies $U\neq\bigcup_{i=1}^{n-1}\mathfrak{r}_{\alpha_i}U=\mathfrak{r}U$. Hence, we obtain 1) from Proposition 3.

3) Let $\mathfrak{a}$ be as above. We assume $\mathfrak{a}\neq\mathfrak{n}$. Then there exists a right ideal $\mathfrak{b}$ properly containing $\mathfrak{n}$ such that $\mathfrak{a}/\mathfrak{n}\oplus\mathfrak{b}/\mathfrak{n}=A/\mathfrak{n}$. Let e be an idempotent element in A such that $\mathfrak{b}/\mathfrak{n}=(eA+\mathfrak{n})/\mathfrak{n}$. Since $\mathfrak{b}\supset\mathfrak{n}$, $\mathfrak{b}U\supset\mathfrak{n}U=\mathfrak{a}U$. Hence, $U=(\mathfrak{a}+\mathfrak{b})U=\mathfrak{b}U$. Put $U_0=eU$. Then $U_0+\mathfrak{n}U=(eA+\mathfrak{n})U=\mathfrak{b}U=U$. Therefore, $U_0=U$ by 3). Hence, $e=I_u$, which is a contradiction.

4) If $\mathfrak{n}\neq\mathfrak{a}$, we obtain the fact $U=U_0+\mathfrak{n}U$ and $U_0\neq U$. Since U/U_0 contains a maximal object from Theorem 7, there exists a maximal subobject $V\supset U_0$. Therefore, $V\not\supset\mathfrak{n}U$.

5) If $\mathfrak{n}\neq\mathfrak{a}$, then there exists a maximal subobject V in U such that $U=V+\mathfrak{n}U$. Since $0\rightarrow[U, V]\rightarrow[U, U]\rightarrow[U, U/V]$ is exact and $[U, U/V]$ is minimal, $\mathfrak{r}_V=[U, V]$ is a maximal right ideal, and hence $\mathfrak{r}_V\supset\mathfrak{n}$, which is a contradiction.

It is clear that there are many examples in which semi-primary generators are not projective.

Corollary 1. *Let U be a semi-primary generator in $\boldsymbol{C}$. If $A/\mathfrak{n}$ is a simple rings, U is projective and small, where $A=[U, U]$ and $\mathfrak{n}$ is its radical.*

Proof. Let $\mathfrak{a}=[U, \mathfrak{n}U]$. Since $U\neq\mathfrak{n}U$, and $\mathfrak{a}$ is a two-sided ideal, $\mathfrak{a}=\mathfrak{n}$.

Corollary 2. *Let B be a semi-primary ring and U be a semi-primary generator in the category of B-right modules. Then $\mathfrak{n}_A U\supset U\mathfrak{n}_B$. $\mathfrak{n}_A U=U\mathfrak{n}_B$ if and only if U is a finitely generated and projective, where, $A=[U, U]$ and $\mathfrak{n}_A$ (resp. $\mathfrak{n}_B$) is the radical of A (resp. B).*

Proof. Let $\varphi(U)=U\mathfrak{n}_B$. Then φ is a functor in mod B satisfying I, II and III. Hence, $\mathfrak{n}_A U\supset U\mathfrak{n}_B$ by Theorem 7. If $\mathfrak{n}_A U=U\mathfrak{n}_B$, a function φ' defined in the proof of Theorem 7 satisfies III. Hence, U is projective and small. The converse is trivial.

EXAMPLE. We shall show that there exists a generator U such that φ^i are defined in U satisfying the following conditions: $U/\varphi(U)$ is a finite directsum of simple object, φ^i satisfies I, II and III for all i and $\varphi^n(U)=0$ for some n, however U is not semi-primary.

Let k be a field and $K=k(x)$. Let $A=\begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ be a tri-angular matrix ring. Then A is semi-primary with radical $\mathfrak{n}$. We define $\varphi(U)=U\mathfrak{n}$ in mod A. Put $\mathfrak{r}=\begin{pmatrix} 0 & 0 \\ k[x] & 0 \end{pmatrix}$. Then $\mathfrak{r}$ is a right ideal in A. Then $[A/\mathfrak{r}, A/\mathfrak{r}]\approx\begin{pmatrix} k & 0 \\ k(x)/k[x] & k[x] \end{pmatrix}$ is not semi-primary. $U=A\oplus A/\mathfrak{r}$ is the desired generator.

3. Abelian category of commutative diagram

We recall the definition of abelian category of commutative diagram over abelian categories $\boldsymbol{C}_i$ (see [4]).

Let $I=(1, 2, \ldots, n)$ be a finite linear ordered set and $\{\boldsymbol{C}_i\}_{i\in I}$ a family of abelian categories. We assume that there are given cokernel preserving functors $T_{ij}\colon \boldsymbol{C}_i \to \boldsymbol{C}_j$ for $i<j$. Furthermore, we assume:

(*) There exist natural transformations

$$\psi_{ijk}\colon T_{jk}T_{ij} \to T_{ik} \qquad \text{for all} \quad i<j<k, \quad \text{and}$$

(**) For any $i<j<k<l$ and V in $\boldsymbol{C}_i$

$$\begin{array}{ccc} T_{kl}T_{jk}T_{ij}(V) & \xrightarrow{T_{kl}(\psi)} & T_{kl}T_{ik}(V) \\ \downarrow{\scriptstyle \psi_{ikl}} & & \downarrow{\scriptstyle \psi_{ikl}} \\ T_{jl}T_{ij}(V) & \xrightarrow{\psi_{ijl}} & T_{il}(V) \end{array}$$

is commutative.

We call a family of morphism $d_{ij}\colon T_{ij}(V_i) \to V_j$ an arrow for $V_i \in \boldsymbol{C}_i$, $V_j \in \boldsymbol{C}_j$ and for all $i<j$, when the diagrams

(***)
$$\begin{array}{ccc} T_{jk}T_{ij}(V_i) & \xrightarrow{T_{jk}(d_{ij})} & T_{jk}(V_j) \\ \downarrow{\scriptstyle \psi_{ijk}} & & \downarrow \\ T_{ik}(V_i) & \xrightarrow{d_{ik}} & V_k \end{array}$$

are commutative.

We define a *commutative diagram* $[I, \boldsymbol{C}_i]$ as follows; Its objects consist of set $\{V_i\}_{i\in I}$ with arrows $\{d_{ij}\}$ and morphisms consist of set $\{(f_i)\}_{i\in I}; f_i\colon V_i \to V_i'$ in $\boldsymbol{C}_i$ such that $d'_{ij}T_{ij}(f_i)=f_jd_{ij}$.

Lemma 9. *Let T_{ij} be functors satisfying* (**). *Then the natural transformation of $T_{i_{n-1}i_n}T_{i_{n-2}i_{n-1}}\cdots T_{i_1i_2} \to T_{i_1i_n}$ does not depend on any choice of combination of $T_{i_ni_{n-1}}, \cdots, T_{i_1i_2}$.*

Proof. We can prove the lemma by using induction on the number of functors and naturality of ψ_{ijk}. Namely, every natural transformation is equal to $T_{i_{n-1}i_n}(T_{i_{n-2}i_{n-1}}(\cdots(T_{i_2i_3}T_{i_1i_2}) \to T_{i_1i_n}$.

We assume that all $\boldsymbol{C}_i$ have projective class $\mathcal{E}_i$. We define a functor $S_i\colon \boldsymbol{C}_i \to [I, \boldsymbol{C}_i]$ by setting $S_i(V_i)=(0, \cdots, 0, V_i, T_{ii+1}(V_i), \cdots, T_{in}(V_i))$. Then the projective objects in $[I, \boldsymbol{C}_i]$ are of the form $\oplus S_i(P_i)$ and their retract, where P_i is $\mathcal{E}_i$-projective for all i, ([4], Prop. 1.2′). If the projective objects in $[I, \boldsymbol{C}_i]$ are only of the former forms, we call $[I, \boldsymbol{C}_i]$ a *good* category of commutative diagram.

Theorem 10. *Let $\boldsymbol{C}_i$ be abelian category with projective class $\mathcal{E}_i$. Then every $[I, \boldsymbol{C}_i]$ with T_{ij} is imbedding in a good category $[I, \boldsymbol{C}_i]$ with T'_{ij}.*

Proof. We shall define new functors T'_{ij}:

$$T'_{ii+1} = T_{ii+1}$$
$$T'_{ij} = T_{j-1j}T_{j-2j-1}\cdots T_{ii+1} \qquad \text{for} \quad i+1<j\,.$$

Then it is clear that T'_{ij} are cokernel preserving and $\psi'_{ijk}=I_{Ck}$ and (**) is trivial. Furthermore, there exist unique natural transformations $\phi_{ij}\colon T'_{ij}\to T_{ij}$ by Lemma 9. Put $\boldsymbol{C}=[I, \boldsymbol{C}_i]$ with T_{ij} and $\boldsymbol{C}'=[I, \boldsymbol{C}_i]$ with T'_{ij}. We define a function F of $\boldsymbol{C}$ into $\boldsymbol{C}'$ as follows: For $V=(V_i)$ with arrows d_{ij} in $\boldsymbol{C}$ we put $F(V)=(V_i)$ with the following arrows d'_{ij}:

$$d'_{ii+1} = d_{ii+1}$$
$$d'_{ij} = d_{ij}\phi_{ij}T'_{ij} \qquad \text{for} \quad i+1<j\,.$$

We have to show that d'_{ij} satisfies (***). We have a diagram for $i<j<k$ and $V_t\in\boldsymbol{C}_t$

$$\begin{array}{ccccc}
T'_{jk}T'_{ij}(V_i) & \xrightarrow{T'(\phi)} & T'_{jk}T_{ij}(V_i) & \xrightarrow{T'(d_{ij})} & T'_{jk}(V_j) \\
\Big\| \psi'_{ijk} & & \Big\downarrow \phi & \text{II} & \Big\downarrow \phi \\
 & \text{I} & T_{jk}T_{ij}(V_i) & \xrightarrow{T(d_{ij})} & T_{jk}(V_j) \\
\Big\| & & \Big\downarrow & \text{III} & \Big\downarrow d_{jk} \\
T'_{ik}(V_i) & \longrightarrow & T_{ik}(V_i) & \xrightarrow{d_{ik}} & V_k\,.
\end{array}$$

I is commutative by Lemma 9, II is commutative by naturality of ϕ and so is III by (**). Hence, d'_{ij} satisfies (***). Define $F((f_i))=(f_i)$ for morphism (f_i) in $\boldsymbol{C}$. Then we can similarly show that F is a functor. It is clear that F is an imbedding functor. Since $\psi'_{ijk}=I_{Ck}$, $K^j(P_i)=0$ in (*) of [4], Lemma 3. 7. Hence, $\boldsymbol{C}'$ is good by [4], Lemma 3. 7.

If every objects in $\boldsymbol{C}$ are projective, $\boldsymbol{C}$ is called a *semi-simple category*.

Corollary. *Let $\boldsymbol{C}_i$ be a semi-simple abelian category. Then $[I, \boldsymbol{C}_i]$ is imbedding in an abelian hereditary category,* (*cf.* [3], *Theorem* 5).

Proof. It is clear from Theorem 10 and [4], Theorem 3. 12.

Finally, we note that if $\boldsymbol{C}_i$ have functor φ_i satisfying I, II and III and $T_{ij}(\varphi_i(V_i))\subseteq\varphi_j T_{ij}(V_i)$ on $V=(V_i)$ in $[I, \boldsymbol{C}_i]$. Then

$$\varphi(V) = (\varphi_1(V_1), \varphi_2(V_2)\cup d_{12}(V_2), \cdots, \varphi_j(V_j) \bigcup_{i<j} d_{ij}(V_i), \cdots)$$

is a functor on $[I, \boldsymbol{C}_i]$ satisfying I, II and III. If $\varphi_t{}^m=0$ for all t then $\varphi^{nm}=0$.

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