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ON GENERALIZED CROSSED PRODUCT AND BRAUER GROUP

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For a commutative ring L wich is a Galois extension of a ring k with Galois group G, Chase, Harrison, and Rosenberg, in [5] and [6] gave a seven terms exact sequence about cohomology groups of G and Brauer group $B(L/k)$ of Azumaya k-algebras split by L, by using the generalized Amiztur cohomology and spectral sequence. In this paper, we give a generalization of the concept of crossed product, and for a commutative Galois extension L of a ring k with Galois group G, we study the generalized crossed product of the commutative ring L and the group G, and concerning the gorup of isomorphism classes of finitely generated projective rank 1 L-modules. Finally, as an application to Brauer group, using the generalized crossed product, we shall derive immediatly the "seven terms exact sequence theorem".

In §1, we define the generalized crossed product $\Delta(f, \Lambda, \Phi, G)$ of a k-algebra Λ and a group G with factor set f related to Φ, where Φ is a group homomorphism of G to the group of isomorphism classes of invertible Λ-Λ-bimodule (see [4], p. 76), and $f=\{f_{\sigma,\tau};\ \sigma, \tau\in G\}$ is a family of isomorphisms of modules satisfying some commutative diagrams. In §2, we suppose that L is a commutative Galois extension of a ring k with fimite Galois group G. Then we shall show that $\Delta(f, L, \Phi, G)$ is an Azumaya k-algebra with a maximal commutative subring L, and conversely, every Azumaya k-algebra with maximal commutative subring L can be written by $\Delta(f, L, \Phi, G)$ for some Φ and f. In §3. using the results of §2, we derive the seven termes exact sequence:

$$(1)\rightarrow H^1(G, L^*)\rightarrow P(k)\rightarrow P(L)^G\rightarrow H^2(G, L^*)\rightarrow B(L/k)\rightarrow H^1(G, P(L))$$
$$\rightarrow H^3(G, L^*)\,.$$

We suppose every ring has identity element and module is unital.

1. Generalized crossed product. Let k be a commutative ring with identity, Λ a k-algebra with identity. A Λ-Λ-bimodule P is called invertible if P is a finitely generated projective and generator (i.e. completely faithful by means of [3]) left Λ-module and $\mathrm{Hom}_\Lambda({}_\Lambda P, {}_\Lambda P)\approx\Lambda^0$, where for $a\in k$ and $x\in P$,

$ax=xa$. Let $Pic_k(\Lambda)$ be the group of isomorphism classes $[P]$ of invertible Λ-Λ-bimodules P with law of composition induced by tensor product over Λ: $[P]\cdot[Q]=[P\otimes_\Lambda Q]$, then $[P]^{-1}=[P^*]$ where $P^*=\mathrm{Hom}_\Lambda(P, \Lambda)$. We define the generalized crossed Product $\Delta(f, \Lambda, \Phi, G)$ of a k-algebra Λ and a group G with factor set $f=\{f_{\sigma,\tau}: \sigma, \tau\in G\}$ as follows: For given group G and k-algebra Λ, let $\Phi: G\to Pic_k(\Lambda)$ be a group homomorphism. Put $\Phi(\sigma)=[J_\sigma]$ for $\sigma\in G$. If $f=\{f_{\sigma,\tau}; \sigma, \tau\in G\}$ which is a family of Λ-Λ-isomorphisms $f_{\sigma,\tau}: J_\sigma\otimes_\Lambda J_\tau\to J_{\sigma\tau}$, $\sigma, \tau\in G$ satisfies the following commutative diagrams:

$$\begin{array}{ccc} J_\sigma\otimes_\Lambda J_\tau\otimes_\Lambda J_\gamma & \xrightarrow{I\otimes f_{\tau,\gamma}} & J_\sigma\otimes_\Lambda J_{\tau\gamma} \\ \downarrow{\scriptstyle f_{\sigma,\tau}\otimes I} & & \downarrow{\scriptstyle f_{\sigma,\tau\gamma}} \\ J_{\sigma\tau}\otimes_\Lambda J_\gamma & \xrightarrow{f_{\sigma\tau,\gamma}} & J_{\sigma\tau\gamma} \end{array}$$

for every $\sigma, \tau, \gamma\in G$, then we call f to be factor set related to Φ. Put $\Delta(f, \Lambda, \Phi, G)=\sum_{\sigma\in G}\oplus J_\sigma$ as Λ-Λ-bimodule. When the multiplication of elements in $\Delta(f, \Lambda, \Phi, G)$ is defined by $x\cdot y=f_{\sigma,\tau}(x\otimes y)$ for $x\in J_\sigma$, $y\in J_\tau$, we call $\Delta(f, \Lambda, \Phi, G)$ a generalized crossed product of Λ and G with factor set f related to Φ.

Proposition 1. *Let G be a group and Λ a k-algebra. For a homomorphism $\Phi: G\to Pic_k(\Lambda)$ and a factor set $f=\{f_{\sigma,\tau}; \sigma, \tau\in G\}$ related to Φ, generalized crossed product $\Delta(f, \Lambda, \Phi, G)$ is an associative k-algebra with identity element, and $\Delta(f, \Lambda, \Phi, G)$ contains a subring isomorphic to Λ, i.e. if $\Phi(\sigma)=[J_\sigma]$ for $\sigma\in G$, $J_1\approx\Lambda$ as k-algebra and Λ-Λ-bimodule.*

Proof. Let $\Phi(\sigma)=[J_\sigma]$, $\sigma\in G$. Since $f_{1,1}: J_1\otimes_\Lambda J_1\to J_1$ is Λ-Λ-isomorphism, J_1 is a subring of $\Delta(f, \Lambda, \Phi, G)$. Since $\Phi(1)=[\Lambda]=[J_1]$, $J_1\approx\Lambda$ as Λ-Λ-bimodules. There exists u in J_1 such that $J_1=\Lambda u=u\Lambda$ and $\lambda u=u\lambda$ for all $\lambda\in\Lambda$. Since $f_{1,1}(J_1\otimes J_1)=J_1$, we can write $f_{1,1}(u\otimes u)=cu$ for some c in Λ, then c is a unit in the center of Λ. If we put $e=c^{-1}u$, then $f_{1,1}(e\otimes e)=e$, so the map $\Lambda\to J_1$: $\lambda\to\lambda e$ is a ring isomorphism. Furthermore, e is identity of $\Delta(f, \Lambda, \Phi, G)$. Because, for any $x\in J_\sigma$, $\sigma\in G$, there is y in J_σ such that $x=f_{1,\sigma}(e\otimes y)$, and $f_{1,\sigma}(e\otimes x)=f_{1,\sigma}(e\otimes f_{1,\sigma}(e\otimes y))=f_{1,\sigma}(f_{1,1}(e\otimes e)\otimes y)=f_{1,\sigma}(e\otimes y)=x$. Similarly, we have $f_{\sigma,1}(x\otimes e)=x$ for every $x\in J_\sigma$, $\sigma\in G$. Therefore, e is identity element of $\Delta(f, \Lambda, \Phi, G)$.

Now, in the following, we may regard $\Lambda=J_1$ in $\Delta(f, \Lambda, \Phi, G)$.

REMARK 1. Let Λ be a k-algebra and G a group. Let $\Phi: G\to Pic_k(\Lambda)$ be a homomorphism, and let the image of Φ consists of $[P]$ in $Pic_k(\Lambda)$ such that P is left Λ-free module. Then for any factor set f related to Φ, $\Delta(f, \Lambda, \Phi, G)$ coincides with an ordinary crossed product $\Delta(\rho, \Lambda, G)$ with a factor set ρ

contained in $Z^2(G, \Lambda^*)$, where Λ^* is the multiplicative group of unit in Λ.

REMARK 2. In Remark 1, in particular, let $\Phi(G)=(1)$, so $\Delta(f, \Lambda, \Phi, G)$ is an ordinary group ring of Λ and G with a factor set in $z^2(G, C^*)$, where C^* is the group of units in the center of Λ.

REMARK 3. Let $\Lambda \supset k$ be a central Galois extension with finite Galois group G (cf. [9]). Then there exists a homomorphism Φ: $G \to Pic_k(k)$ and a factor set f related to Φ such that $\Delta(f, k, \Phi, G) \approx \Lambda$ as k-algebras (see [9]).

2. Generalized crossed product for a Galois extension

Let L be a commutative k-algebra with identity, $Aut_k(L)$ the group of all k-algebra automorphisms of L. Then we have the homomorphism Ψ: $Pic_k(L) \to Aut_k(L)$ defined by $\Psi([P])=\sigma_P$ for $[P]\in Pic_k(L)$, where σ_P is defined by $\sigma_P(a)x=xa$ for all $a\in L$, $x\in P$ (cf. [4], p. 80). We put $Pic_L(L)=P(L)$. Then for $[P]\in P(L)$, P is regarded as new L-L-bimodule by new operation $*$ defined by $a*x=\sigma^{-1}(a)x=x\sigma^{-1}(a)$ and $x*a=xa$ (or $a*x=ax$, $x*a=x\sigma^{-1}(a)=\sigma^{-1}(a)x$) for all $a\in L$ and $x\in P$. We denote it by ${}_\sigma P_I$ (or ${}_I P_\sigma$). If $[P]\in P(L)$ and $\sigma\in Aut_k(L)$, then $[{}_\sigma P_I]$ is in $Pic_k(L)$ and $\Psi([{}_\sigma P_I])=\sigma$. Since the map Φ_0:$Aut_k(L)\to Pic_k(L)$ defined by $\Phi_0(\sigma)=[{}_\sigma L_I]$ is a homorphism and satisfies $\Psi\circ\Phi_0=I_{Aut_k(L)}$, we have the following right split exact sequence;

$$(1) \to P(L) \to Pic_k(L) \to Aut_k(L) \to (1)\,, \qquad \text{(cf. [4], p. 80).}$$

Now, we assume that $L\supset k$ is a Galois extension with finite Galois group G. Then $G\subset Aut_k(L)$. Since $P(L)$ is an abelian and normal subgroup of $Pic_k(L)$, for each $\sigma\in G$, σ defines the automorphism of $P(L)$ by $[P]^\sigma=[{}_\sigma L_I]\cdot[P]\cdot[{}_\sigma L_I]^{-1}$. If we put $P^\sigma={}_\sigma L_I\otimes_L P\otimes_L {}_{\sigma^{-1}}L_I$, $[P^\sigma]=[P]^\sigma$ in $P(L)$ for $\sigma\in G$. Let $\mathfrak{G}$ be the set of all homomorphisms Φ: $G\to Pic_k(L)$ such that $\Psi\circ\Phi=I_G$. Since $\Phi_0\in\mathfrak{G}$, each Φ in $\mathfrak{G}$ determines a function φ of G into $P(L)$ such that $\Phi(\sigma)=\varphi(\sigma)\cdot\Phi_0(\sigma)$ for all $\sigma\in G$. Using Φ and Φ_0 to be group homomorphisms, we can easily check that $\varphi(\sigma\tau)=\varphi(\sigma)\cdot\varphi(\tau)^\sigma$ for every σ, $\tau\in G$. This means that φ is contained in 1–cocycle group $Z^1(G, P(L))$. Conversely, for any φ in $Z^1(G, P(L))$, putting $\Phi=\varphi\Phi_0$, i.e. $\Phi(\sigma)=\varphi(\sigma)\cdot\Phi_0(\sigma)$ for all $\sigma\in G$, we see that Φ is a group homomorphism of G into $Pic_k(L)$ and Φ is in $\mathfrak{G}$. Therefore, between $\mathfrak{G}$ and $Z^1(G, P(L))$ there exists the one to one correspondence $\Phi=\varphi\Phi_0\longleftrightarrow\varphi$. For $\Phi=\varphi\Phi_0$ and $\Phi'=\varphi'\Phi_0$ in $\mathfrak{G}$, we denote $(\varphi\cdot\varphi')\Phi_0$ by $\Phi\cdot\Phi'$. Then under this multiplication in $\mathfrak{G}$, $\mathfrak{G}$ is isomorphic to $Z^1(G, P(L))$.

REMARK 4. For any factor set f related to Φ_0, by Remark 1 $\Delta(f, L, \Phi_0, G)$ is an ordinary crossed product $\Delta(\rho, L, G)$ with a factor set ρ in $Z^2(G, L^*)$, i.e. $\Phi_0(\sigma)=[{}_\sigma L_I]$ and it has some L-free base $\{u_\sigma;\ \sigma\in G\}$ such that ${}_\sigma L_I=Lu_\sigma$, $\sigma(x)u_\sigma=u_\sigma x$ for all $x\in L$ and $u_\sigma u_\tau=\rho(\sigma, \tau)u_{\sigma\tau}$.

Proposition 2. *Let $L\supset k$ be a Galois extension with Galois group G. For any $\Phi\in\mathfrak{G}$ such that there is a fator set f rerated to Φ, $\Delta(f, L, \Phi, G)$ is an Azumaya k-algebra (i.e. central separable), with maximal commutative subalgebra L.*

Proof. We put $\Delta=\Delta(f, L, \Phi, G)=\sum_{\sigma\in G}\oplus J_{\sigma}$, where $[J_{\sigma}]=\Phi(\sigma)$, $\sigma\in G$. At first, we shall show that $L=J_1$ is a maximal commutative subring of $\Delta(f, L, \Phi, G)$. The commutor ring $V_{\Delta}(L)$ of L in Δ contains L. On the other hand, if z is in $V_{\Delta}(L)$, then z can be written as $z=\sum_{\sigma\in G}z_{\sigma}$ for some z_{σ} in J_{σ}, and so $\sum_{\sigma\in G}az_{\sigma}=az$ $=za=\sum_{\sigma\in G}z_{\sigma}a=\sum_{\sigma\in G}\sigma(a)z_{\sigma}$, for all $a\in L$. Therefore, we have $az_{\sigma}=\sigma(a)z_{\sigma}$ for every $a\in L$ and $\sigma\in G$. But, since $L\supset k$ is Galois extension, there exist $a_1, a_2,\cdots, a_n$ and $b_1, b_2,\cdots, b_n$ in L such that $\sum_{i=1}^{n}a_i\sigma(b_i)=\begin{cases}1, & \sigma=I\\ 0, & \sigma\neq I\end{cases}$. Accordingly, $z_{\sigma}=\sum_i a_ib_iz_{\sigma}=$ $\sum_i a_i\sigma(b_i)z_{\sigma}=0$ for $\sigma\neq I$. Therefore, we have $z\in J_I=L$ and $V_{\Delta}(L)=L$. In other words, L is a maximal commutative subalgebra of $\Delta(f, L, \Phi, G)$. Secondly, we shall show that k is the center of $\Delta(f, L, \Phi, G)$. Since $V_{\Delta}(\Delta)\subset V_{\Delta}(L)=L$, for any $a\in V_{\Delta}(\Delta)$, we have $ax=\sigma(a)x$ for every $x\in J_{\sigma}$ and every $\sigma\in G$. Since J_{σ} is faithful L-module, $a=\sigma(a)$ for every $\sigma\in G$, therefore $a\in L^G=k$. Accordingly, k is the center of Δ. Finally, we shall show that $\Delta(f, L, \Phi, G)$ is separable over k. Since Δ is a finitely generated projective k-module, by [7], Proposition 1.1 Δ is separable over k if and only if $\Delta\otimes_k k_{\mathfrak{m}}$ is separable over $k_{\mathfrak{m}}$ for all maximal ideal $\mathfrak{m}$ of k. Therefore, we may work with $\Delta(f_{\mathfrak{m}}, L_{\mathfrak{m}}, \Phi_{\mathfrak{m}}, G)=\Delta\otimes_k k_{\mathfrak{m}}$, i.e. we may assume that k is local, so L is semi-local. Then every finitely generated rank 1 projective L-module is free, so Φ coincides with Φ_0. Therefore, $\Delta(f, L, \Phi_0, G)$ is an ordinary crossed product, hence by [1], Theorem A. 12, $\Delta(f, L, \Phi, G)$ is separable over k. This completes the proof.

Proposition 3. *Let $L\supset k$ be a Galois extension with Calois group G, and let Λ be an Azumaya k-algebra containing L as a maximal commutative subalgebra. Then Λ is L-isomorphic to a generalized crossed product of L and G with some $\Phi\in\mathfrak{G}$ and some factor set f related to Φ, as k-algebra.*

Proof. For each $\sigma\in G$, we put $J_{\sigma}={}_{\sigma^{-1}}\Lambda_I{}^L=\{a\in\Lambda;\ \sigma(x)a=ax, \text{ for all } x\in L\}$, then, regarding Λ and ${}_{\sigma^{-1}}\Lambda_I$ as $L\otimes_k\Lambda^0$-left module, $J_{\sigma}\approx\mathrm{Hom}_{L\otimes_k\Lambda^0}(\Lambda, {}_{\sigma^{-1}}\Lambda_I)$. Since Λ is a faithful $L\otimes_k\Lambda^0$-left module and $L\otimes_k\Lambda^0$ is a separable k-algebra, it follows from [8], Theorem 1 that Λ is finitely generated projestive generator as an $L\otimes_k\Lambda^0$-left module, and $\mathrm{Hom}_{L\otimes_k\Lambda^0}(\Lambda, \Lambda)=L$. Accordingly, we have $J_{\sigma}\otimes_L\Lambda\approx\mathrm{Hom}_{L\otimes_k\Lambda^0}(\Lambda, {}_{\sigma^{-1}}\Lambda_I)\otimes_L\Lambda\approx{}_{\sigma^{-1}}\Lambda_I$ as left L- and right Λ-modules. Therefore, we obtain $[J_{\sigma}]\in P_{ick}(L)$ and $J_{\sigma}\Lambda=\Lambda$. Using the inclusion map $J_{\sigma}\rightarrow\Lambda$, we define the L-L-homomorphism $\theta:\sum_{\sigma\in G}\oplus J_{\sigma}\rightarrow\Lambda$; $\theta(\sum_{\sigma\in G}x_{\sigma})=\sum x_{\sigma}$ in Λ, for $x_{\sigma}\in J_{\sigma}$. In order to show that θ is an isomorphism it suffices to show that for every maximal ideal $\mathfrak{m}$ of k, the localized map $\theta_{\mathfrak{m}}: \sum\oplus(J_\sigma)_{\mathfrak{m}} \to \Lambda_{\mathfrak{m}}$ is isomorphism. Therefore, we may suppose that k is a local ring, so L is a semi-local ring. Then J_σ is a free L-module of rank 1; there is u_σ in J_σ such that $J_\sigma=u_\sigma L=Lu_\sigma$. Since $\Lambda=u_\sigma\Lambda$, and $u_\sigma\Lambda$ is Λ-free, u_σ is a unit in Λ, and σ is extended to an inner automorphism induced by u_σ. Therefore, we obtain from [1], Theorem A. 13 that Λ is isomorphic to an ordinary crossed product $\Delta(\rho, \Lambda, G)=\sum_{\sigma\in G}\oplus\Lambda u_\sigma$. Consequently, θ is an isomorphism. Since $J_\sigma\cdot J_\tau\subset J_{\sigma\tau}$ and for every maximal ideal $\mathfrak{m}$ of k $(J_\sigma J_\tau)_{\mathfrak{m}}=(J_\sigma)_{\mathfrak{m}}(J_\tau)_{\mathfrak{m}}=(J_{\sigma\tau})_{\mathfrak{m}}$, we obtain $J_\sigma\otimes_L J_\tau\approx J_\sigma J_\tau=J_{\sigma\tau}$. If we define Φ: $G\to Pic_k(L)$ by $\Phi(\sigma)=[J_\sigma]$ for each $\sigma\in G$, and $f_{\sigma,\tau}$: $J_\sigma\otimes_L J_\tau\to J_{\sigma\tau}$ by $f_{\sigma,\tau}(x\otimes y)=xy$ for each σ, $\tau\in G$, then Φ is in $\mathfrak{G}$ and $f=\{f_{\sigma,\tau};\ \sigma,\ \tau\in G\}$ is a factor set related to Φ, and we obtain that Λ and $\Delta(f, L, \Phi, G)$ are k-algebra isomorphic and L-isomorphic.

Proposition 4. *Let $L\supset k$ be a Galois extension with Galois group G, and let Φ be an element in $\mathfrak{G}$. If $f=\{f_{\sigma,\tau};\ \sigma,\ \tau\in G\}$ and $g=\{g_{\sigma,\tau};\ \sigma,\ \tau\in G\}$ are factor sets related to Φ, then there is a cocycle ρ in $Z^2(G, L^*)$ such that $g=\rho f$, i.e. $g_{\sigma,\tau}(x\otimes y)=\rho(\sigma, \rho)\cdot f_{\sigma,\tau}(x\otimes y)$ for $x\otimes y\in J_\sigma\otimes_L J_\tau$, $\sigma, \tau\in G$, where L^* is a multiplicative group of units in L, and $\Phi(\sigma)=[J_\sigma]$ for $\sigma\in G$. Furthermore, $\Delta(f, L, \Phi, G)$ is L-isomorphic to $\Delta(\rho f, L, \Phi, G)$ as k-algebra if and only if ρ is in $B^2(G, L^*)$.*

Proof. Let $\Phi(\sigma)=[J_\sigma]$, $\sigma\in G$. Since $f_{\sigma,\tau}$ and $g_{\sigma,\tau}$ are isomorphisms of $J_\sigma\otimes_L J_\tau$ to $J_{\sigma\tau}$ for $\sigma, \tau\in G$, $g_{\sigma,\tau}\circ f^{-1}_{\tau,\sigma}$ is an automorphism of $J_{\sigma\tau}$, so there exists a unit $\rho(\sigma, \tau)$ in $\mathrm{Hom}_L(J_{\sigma\tau}, J_{\sigma\tau})=L$ such that $g_{\sigma,\tau}(x\otimes y)=\rho(\sigma, \tau)\cdot f_{\sigma,\tau}(x\otimes y)$ for every $x\otimes y\in J_\sigma\otimes_L J_\tau$. Since f and g are factor set related to Φ, we can check easily that ρ is in $Z^2(G, L^*)$. We write $g=\rho f$. If h: $\Delta(f, L, \Phi, G)\to\Delta(\rho f, L, \Phi, G)$ is a L-isomorphism as k-algebra, then $h(J_\sigma)=J_\sigma$ for each $\sigma\in G$. Because for any $x\in J_\sigma$, one can write $h(x)=\sum_{\tau\in G} z_\tau$ for z_τ in J_τ, so

$$\sum_{\tau\in G}\tau(a)z_\tau=\sum_\tau z_\tau a=h(x)a=h(\sigma(a)x)=\sigma(a)h(x)=\sum_{\tau\in G}\sigma(a)z_\tau\,.$$

Therefore, $\tau(a)z_\tau=\sigma(a)z_\tau$ for all $a\in L$ and each $\tau\in G$. If we take $a_1, a_2,\cdots, a_n$, $b_1, b_2,\cdots, b_n$ in L such that $\sum_i a_i\gamma(b_i)=\begin{cases}1;\ \gamma=I\\0;\ \gamma\neq I,\end{cases}$ $\gamma\in G$, then $z_\tau=\sum_i a_ib_iz_\tau=\sum_i\tau(a_i)\tau(b_i)z_\tau=\sum_i\tau(a_i)\sigma(b_i)z_\tau=\tau(\sum_i a_i\tau^{-1}\sigma(b_i))z_\tau=0$ for $\tau\neq\sigma$. Thus we have $h(x)\in J_\sigma$. Therefore $h(J_\sigma)=J_\sigma$ and so, for each $\sigma\in G$, the isomorphism h determies the element d_σ in L^* such that $h(x)=d_\sigma x$ for all $x\in J_\sigma$. Since h is L-isomorphism, $d_I=1$. Since h is ring-isomorphism, $h(f_{\sigma,\tau}(x\otimes y))=d_{\sigma,\tau}\cdot f_{\sigma,\tau}(x\otimes y)=\rho(\sigma, \tau)\cdot f_{\sigma,\tau}(h(x), h(y))=\rho(\sigma, \tau)\cdot d_\sigma\cdot\sigma(d_\tau)f_{\sigma,\tau}(x\otimes y)$ for all $x\otimes y\in J_\sigma\otimes J_\tau$. Accordingly, $\rho(\sigma, \tau)=d_{\sigma\tau}\cdot d_\sigma^{-1}\cdot\sigma(d_\tau)^{-1}$ for $\sigma, \tau\in G$, hence ρ is in $B^2(G, L^*)$. Conversely, if ρ is in $B^2(G, L^*)$, there exists $\{d_\sigma;\ \sigma\in G\}$ in L^* such that $\rho(\sigma, \tau)=d_{\sigma\tau}\cdot d_\sigma^{-1}\cdot\sigma(d_\tau)^{-1}$ for $\sigma, \tau\in G$. If one take $d_I=1$, the map

$h\colon \Delta(f, L, \Phi, G)=\sum_{\sigma\in G}\oplus J_\sigma\to\Delta(\rho f, L, \Phi, G)=\sum_{\sigma\in G}\oplus J_\sigma$ defined by $h(x)=d_\sigma x$ for $x\in J_\sigma$ and $\sigma\in G$, is L-isomorphism as k-algebra.

Lemma 1. *Let $L\supset k$ be a Galois extension with Galois group G, $[P]$ an element of $P(L)$. Then the following conditions are equivalemt;*

1) *$\mathrm{Hom}_k(P,P)$ is L-isomorphic to $\Delta(L,G)$ as k-algebra, where $\Delta(L,G)$ means the ordinary crossed product with trivial factor set.*

2) *There is an element $[P_0]$ in $P(k)$ such that $[P]=[P_0\otimes_k L]$ in $P(L)$.*

Proof. 1)$\to$2); Since L is a Galois extension of k, L is finitely generated projective generator as a $\Delta(L, G)$-module, and $\mathrm{Hom}_{\Delta(L,G)}(L, L)=k$. Regarding P as $\Delta(L, G)$-module, we have $P\approx\mathrm{Hom}_{\Delta(L,G)}(L, P)\otimes_k L$. Since P is a finitely generated projective L-module of rank 1, $P_0=\mathrm{Hom}_{\Delta(L,G)}(L, P)$ is a finitely generated projective k-module of rank 1, so $[P_0]\in P(k)$ and $[P_0\otimes_k L]=[P]$.

2)$\to$1); If $[P_0]\in P(k)$ and $[P]=[P_0\otimes_k L]$, then $\mathrm{Hom}_k(P, P)\approx\mathrm{Hom}_k(P_0\otimes_k L, P_0\otimes_k L)\approx\mathrm{Hom}_k(P_0, P_0)\otimes_k\mathrm{Hom}_k(L, L)\approx k\otimes_k\Delta(L, G)\approx\Delta(L, G)$ as L-modules and k-algebras.

REMRAK 5. Let $L\supset k$ be a trivial Galois extension with Galois group G, i.e. $L=\sum_{\sigma\in G}\oplus ke_\sigma$, $\sum_\sigma e_\sigma=1$, $e_\sigma\cdot e_\tau=\begin{cases}e_\sigma; & \sigma=\tau\\ 0; & \sigma\neq\tau,\end{cases}$ and $\sigma(e_1)=e_\sigma$, $ke_\sigma\approx k$ as k-algebra, for $\sigma\in G$. Then $P(L)^G=Im(P(k)\to P(L))$ where $P(k)\to P(L)$ is defined by $[P_0]\leadsto[P_0\otimes_k L]$, and $P(L)^G=\{[P]\in P[L];\ [P]^\sigma=[P]$ forall $\sigma\in G\}$.

Proof. Let $[P]\in P(L)^G$, so ${}_\sigma L_I\otimes_L P\approx P\otimes_L{}_\sigma L_I$ as L-L-bimodule, for all $\sigma\in G$. Since $L=\sum_{\sigma\in G}\oplus e_\sigma k$, we have $P=\sum_{\sigma\in G}\oplus e_\sigma P$. Then $e_\sigma P$ and $e_\tau P$ are k-isomorphic for every $\sigma,\tau\in G$. Because, from the L-L-isomorphism $h_\sigma\colon {}_\sigma L_I\otimes_L P=\sum_{\tau\in G}\oplus\sigma(e_\tau){}_\sigma L_I\otimes_L P\to P\otimes_L{}_\sigma L_I=\sum_{\tau\in G}\oplus e_\tau P\otimes_L{}_\sigma L_I$, we obtain the L-L-isomorphism $\sigma(e_\tau){}_\sigma L_I\otimes_L P={}_\sigma L_I\otimes_L e_\tau P\to e_{\sigma\tau}P\otimes_L{}_\sigma L_I$, for each σ and τ in G. Since ${}_\sigma L_I\otimes_L e_\tau P$ and $e_\tau P$ are k-isomorphic, and $e_{\sigma\tau}P$ and $e_{\sigma\tau}P\otimes_L{}_\sigma L_I$ are k-isomorphic, therefore $e_\tau P$ and $e_{\sigma\tau}P$ are k-isomorphic for every $\sigma,\tau\in G$. Since $[P]\in P(L)$, $P=\sum_{\sigma\in G}\oplus e_\sigma P$ and $(e_1P)_{\mathfrak{m}}\approx(e_\sigma P)_{\mathfrak{m}}$ for all maximal ideal $\mathfrak{m}$ of k, we obtain $[e_1P]\in P(k)$. Now, we shall show $L\otimes_k e_1P\approx P$ as L-module. Let h_σ' be the k-isomorphism of $e_\sigma P$ to e_1P obtained above, for each $\sigma\in G$. We defined the map $h\colon P\to L\otimes_k e_1P=\sum_{\sigma\in G}\oplus e_\sigma k\otimes_k e_1P$ by $h(x)=\sum_{\sigma\in G}e_\sigma\otimes h_\sigma'(e_\sigma x)$. Then $h(e_\tau x)=\sum_{\sigma\in G}e_\sigma\otimes h_\sigma'(e_\sigma e_\tau x)=e_\tau\otimes h_\tau'(e_\tau x)=e_\tau(\sum_{\sigma\in G}e_\sigma\otimes h_\sigma'(e_\sigma x))=e_\tau h(x)$, therefore h is L-isomorphism. We obtain $[e_1P]\in P(k)$ and $[P]=[L\otimes_k e_1P]$.

Proposition 5. *Let $L\supset k$ be a Galois extension with Galois group G. Let Φ be an element in $\mathfrak{G}$ such that there exists a factor set f related to Φ and there is* *a finitely generated faithful projective k-module P which satisfies* $\Delta(f, L, \Phi, G) \approx \mathrm{Hom}_k(P, P)$ *as k-algebras. Then,* 1) $[P]$ *is in* $P(L)$, 2) *we have* $\Phi(\sigma)\cdot[P] = [P]\cdot\Phi_0(\sigma)$ *for all* $\sigma \in G$ *i.e.* $\Phi = \varphi\cdot\Phi_0$ *and* $\varphi(\sigma) = [P]\cdot([P]^{-1})^\sigma$ *for all* $\sigma \in G$.

Proof. Since L is a maximal commutative subalgebra of $\Delta(f, L, \Phi, G)$, regarding P as L-module, $L = V_{\mathrm{Hom}_k(P,P)}(L) = \mathrm{Hom}_L(P, P)$. Since L is separable over k, P is a finitely generated projective L-module, so $[P]$ is contained in $P(L)$. We put $\Phi(\sigma) = [J_\sigma]$ for $\sigma \in G$. Then from the proof of Proposition 3 we obtain $J_\sigma = {}_{\sigma^{-1}}(\mathrm{Hom}_k(P, P))_I{}^L = \{f \in \mathrm{Hom}_k(P, P);\ \sigma(a)f(x) = f(ax)$ for all $x \in P,\ a \in L\}$. We shall show the map θ; ${}_{\sigma^{-1}}(\mathrm{Hom}_k(P, P))_I{}^L \otimes_L P \to P \otimes_L {}_\sigma L_I = P \otimes L u_\sigma$, defined by $\theta(f \otimes x) = f(x) \otimes u_\sigma$, is an L-L-isomorphism, where u_σ is a base of ${}_\sigma L_I$. Since $\theta(f \otimes xa) = f(xa) \otimes u_\sigma = f(ax) \otimes u_\sigma = \sigma(a)f(x) \otimes u_\sigma = f(x) \otimes \sigma(a)u_\sigma = f(x) \otimes u_\sigma a$ and $\theta(af \otimes x) = af(x) \otimes u_\sigma$ for $a \in L$, $x \in P$, so θ is a L-L-homomorphism. In order to show that is θ isomorphism, it suffices to show that for every maximal ideal $\mathfrak{m}$ of k $\theta_\mathfrak{m}$: $({}_{\sigma^{-1}}(\mathrm{Hom}_k(P, P))_I{}^L \otimes_L P)_\mathfrak{m} \to (P \otimes_L {}_\sigma L_I)_\mathfrak{m}$ is an isomorphism. But, $L_\mathfrak{m} = L \otimes_k k_\mathfrak{m}$ is semi-local and $({}_{\sigma^{-1}}(\mathrm{Hom}_k(P, P))_I{}^L)_\mathfrak{m} = {}_{\sigma^{-1}}(\mathrm{Hom}_{k_\mathfrak{m}}(P_\mathfrak{m}, P_\mathfrak{m}))_I{}^{L_\mathfrak{m}}$ is free $L_\mathfrak{m}$-module generated by a unit f in $\mathrm{Hom}_{k_\mathfrak{m}}(P_\mathfrak{m}, P_\mathfrak{m})$. Therefore $\theta_\mathfrak{m}$ is a homomorphism of $L_\mathfrak{m} f \otimes_{L_\mathfrak{m}} P_\mathfrak{m}$ to $P_\mathfrak{m} \otimes_{L_\mathfrak{m}} L_\mathfrak{m} u_\sigma$defined by $\theta_\mathfrak{m}(f \otimes x) = f(x) \otimes u_\sigma$. Since f is an automorphism of $P_\mathfrak{m}$, we obtain that $\theta_\mathfrak{m}$ is isomorphism. Thus, we obtain $J_\sigma \otimes_L P \approx P \otimes_L {}_\sigma L_I$, so $\Phi(\sigma)\cdot[P] = [P]\cdot\Phi_0(\sigma)$, $\sigma \in G$.

Corollary 1. *Let* $L \supset k$ *be a Galois extension with Galois group* G, *and* $[P]$ *an elemant of* $P(L)$. *Then* $\mathrm{Hom}_k(P, P)$ *is L-isomorphic to a generalized crossed product* $\Delta(f, L, \Phi_0, G)$ *of L and G with some factor set f related to* Φ_0 *as k-algebra, if and only if* $[P]$ *is contained in* $P(L)^G$.

Proof. If $\mathrm{Hom}_k(P, P) \approx \Delta(f, L, \Phi_0, G)$, then by Proposition 5, 2) we obtain $[P] = [P]^\sigma$ for all $\sigma \in G$, so $[P] \in P(L)^G$. Conversely, let $[P] \in P(L)^G$. Since $\mathrm{Hom}_k(P, P)$ is an Azumaya k-algebra with maximal commutative subalgebra L, $\mathrm{Hom}_k(P, P)$ is written by $\Delta(f, L, \Phi, G)$ for some Φ and f. Therefore, by Proposition 5, 2) we have $\Phi(\sigma)\cdot[P] = [P]\Phi_0(\sigma)$ and so $[P]^\sigma\Phi(\sigma) = [P]\Phi_0(\sigma)$. Accordingly $\Phi(\sigma) = \Phi_0(\sigma)$ for all $\sigma \in G$, i.e. $\Phi = \Phi_0$.

Proposition 6. *Let* $L \supset k$ *be a Galois extension with Galois group* G. *For any* $\Phi = \varphi\Phi_0 \in \mathfrak{G}$ *with some factor set f related to* Φ, $\Delta(f, L, \Phi, G)$ *has an opposite k-algebra* $\Delta(f, L, \Phi, G)^0 = \Delta(f^0, L, \Phi^0, G)$ *where* $\Phi^0 = \varphi^{-1}\Phi_0$ *and* f^0 *is some factor set related to* Φ^0.

Proof. Put $\Phi(\sigma) = [J_\sigma]$, $\varphi(\sigma) = [P_\sigma]$ and $\Phi^0(\sigma) = \varphi(\sigma)^{-1}\cdot\Phi_0(\sigma) = [P_\sigma{}^* \otimes_L {}_\sigma L_I] = [J_\sigma']$ for $\sigma \in G$, where $P_\sigma{}^* = \mathrm{Hom}_L(P_\sigma, L)$. Since $1 = \varphi(1) = \varphi(\sigma\sigma^{-1}) = \varphi(\sigma)\cdot\varphi(\sigma^{-1})^\sigma$, we have $[P_\sigma] = \varphi(\sigma) = (\varphi(\sigma^{-1})^{-1})^\sigma = [P_{\sigma^{-1}}^*]^\sigma$. Thus P_σ and $(P_{\sigma^{-1}}^*)^\sigma = {}_\sigma L_I \otimes_L P_{\sigma^{-1}}^* \otimes_L {}_{\sigma^{-1}} L_I$ are L-L-isomorphic. Let h_σ: $P_\sigma \to (P_{\sigma^{-1}}^*)^\sigma$ be the L-L-isomorphism, and let g_σ: $(P_{\sigma^{-1}}^*)^\sigma = Lu_\sigma \otimes_L P_{\sigma^{-1}}^* \otimes_L Lu_{\sigma^{-1}} \to P_{\sigma^{-1}}^*$ be a k-isomor-

phism defined by $g_\sigma(u_\sigma \otimes x \otimes u_{\sigma^{-1}}) = x$. Then $g_\sigma \circ h_\sigma$ is a k-isomorphism satisfying $g_\sigma \circ h_\sigma(ax) = \sigma^{-1}(a) g_\sigma \circ h_\sigma(x)$ for all $x \in P_\sigma$ and $a \in L$. For each $\sigma \in G$, we define the map g: $J_\sigma = P_\sigma \otimes_L {}_\sigma L_I \to J'_{\sigma^{-1}} = P^*_{\sigma^{-1}} \otimes_L {}_{\sigma^{-1}} L_I$ as follows: For $x \otimes au_\sigma \in P_\sigma \otimes_L {}_\sigma L_I = P_\sigma \otimes_L Lu_\sigma$, $g(x \otimes au_\sigma) = g_\sigma \circ h_\sigma(x) \otimes \sigma^{-1}(a) u_{\sigma^{-1}}$. It is easily checked that g is well defined. Then the map g induces the k-isomorphism of $\sum_{\sigma \in G} \oplus J_\sigma$ to $\sum_{\sigma \in G} \oplus J_\sigma'$, and satisfies $g(x \otimes ya) = g(x \otimes \sigma(a) y) = g(\sigma(a) x \otimes y) = ag(x \otimes y)$ and $g(ax \otimes y) = g(x \otimes ay) = g(x \otimes y)a$ for all $x \otimes y \in P_\sigma \otimes_L {}_\sigma L_I = J_\sigma$ and $a \in L$.
Now, we define the map $f^0_{\sigma,\tau}$: $J_\sigma' \otimes J_\tau' \to J_{\sigma\tau}'$ as follows:

$$f^0_{\sigma,\tau}(x' \otimes y') = g(f_{\tau^{-1}\sigma^{-1}}(g^{-1}(y') \otimes g^{-1}(x'))) \qquad \text{for} \quad x' \otimes y' \in J_\sigma' \otimes J_\tau' .$$

Then $f^0_{\sigma,\tau}$ is L-L-isomorphism. Because, $f^0_{\sigma,\tau}(ax' \otimes y') = g(f_{\tau^{-1},\sigma^{-1}}(g^{-1}(y') \otimes g^{-1}(ax'))) = g(f_{\tau^{-1},\sigma^{-1}}(g^{-1}(y') \otimes g^{-1}(x')a)) = g(f_{\tau^{-1},\sigma^{-1}}(g^{-1}(y') \otimes g^{-1}(x'))a) = a \cdot g(f_{\tau^{-1},\sigma^{-1}}(g^{-1}(y') \otimes g^{-1}(x'))) = af^0_{\sigma,\tau}(x \otimes y)$, similarly, $f^0_{\sigma,\tau}(x' \otimes y'a) = f^0_{\sigma,\tau}(x' \otimes y')a$ for all $x' \otimes y' \in J_\sigma' \otimes J_\tau'$, $a \in L$. Furthermore, $f^0 = \{f^0_{\sigma,\tau};\ \sigma, \tau \in G\}$ is a factor set related to $\Phi^0 = \varphi^{-1} \Phi_0$;

$$\begin{aligned}
f^0_{\sigma\tau,\gamma}(f^0_{\sigma,\tau}(x' \otimes y') \otimes z') &= g(f_{\gamma^{-1},(\sigma\tau)^{-1}}(g^{-1}(z') \otimes g^{-1}(f^0_{\sigma,\tau}(x' \otimes y')))) \\
&= g(f_{\gamma^{-1},\tau^{-1}\sigma^{-1}}(g^{-1}(z') \otimes f_{\tau^{-1},\sigma^{-1}}(g^{-1}(y') \otimes g^{-1}(x'))) \\
&= g(f_{\gamma^{-1}\tau^{-1},\sigma^{-1}}(f_{\gamma^{-1},\tau^{-1}}(g^{-1}(z) \otimes g^{-1}(y')) \otimes g^{-1}(x'))) \\
&= g(f_{(\tau\gamma)^{-1},\sigma^{-1}}(g^{-1}(f^0_{\tau,\gamma}(y' \otimes z')) \otimes g^{-1}(x'))) \\
&= f^0_{\sigma,\tau\gamma}(x' \otimes f^0_{\tau,\gamma}(y' \otimes z')), \qquad \text{for} \quad x' \otimes y' \otimes z' \in J_\sigma' \otimes J_\tau' \otimes J_\gamma' .
\end{aligned}$$

Therefore, Φ^0 and f^0 define a generalized crossed product $\Delta(f^0, L, \Phi^0, G) = \sum_{\sigma \in G} \oplus J_\sigma'$ of L and G. Since $g(f_{\sigma,\tau}(x \otimes y)) = f^0_{\tau^{-1},\sigma^{-1}}(g(y) \otimes g(x))$, for $x \otimes y \in J_\sigma \otimes J_\tau$, g is an opposite k-algebraisomorphism of $\Delta(f, L, \Phi, G)$ to $\Delta(f^0, L, \Phi^0, G)$.

3. Application to Brauer group. The purpose of this section is to derive the seven terms exact sequence, using the results in §2. We define the maps θ_i in the sequence

$$H^1(G, L^*) \overset{\theta_1}{\to} P(k) \overset{\theta_2}{\to} P(L)^G \overset{\theta_3}{\to} H^2(G, L^*) \overset{\theta_4}{\to} B(L/k) \overset{\theta_5}{\to} H^1(G, P(L)) \overset{\theta_6}{\to} H^3(G, L^*)$$

in the following way: We suppose that L is a Galois extension of k with finite Galois group G.

(1) θ_1: $H^1(G, L^*) \to P(k)$;

Let $\rho \in Z^1(G, L^*)$. We define the new operation of element σ of G on L; for $\sigma \in G$, $x \in L$, $\sigma * x = \rho(\sigma) \cdot \sigma(x)$. Under this operation, we may regard L as $\Delta(L, G)$-left module, then we denote L by ${}_\rho L$. We put $P_0 = {}_\rho L^G = \{a \in L;\ \sigma * a = \rho(\sigma) \cdot \sigma(a) = a \text{ for all } \sigma \in G\} \approx \mathrm{Hom}_{\Delta(L,G)}(L, {}_\rho L)$. Since $L \supset k$ is a Galois extension, L is finitely generated projective generator as a $\Delta(L, G)$-module, so we have ${}_\rho L \approx \mathrm{Hom}_{\Delta(L,G)}(L, {}_\rho L)\otimes_k L$. Since $L\supset k$ is a finitely generated projective k-module, $[P_0]=[{}_\rho L^G]$ is in $P(k)$. We define the map θ_1 by $\theta_1(\bar{\rho})=[P_0]=[{}_\rho L^G]$ for $\bar{\rho}\in H^1(G, L^*)$. It is well defined. Because, if $\rho'=\rho_0\rho$ for $\rho_0\in B^1(G, L^*)$, then there is $\alpha\in L^*$ such that $\rho_0(\sigma)=\alpha^{-1}\cdot\sigma(\alpha)$ for all $\sigma\in G$. Then $P_0'={}_{\rho'}L^G=\{x\in L; x=\alpha^{-1}\sigma(\alpha)\rho(\sigma)\sigma(x)$, for all $\sigma\in G\}=\alpha^{-1}\cdot{}_\rho L^G$. Thus $P_0'\approx P_0$ as k-module, therefore $[P_0']=[P_0]$ in $P(k)$.

Lemma 2. *The map θ_1: $H^1(G, L^*)\to P(k)$ is a monomorphism.*

Proof. In order to show that θ_1 is a homomorphism, it suffices to show that for $\bar{\rho}_1$, $\bar{\rho}_2$ in $H^1(G, L^*)$, ${}_{\rho_1}L^G\otimes_k {}_{\rho_2}L^G\approx {}_{\rho_1\rho_2}L^G$ as k-module. It is easily seen that ${}_{\rho_1}L^G\cdot{}_{\rho_2}L^G\subset{}_{\rho_1\rho_2}L^G$. We consider the map η: ${}_{\rho_1}L^G\otimes_k{}_{\rho_2}L^G\to{}_{\rho_1\rho_2}L^G$ defined by $\eta(x\otimes y)=xy$ for $x\in{}_{\rho_1}L^G$, $y\in{}_{\rho_2}L^G$. Since ${}_{\rho_i}L^G\otimes_k L\approx{}_{\rho_i}L^G\cdot L={}_{\rho_i}L$, for every maximal ideal $\mathfrak{m}$ of k, the localization $({}_{\rho_1}L^G)_\mathfrak{m}$, $({}_{\rho_2}L^G)_\mathfrak{m}$ and $({}_{\rho_1\rho_2}L^G)_\mathfrak{m}$ are rank 1 $k_\mathfrak{m}$-free module and generated by units in $L_\mathfrak{m}$. Therefore, $({}_{\rho_1}L^G)_\mathfrak{m}=k_\mathfrak{m}u_1$, $({}_{\rho_2}L^G)_\mathfrak{m}=k_\mathfrak{m}u_2$ and $({}_{\rho_1}L^G\cdot{}_{\rho_2}L^G)_\mathfrak{m}=({}_{\rho_1}L^G)_\mathfrak{m}\cdot({}_{\rho_2}L^G)_\mathfrak{m}=k_\mathfrak{m}u_1u_2\subset({}_{\rho_1\rho_2}L^G)_\mathfrak{m}=k_\mathfrak{m}u_3$. Since $u_3=(u_3u_2^{-1}u_1^{-1})u_1u_2$ and $u_3u_2^{-1}u_1^{-1}\in L_\mathfrak{m}{}^G=k_\mathfrak{m}$, we have $({}_{\rho_1}L^G\cdot{}_{\rho_2}L^G)_\mathfrak{m}=k_\mathfrak{m}u_1u_2=k_\mathfrak{m}u_3=({}_{\rho_1\rho_2}L^G)_\mathfrak{m}$, so $\eta_\mathfrak{m}:{}_{\rho_1}L^G{}_\mathfrak{m}\otimes_{k_\mathfrak{m}}{}_{\rho_2}L^G{}_\mathfrak{m}\to{}_{\rho_1\rho_2}L^G{}_\mathfrak{m}$ is a $k_\mathfrak{m}$-isomorphism. Accordingly, η is a k-isomorphism, and so θ_1 is a homomorphism. Let $\bar{\rho}\in H^1(G, L^*)$ and $\theta_1(\bar{\rho})=[{}_\rho L^G]=[k]$, i.e. ${}_\rho L^G=k\cdot u$ where u is a free base in ${}_\rho L^G$, and so u is a unit in L. Therefore, $u=\rho(\sigma)\cdot\sigma(u)$ for every $\sigma\in G$, i.e. $\rho(\sigma)=u\cdot\sigma(u)^{-1}$ so ρ is in $B^1(G, L^*)$. Accordingly, θ_1 is a monomorphism.

(2). θ_2: $P(k)\to P(L)^G$;

We put $\theta_2([P_0])=[L\otimes_k P_0]$ for $[P_0]\in P(k)$. Then θ_2 is a homorphism of $P(k)$ to $P(L)^G$ by Lemma 1 and Corollary 1.

Lemma 3. $H^1(G, L^*)\overset{\theta_1}{\to}P(k)\overset{\theta_2}{\to}P(L)^G$ *is exact.*

Proof. For any $\bar{\rho}$ in $H^1(G, L^*)$, $\theta_2\theta_1(\bar{\rho})=\theta_2([{}_\rho L^G])=[{}_\rho L^G\otimes_k L]=[{}_\rho L]=[L]$ in $P(L)$. Let $[P_0]$ be in $P(k)$ and $[P_0\otimes_k L]=[L]$, i.e. there is an L-isomorphism h: $L\to P_0\otimes_k L$. Since $\mathrm{Hom}_k(P_0\otimes_k L, P_0\otimes_k L)\approx\mathrm{Hom}_k(L, L)=\Delta(L, G)=\sum_{\sigma\in G}\oplus Lu_\sigma$, we can regard $P\otimes_k L$ as a faithful $\Delta(L, G)$-module by the isomorphism h. Then Lu_σ is described as $J_\sigma=\{g\in\mathrm{Hom}_k(P\otimes_k L, P\otimes_k L); g\cdot a=\sigma(a)g$, for all $a\in L\}$. The k-isomorphism $\bar{\sigma}=I\otimes\sigma$: $P_0\otimes_k L\to P_0\otimes_k L$ is a unit element in $\mathrm{Hom}_k(P_0\otimes_k L, P_0\otimes_k L)$ and is contained in J_σ for each $\sigma\in G$. Therefore, there exists d_σ in L^* such that $\bar{\sigma}=d_\sigma u_\sigma$ for each $\sigma\in G$. Then, the map ρ: $G\to L^*$ defined by $\rho(\sigma)=d_\sigma$ is in $Z^1(G, L^*)$. Because, $d_{\sigma\tau}=\overline{\sigma\tau}\cdot u_{\sigma\tau}^{-1}=\bar{\sigma}\cdot\bar{\tau}u_\tau^{-1}\cdot u_\sigma^{-1}=\bar{\sigma}\cdot d_\tau u_\sigma^{-1}=\bar{\sigma}u_\sigma^{-1}\sigma(d_\tau)=d_\sigma\cdot\sigma(d_\tau)$. It follows that $\theta_1(\bar{\rho})=[{}_\rho L^G]$, and ${}_\rho L^G=\{x\in L; x=\rho(\sigma)\cdot\sigma(x)$, for all $\sigma\in G\}\approx\{y\in P_0\otimes_k L; y=\rho(\sigma)\cdot u_\sigma y\}$
$=\{y\in P_0\otimes_k L; y=\rho(\sigma)\cdot d_\sigma^{-1}\cdot\bar{\sigma}(y)$, for all $\sigma\in G\}=\{y\in P_0\otimes_k L; y=I\otimes\sigma(y)$ for all $\sigma\in G\}=(P_0\otimes_k L)^{I\times G}$. Since $L\supset k$ is a Galois extension, there is an element c in L such that $\sum_{\sigma\in G}\sigma(c)=1$, therefore any element $y=\sum x_i\otimes a_i\in(P_0\otimes L)^{I\times G}$,

$y=\sum_{\sigma\in G} y\sigma(c)=\sum_{\sigma\in G} I\otimes\sigma(yc)=\sum_{C} x_i\otimes\sum_{\sigma\in G}\sigma(a_ic)=\sum_{C} x_i\cdot\sum_{\sigma\in G}\sigma(a_ic)\otimes 1$, so y is contained in $P_0\otimes_k L^G=P_0\otimes k=P_0$. Accordingly, we have $\theta_1(\bar{\rho})=[P_0]$.

(3). θ_3: $P(L)^G\to H^2(G, L^*)$;

Let $[P]\in P(L)^G$. By Corollary 1, there exists a factor set f related to Φ_0, i.e. $f=\rho\in Z^2(G, L^*)$, such that $\mathrm{Hom}_k(P, P)$ is L-isomrphic to $\Delta(f, L, \Phi_0, G)$ $=\Delta(\rho, L, G)$ as k-algebra. We define the map θ_3: $P(L)^G\to H^2(G, L^*)$ by $\theta_3([P])=\bar{\rho}$ for $[P]\in P(L)^G$. Then θ_3 is a homomorphism. Because, for $[P]$, $[P']\in P(L)^G$, we have $\mathrm{Hom}_k(P, P)=\Delta(\rho, L, G)=\sum_{\sigma\in G}\oplus Lf_\sigma$ and $\mathrm{Hom}_k(P', P')$ $=\Delta(\rho', L, G)=\sum_{\sigma\in G}\oplus Lf_\sigma'$ where $\bar{\rho}=\theta_3([P])$, $\bar{\rho}'=\theta_3([P'])$, and $\{f_\sigma\}_{\sigma\in G}$ and $\{f_\sigma'\}_{\sigma\in G}$ are L-free basis in $\mathrm{Hom}_k(P, P)$ and $\mathrm{Hom}_k(P', P')$, repsectively. Then the k-isomorphism $f_\sigma\otimes f_\sigma'$: $P\otimes_L P'\to P\otimes_L P'$ defined by $f_\sigma\otimes f_\sigma'(x\otimes y)=f_\sigma(x)\otimes f_\sigma'(y)$ for $x\otimes y\in P\otimes_L P'$, (it is well defined), satisfies $\sigma(a)\cdot f_\sigma\otimes f_\sigma'=f_\sigma\otimes f_\sigma'\cdot a$ for all a in L and $f_\sigma\otimes f_\sigma'\cdot f_\tau\otimes f_\tau'=\rho(\sigma,\tau)\cdot\rho'(\sigma,\tau)\cdot f_{\sigma\tau}\otimes f_{\sigma\tau}'$. Therefore, we can write $\mathrm{Hom}_k(P\otimes_L P', P\otimes_L P')=\Delta(\rho\cdot\rho', L, G)=\sum\oplus L f_\sigma\otimes f_\sigma'$. Accordingly, $\theta_3([P]\cdot[P'])=\theta_3([P])\cdot\theta_3([P'])$.

Lemma 4. *$P(k)\overset{\theta_2}{\to}P(L)^G\overset{\theta_3}{\to}H^2(G, L^*)$ is exact.*

Proof. If $[P_0]\in P(k)$ then $\theta_2([P_0])=[P_0\otimes_k L]$ and $\mathrm{Hom}_k(P_0\otimes_k L, P_0\otimes_k L)$ $\approx\mathrm{Hom}_k(L, L)=\Delta(L, G)$ and so $\theta_3(\theta_2([P_0])=1$. Let $[P]\in P(L)^G$ and $\theta_3([P])=1$. so $\mathrm{Hom}_k(P, P)\approx\Delta(L, G)$. By Lemma 1, there is $[P_0]$ in $P(k)$, and $P\approx P_0\otimes_k L$, therefore $\theta_2([P_0])=[P]$.

(4). θ_4: $H^2(G, L^*)\to B(L/k)$;

$B(L/k)$ denotes the Brauer group of k-Azumaya algebras split by L. θ_4: $H^2(G, L^*)\to B(L/k)$ is defined by $\theta_4(\bar{\rho})=[\Delta(\rho, L, G)]$ in $B(L/k)$ for $\bar{\rho}\in H^2(G, L^*)$, then θ_4 is a homomorphism by [1], Theorem A. 12.

Lemma 5. *$P(L)^G\overset{\theta_3}{\to}H^2(G, L^*)\overset{\theta_4}{\to}B(L/k)$ is exact.*

Proof. Let $[P]\in P(L)^G$ and $\mathrm{Hom}_k(P, P)\approx\Delta(\rho, L, G)$. Then $\theta_4\theta_3([P])=$ $[\Delta(\rho, L, G)]=[\mathrm{Hom}_k(P, P)]=1$ in $B(L/k)$. On the other hand, if $\bar{\rho}$ is an element in $H^2(G, L^*)$ such tha $\theta_4(\bar{\rho})=[\Delta(\rho, L, G)]=[k]$, then there is a finitely generated faithful projective k-module P such that $\Delta(\rho, L, G)\cong\mathrm{Hom}_k(P, P)$. By Proposition 5, $[P]\in P(L)$ and by Corollary 1 $[P]\in P(L)^G$, and so $\bar{\rho}=\theta_3([P])$.

(5). θ_5: $B(L/k)\to H^1(G, P(L))$;

For any $[A]\in B(L/k)$, there is an Azumaya k-algebra Λ in $[A]$ such that Λ contains L as maximal commutative subalgebra (cf. [1], Theorem 5. 7). By Proposition 3, Λ is written by $\Delta(f, L, \Phi, G)$ for some Φ and f, and then $\Phi=\varphi\Phi_0$. for some φ in $Z^1(G, P(L))$. We put $\theta_5([A])=\bar{\varphi}$. From the following lemma, it is shown that θ_5 defines the map $B(L/k)\to H^1(G, P(L))$.

Lemma 6. *Let $\Phi=\varphi\Phi_0$ and $\Phi'=\varphi'\Phi_0$ be elements in $\mathfrak{G}$, and f and f' factor set related to Φ and Φ', respectively. If $[\Delta(f, L, \Phi, G)]=[\Delta(f', L, \Phi', G)]$ in $B(L/k)$, then $\varphi'\varphi^{-1}$ is in $B^1(G, P(L))$.*

Proof. If $[\Delta(f, L, \Phi, G)]=[\Delta(f', L, \Phi', G)]$, then there is a finitely generated projective and faithful k-module P such that

$$\begin{aligned}\mathrm{Hom}_k(P, P)&\approx\Delta(f', L, \Phi', G)\otimes_k\Delta(f, L, \Phi, G)^0\\&=\Delta(f, L, \Phi', G)\otimes_k\Delta(f^0, L, \Phi^0, G)\\&\approx\Delta(f'\otimes f^0, L\otimes_k L, \Phi'\otimes\Phi^0\ G\times G),\end{aligned}$$

where $\Phi'(\sigma)=[J_\sigma']$, $\Phi^0(\sigma)=[J_\sigma^0]$ and $\Phi'\otimes\Phi^0(\sigma\times\tau)=[J_\sigma'\otimes_k J_\tau^0]$ in $Pic_k(L\otimes_k L)$, and $(f'\otimes f^0)_{\sigma\times\tau,\sigma'\times\tau'}\approx f'_{\sigma,\tau'}\otimes f_{\tau,\tau'}$. Regarding P as $L\otimes_k L$-module, by Proposition 5, $[P]\in P(L\otimes_k L)$ and $(\Phi(\sigma)\otimes\Phi^0(\tau))\cdot[P]=[P]\cdot(\Phi_0(\sigma)\otimes\Phi_0(\tau))$ for $\sigma, \tau\in G$. Since $\Phi'=\varphi'\Phi_0$, $\Phi^0=\varphi^{-1}\Phi_0$, we have $\varphi'(\sigma)\otimes\varphi^{-1}(\tau)=[P]\cdot([P]^{-1})^{\sigma\times\tau}$ in $P(L\otimes_k L)$. In particular, if one put $\Phi=\Phi'$, then obtain similarly $\varphi'(\sigma)^{-1}\otimes\varphi'(\tau)=[Q]\cdot([Q]^{-1})^{\sigma\times\tau}$ for some $[Q]$ in $P(L\otimes_k L)$. From $\varphi'(\sigma)\otimes\varphi^{-1}(\tau)=[P]\cdot([P]^{-1})^{\sigma\times\tau}$ and $\varphi'^{-1}(\sigma)\otimes\varphi'(\tau)=[Q]\cdot([Q]^{-1})^{\sigma\times\tau}$, we obtain $[L]\otimes\varphi'(\tau)\varphi(\tau)^{-1}=[P\otimes_{L\otimes_k L}Q]\cdot([P\otimes_{L\otimes_k L}Q]^{-1})^{\sigma\times\tau}$. We put $[R]=[P\otimes_{L\otimes_k L}Q]$ and $[P_\tau]=\varphi'\varphi^{-1}(\tau)=\varphi'(\tau)\cdot\varphi^{-1}(\tau)$, so we have $[L\otimes_k P_\tau]=[R]\cdot([R]^{-1})^{\sigma\times\tau}$. If one takes $\tau=1$, then from $\varphi'\varphi^{-1}(1)=[P_1]=[L]$, we have $[L\otimes_k L]=[R]\cdot([R]^{-1})^{\sigma\times I}$ and so $[R]=[R]^{\sigma\times I}$ for all $\sigma\in G$. Regarding $L\otimes_k L$ as a Galois extension of L with Galois groop $G\times I$, it is known that $L\otimes_k L$ is a trivlal Galois extension of L with Galois group $G\times I$. From Remark 5, there is an element $[R_0]$ in $P(L)$ such that $[R]=[(L\otimes_k L)\otimes_L R_0]=[L\otimes_k R_0]$ in $P(L\otimes_k L)$. Therefore, $[L\otimes_k P_\tau]=[L\otimes_k R_0]\cdot([L\otimes_k R_0]^{-1})^{\sigma\times\tau}$, and so it can be computed that $L\otimes_k P_\tau\approx L\otimes_k(R_0\otimes_L {}_\tau L_1\otimes_L R_0{}^*\otimes_{\tau^{-1}}L_1)$ as $L\otimes_k L$-module for every $\tau\in G$. Therefore, $L\otimes_k L\otimes_L P_\tau\approx L\otimes_k L\otimes_L(R_0\otimes_L R_0^{*\tau})$ as $L\otimes_k L$-moduoe. Since $L\otimes_k L=\sum_{\sigma\in G}\oplus e_\sigma L$ is a trivial Galois extension of L, we have $\sum_{\sigma\in G}\oplus e_\sigma L\otimes_L P_\tau\approx\sum_{\sigma\in G}\oplus e_\sigma L\otimes_L(R_0\otimes_L R_0^{*\tau})$ as $L\otimes_k L=\sum e_\sigma L$-modules, and so $e_\sigma L\otimes_L P_\tau\approx e_\sigma L\otimes_L(R_0\otimes_L R_0^{*\tau})$ as $L\otimes_k L$-module for each $\sigma\in G$. On the other hand, $e_\sigma L\otimes_L P_\tau$ and P_τ are L-isomorphic, and $e_\sigma L\otimes_L(R_0\otimes_L R_0^{*\tau})$ and $R_0\otimes_L R_0^{*\tau}$ are so. Therefore, we have $P_\tau\approx R_0\otimes_L R_0^{*\tau}$ as L-module for every $\tau\in G$, i.e. $[P_\tau]=\varphi'\varphi^{-1}(\tau)=[R_0]\cdot([R_0]^{-1})^\tau$ in $P(L)$ for every $\tau\in G$. Accordingly, $\varphi'\varphi^{-1}$ is in $B^1(G, P(L))$.

Lemma 7. *$H^2(G, L^*)\overset{\theta_4}{\to}B(L/k)\overset{\theta_5}{\to}H^1(G, P(L))$ is exact.*

Proof. If $\bar{\rho}$ is in $H^2(G, L^*)$, then $\theta_4(\bar{\rho})=[\Delta(\rho, L, G)]=[\Delta(\rho, L, \Phi_0, G)]$, so $\theta_5\theta_4(\bar{\rho})=1$. Let $[A]=[\Delta(f, L, \Phi, G)]\in B(L/k)$ and $\theta_5([A])=\bar{\varphi}=1$. Since

$\varphi\in B^1(G, P(L))$, there is $[P]$ in $P(L)$ and $\varphi(\sigma)=[P]\cdot([P]^{-1})^\sigma$ for all $\sigma\in G$. Since $\mathrm{Hom}_k(P, P)$ is an Azumaya k-algebra with maximal commutative subalgebra L, by Proposition 3 $\mathrm{Hom}_k(P, P)$ is L-isomorphic to $\Delta(g. L, \varphi'\Phi_0, G)$ with some φ' and g, as k-algebra. From Proposition 5, we have $\varphi'(\sigma)\Phi_0(\sigma)\cdot[P]=[P]\cdot\Phi_0(\sigma)$ for all $\sigma\in G$, and so $\varphi'(\sigma)=[P]\cdot([P]^{-1})^\sigma=\varphi(\sigma)$ for all $\sigma\in G$, i.e. $\varphi=\varphi'$. We put $\Phi=\varphi\Phi_0=\varphi'\Phi_0$. By Proposition 4, there exists an element ρ in $Z^2(G, L^*)$ such the $f=\rho g$. Since $\rho\otimes\rho^{-1}$ is in $B^2(G\times G, (L\otimes_k L)^*)$ (cf. [1], Proposition A. 11), by Proposition 4, $\Delta((\rho\otimes\rho^{-1})(I\otimes\rho)(g\otimes I), L\otimes_k L, \Phi\otimes\Phi_0, G\times G)$ and $\Delta((I\otimes\rho)(g\otimes I), L\otimes_k L, \Phi\otimes\Phi_0, G\times G)$ are $L\otimes_k L$-isomorphic as k-algebra. On the other hand,

$$\begin{aligned}&\Delta((\rho\otimes\rho^{-1})(I\otimes\rho)(g\otimes I), L\otimes_k L, \Phi\otimes\Phi_0, G\times G)\\ &\approx\Delta(\rho g, L, \Phi, G)\otimes_k\Delta(I, L, \Phi_0, G) = \Delta(f, L, \Phi, G)\otimes_k\Delta(L, G)\,,\\ &\text{and } \Delta((I\otimes\rho)(g\otimes I), L\otimes_k L, \Phi\otimes\Phi_0, G\times G)\\ &\approx\Delta(g, L, \Phi, G)\otimes_k\Delta(\rho, L, \Phi_0, G)=\mathrm{Hom}_k(P, P)\otimes_k\Delta(\rho, L, G)\,.\end{aligned}$$

Accordingly, $[A]=[\Delta(f, L, \Phi, G)]=[\Delta(\rho, L, G)]=\theta_4(\bar{\rho})$.

(6). θ_6; $H^1(G, P(L))\to H^3(G, L^*)$;

Let $\varphi\in Z^1(G, P(L))$. We put $\Phi=\varphi\Phi_0$ and $\Phi(\sigma)=[J_\sigma]$ for each $\sigma\in G$. One takes a family $\{f_{\sigma,\tau};\sigma,\tau\in G\}$ of L-L-isomorphism $f_{\sigma,\tau}\colon J_\sigma\otimes_L J_\tau\to J_{\sigma\tau}$. Put $\omega(\sigma,\tau,\gamma)=f_{\sigma\tau,\gamma}\circ(f_{\sigma,\tau}\otimes I)\circ(I\otimes f_{\tau,\gamma})^{-1}\circ f_{\sigma,\tau\gamma}{}^{-1}$ for each $\sigma,\tau,\gamma\in G$. Since $\omega(\sigma,\tau,\gamma)$ is a unit in $\mathrm{Hom}_L(J_{\sigma\tau\gamma}, J_{\sigma\tau\gamma})=L$, we have a function $\omega\colon G\times G\times G\to L^*$; $(\sigma,\tau,\gamma)\rightsquigarrow\omega(\sigma,\tau,\gamma)$. We shall show that ω is in $Z^3(G, L^*)$ i.e. $\delta(\omega)=1$ where δ is coboundary operator. Since $\delta(\omega)(\sigma,\tau,\gamma,\varepsilon)$ is a unit in L, $\delta(\omega)(\sigma,\tau,\gamma,\varepsilon)=1$ for every $\sigma,\tau,\gamma,\varepsilon$ in G, if and only if for any maximal ideal $\mathfrak{m}$ of L, the image of $\delta(\omega)(\sigma,\tau,\gamma,\varepsilon)$ in $L_\mathfrak{m}$ equals to 1 for every $\sigma,\tau,\gamma,\varepsilon$ in G. But, if L is local, then, from that J_σ is a free L-module, there is a map ρ of $G\times G$ to L^* such that $\omega=\delta(\rho)$. Therefore, $\delta(\omega)=\delta^2(\rho)=1$, i.e. $\delta(\omega)(\sigma,\tau,\gamma,\varepsilon)=1$ for every $\sigma,\tau,\gamma,\varepsilon$ in G. Accordingly ω is in $Z^3(G, L^*)$. For Φ in $\mathfrak{G}$, if one takes another family $\{f'_{\sigma,\tau};\sigma,\tau\in G\}$, then there is a map $\rho\colon G\times G\to L^*$ such that $f'_{\sigma,\tau}=\rho(\sigma,\tau)\cdot f_{\sigma,\tau}$ for every $\sigma,\tau\in G$. Then. it is easily computed that

$$\begin{aligned}\omega'(\sigma,\tau,\gamma)&=f'_{\sigma\tau,\gamma}\circ(f'_{\sigma,\tau}\otimes I)\circ(I\otimes f'_{\tau,\gamma})^{-1}\circ f'^{-1}_{\sigma,\tau\gamma}\\ &=\sigma(\rho(\sigma\tau,\gamma))\cdot\rho(\sigma\tau,\gamma)^{-1}\cdot\rho(\sigma,\tau\gamma)\cdot\rho(\sigma,\tau)\cdot f_{\sigma\tau,\gamma}\circ(f_{\sigma,\tau}\otimes I)\circ(I\otimes f_{\tau,\gamma})^{-1}\circ f^{-1}_{\sigma,\tau\gamma}\\ &=\delta(\rho)(\sigma,\tau,\gamma)\cdot\omega(\sigma,\tau,\gamma).\end{aligned}$$

If $\varphi'=\varphi_0\cdot\varphi$ for some φ_0 in $B^1(G, L^*)$, then there is $[P]\in P(L)$ such that $\varphi'\Phi_0(\sigma)=[P]\cdot\Phi(\sigma)\cdot[P^*]$. If $f_{\sigma,\tau}\colon J_\sigma\otimes_L J_\tau\to J_{\sigma\tau}$ and $I\otimes f_{\sigma,\tau}\otimes I\colon (P\otimes_L J_\sigma\otimes_L P^*)\otimes_L(P\otimes J_\tau\otimes P^*)=P\otimes_L J_\sigma\otimes_L J_\tau\otimes_L P^*\to P\otimes_L J_{\sigma\tau}\otimes_L P^*$ identify, then we can consider that $\omega(\sigma,\tau,\gamma)$ is in $\mathrm{Hom}_L(P\otimes_L J_{\sigma\tau\gamma}\otimes_L P^*, P\otimes_L J_{\sigma\tau\gamma}\otimes_L P^*)$. Therefore, a element $\bar{\omega}$ in $H^3(G, L^*)$ is determined by an element $\bar{\varphi}$ in $H^1(G, L^*)$. We can define the map θ_6: $H^1(G, P(L))\to H^3(G, L^*)$ by $\theta_6(\bar{\varphi})=\bar{\omega}$, for $\bar{\varphi}\in H^1(G, P(L))$.

Lemma 8. $B(L/k)\xrightarrow{\theta_5} H^1(G, P(L))\xrightarrow{\theta_6} H^3(G, L^*)$ *is exact.*

Proof. For $\bar{\varphi}$ in $H^1(G, P(L))$, we put $\Phi=\varphi\Phi_0$ and $\Phi(\sigma)=[J_\sigma]$ for $\sigma\in G$. Then it is easily seen that $\theta_6(\bar{\varphi})=1$ if and only if there is a family $\{f_{\sigma,\tau}: J_\sigma J_\tau\otimes_L\to J_{\sigma\tau};$ L-L-isomorphism, $\sigma, \tau\in G\}$ such that $\{f_{\sigma,\tau}; \sigma, \tau\in G\}$ is a factor set related to Φ. Therefore $\theta_6(\bar{\varphi})=1$ if and only if there is $\Delta[(f, L, \Phi, G)]$ in $B(L/k)$ such that $\theta_5([\Delta(f, L, \Phi, G)])=\bar{\varphi}$.

We have obtained the following seven terms exact sequence.

Theorem (Chase, Harrison and Rosenberg).

$$(1)\longrightarrow H^1(G, L^*)\xrightarrow{\theta_1} P(k)\xrightarrow{\theta_2} P(L)^G\xrightarrow{\theta_3} H^2(G, L^*)\xrightarrow{\theta_4}$$
$$B(L/k)\xrightarrow{\theta_5} H^1(G, P(L))\xrightarrow{\theta_6} H^3(G, L^*)$$

is exact.

From Remark 5 and Therorm, we have

Corollary 2. *If $L\supset k$ is a trivial Galois extension, then*

$$(1)\longrightarrow H^1(G, L^*)\xrightarrow{\theta_1} P(k)\xrightarrow{\theta_2} P(L)^G\longrightarrow(1) \quad \textit{and}$$
$$(1)\longrightarrow H^2(G, L^*)\xrightarrow{\theta_4} B(L/k)\xrightarrow{\theta_5} H^1(G, P(L))\xrightarrow{\theta_6} H^3(G, L^*)$$

are exact.

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