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ON MULTIPLY TRANSITIVE GROUPS VII

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1. Introduction

Let G be a 4-fold transitive group on $\Omega=\{1,2,\cdots,n\}$, and let P be a Sylow 2-subgroup of a stabilizer of four points in G. If $P=1$, then by a theorem of M. Hall [1. Theorem 5.8.1] G must be one of the following groups: S_4, S_5, A_6, A_7 or M_{11}. From a recent result of H. Nagao [7] it follows that, if $P\neq 1$ is semi-regular and leaves exactly four or five points fixed, then G must be one of the following groups: S_6, S_7, A_8, A_9 or M_{12}.

The purpose of this paper is to extend the result of H. Nagao. Namely we shall prove the following

Theorem. *Let G be a 4-fold transitive group. If a Sylow 2-subgroup of a stabilizer of four points in G is semi-regular and not identity, then G must be S_6, S_7, A_8, A_9, M_{12} or M_{23}.*

DEFINITION AND NOTATION. A permutation x is called semi-regular if there is no point fixed by x. A permutation group H is called semi-regular if every nonidentity element of H is semi-regular on the points actually moved by H. For a permutation group G on Ω, let $G_{ij\cdots r}$ denote the stabilizer of the points $i, j,\cdots, r$ in G. For a subset S of G we denote the normalizer (or centralizer) of S in G by $N_G(S)$ (or $C_G(S)$). Let $\alpha_i(x)$ denote the number of i-cycles of a permutation x. The totality of points left fixed by a set X of permutations is denoted by $I(X)$, and if a subset Δ of Ω is a fixed block of X, then the restriction of X on Δ is denoted by X^{Δ}.

2. Proof of the theorem

To prove the theorem we may assume that a stabilizer of four points in G fixes exactly four points (See [6]). In the proof of the theorem, we shall also make use of the fact [1. p. 80] that a 4-fold transitive group of degree less than 35 is one of the known groups.

Let P be a Sylow 2-subgroup of G_{1234}. Then $|I(P)|$ is four, five, six, seven or eleven, and $N_G(P)^{I(P)}$ is S_4, S_5, A_6, A_7 or M_{11} (cf. [5]. Lemma 1). By the theorem of H. Nagao, we may treat only the last three cases.

Case I. $\quad |I(P)| = 11\,,\quad N_G(P)^{I(P)} = M_{11}\,.$

Let a be a central involution of P, and suppose that P has an involution b different from a. Then a and b fix the same eleven points and generate a four group. Therefore we may assume that

$$\begin{aligned} a &= (1)\,(2)\cdots(11)\,(i\,j)\,(k\,l)\cdots\,, \\ b &= (1)\,(2)\cdots(11)\,(i\,k)\,(j\,l)\cdots\,. \end{aligned}$$

Then $\langle a, b\rangle$ normalizes G_{ijkl} and hence it normalizes some Sylow 2–subgroup P' of G_{ijkl}. Since $a^{I(P')}$ is an involution of M_{11}, it fixes three points. Now $I(a) = I(b) = \{1, 2, \cdots, 11\}$. Hence $b^{I(P')}$ fixes these three points and the four group $\langle a, b\rangle^{I(P')}$ is contained in a stabilizer of three points of M_{11}. But this is impossible, because a stabilizer of three points of M_{11} is a quaternion group.

Thus P has only one involution, and hence P must be a cyclic group or a generalized quaternion group [1. Theorem 12.5.2.]. By Theorem 1 in [8] P is not cyclic, and by the following unpublished result of H. Nagao we have a contradiction.

Lemma 1. (H. Nagao) *Let G be a 4–fold transitive group, and $P \neq 1$ a Sylow 2–subgroup of G_{1234}. If P fixes eleven points, then P is not a generalized quaternion group.*

Thus we have no group in this case.

Case II. $\quad |I(P)| = 6$ or $7\,,\quad N_G(P)^{I(P)} = A_6$ or $A_7\,.$

In the proofs of the following series from i) to v) we assume that $N_G(P)^{I(P)} = A_6$, and we need the following result [8. Theorem 2] that any involution of G fixes exactly six points. The proofs in the case $N_G(P)^{I(P)} = A_7$ are similar.

i) *If an element a of G has a 4–cycle, then it's order is an odd multiply of 4 or 8.*

1) *If a is of order 4, then $\alpha_2(a)=2$ and $\alpha_1(a)=2$. (When $N_G(P)^{I(P)} = A_7$, $\alpha_2(a)=2$ and $\alpha_1(a)=3$).*

2) *If a is of order $4t$ with t odd, then $\alpha_2(a)=2$ and $\alpha_1(a)=2$. (When $N_G(P)^{I(P)} = A_7$, $\alpha_2(a)=2$ and $\alpha_1(a)=0$ or 3).*

3) *If a is of order $8t$ with t odd, then $\alpha_4(a)=1$ and $\alpha_2(a)=1$.*

Proof. 1) Let a be an element of order 4. Then we may assume that

$$a = (1\ 2\ 3\ 4)\cdots\,.$$

Since a normalizes G_{1234}, a normalizes some Sylow 2–subgroup of G_{1234}. We may assume that a normalizes P. Since $N_G(P)^{I(P)} = A_6$, a must be of the following form

$$a=(1\ 2\ 3\ 4)\,(5\ 6)\cdots,$$

where $I(P)=\{1, 2, 3, 4, 5, 6\}$. Since a^2 is an involution and $\alpha_1(a^2)=6, \alpha_1(a)=0$. 2 or 4. If $\alpha_1(a)=0$, then $\alpha_2(a)=3$ and a is of the form

$$a=(1\ 2\ 3\ 4)\,(5\ 6)(i_1 j_1)\,(i_2 j_2)\cdots.$$

From this form a normalizes some Sylow 2–subgroup P' of $G_{56i_1j_1}$ and $a^{I(P')}=(5\ 6)\,(i_1 j_1)\,(i_2 j_2)$, which is contrary to $N_G(P')^{I(P')}=A_6$. Therefore $\alpha_1(a)\neq 0$. Since P is semi-regular, P is elementary abelian by Lemma 2 in [8]. Therefore $\alpha_1(a)\neq 4$. Hence we have that $\alpha_1(a)=2$ and consequently $\alpha_2(a)=2$.

2) Let a be of order $4t$ with t odd, then $\alpha_2(a^t)=2$ and $\alpha_1(a^t)=2$. Therefore $\alpha_2(a)=2$ and $\alpha_1(a)=2$.

3) Let a be of order 8. Then from 1) $\alpha_2(a^2)=2$. Hence $\alpha_4(a)=1$. Thus we may assume that a is of the following form

$$a=(1\ 2\ 3\ 4)\cdots.$$

Then a normalizes some Sylow 2–subgroup of G_{1234}. We may assume that a normalizes P. Since $N_G(P)^{I(P)}=A_6$, a must be of the following form

$$a=(1\ 2\ 3\ 4)\,(5\ 6)\cdots,$$

where $I(P)=\{1, 2, 3, 4, 5, 6\}$. Since $\alpha_1(a^2)=2$, $\alpha_2(a)=1$. This is also true for an element of order $8t$ with t odd.

Since an element of order 8 has only one 4–cycle, G has no element of order 16.

ii) *P is an elementary abelian group of order* 16.

Proof. By Lemma 2 in [8] P is elementary abelian. Therefore it suffices to prove that the order of P is 16. Let $a=(1\ 2)\,(3\ 4)\cdots$ be an involution of G. Then a normalizes a Sylow 2–subgroup of G_{1234}. We may assume that a normalizes P. Let $I(P)=\{1, 2, 3, 4, 5, 6\}$. Since $a^{I(P)}$ must be an even permutation, a is of the following form

$$a=(1\ 2)\,(3\ 4)\,(5)\,(6)\cdots.$$

Let a fixes the point 7 and let Δ be the P-orbit containing 7. Then a fixes at most four points of Δ and P is regular on Δ. Therefore by Lemma in [4] we have that the order of P is at most 16.

Now let K be the kernel of the natural homomorphism $N_G(P)\rightarrow N_G(P)^{I(P)}$. Then $KC_G(P)/C_G(P)\cong K/K\cap C_G(P)$, and $N_G(P)\trianglerighteq K\cdot C_G(P)\trianglerighteq K$. Since $N_G(P)/K=N_G(P)^{I(P)}$, $N_G(P)/K$ is a simple group. Therefore $N_G(P)=K\cdot C_G(P)$ or $K\cdot C_G(P)=K$. Since $G_{1234}\geq K\geq K\cap C_G(P)\geq P$ and P is a Sylow 2–subgroup of G_{1234}, $K/K\cap C_G(P)$ is of odd order. If $N_G(P)=K\cdot C_G(P)$, then from $KC_G(P)/C_G(P)\cong K/K\cap C_G(P)$, $N_G(P)/C_G(P)$ is of odd order. Hence any 2-element of $N_G(P)$ belongs to $C_G(P)$. On the other hand there is an element x of order four, which is of the following form

$$x=(1\ 2\ 3\ 4)\cdots .$$

Then x normalizes G_{1234}, and hence we may assume that x normalizes P. From $N_G(P)^{I(P)}=A_6$, x must be of the form

$$x=(1\ 2\ 3\ 4)\,(5\ 6)\cdots ,$$

where $I(P)=\{1, 2, 3, 4, 5, 6\}$. By 1) of i) x fixes two points of $\Omega - I(P)$. Since P is semi-regular on $\Omega - I(P)$, x commutes with exactly two elements of P. But by Theorem 1 in [8] $|P|>2$. Therefore $x\notin C_G(P)$, which is a contradiction. Thus $K\cdot C_G(P)=K$.

Now $N_G(P)/C_G(P)$ is a subgroup of the automorphism group of P. From $N_G(P)/K\cong A_6$ and $K\geq C_G(P)$, the order of the automorphism group of P is not smaller than the order of A_6. Since P is elementary abelian and it's order is at most 16, the order of P must be 16.

Next we also need a theorem of G. Frobenius (See [3]. Proposition 14.5), which will be stated here as Lemma 2.

Lemma 2. (G. Frobenius) *Let $G\leq S_n$, then*

$$\sum_{x\in G}\binom{\alpha_1(x)}{\kappa}\binom{\alpha_2(x)}{\lambda}\cdots=\frac{m\cdot|G|}{1^\kappa\cdot\kappa!\cdot 2^\lambda\cdot\lambda!\cdots}.$$

Here m is an integer obtained in the following way. Let $\Omega^{(t)}=\{(i_1,\cdots, i_\kappa, j_1, j_1{}', \cdots, j_\lambda, j_\lambda{}', \cdots)\}$ be a family of ordered sets consisting of t $(=\kappa+2\lambda+\cdots)$ points of Ω such that there is at least one element x of G of the form

$$x=(i_1)\cdots(i_\kappa)(j_1j_1{}')\cdots(j_\lambda j_\lambda{}')\cdots .$$

When G is regarded as a permutation group on $\Omega^{(t)}$ by setting

$$(a_1,\cdots, a_t)^x=(a_1^x,\cdots, a_t^x)$$

for $x\in G$ and $(a_1,\cdots, a_t)\in\Omega^{(t)}$, m is the number of G-orbits in $\Omega^{(t)}$.

iii) *Let x be an involution of $N_G(P)-P$. Then any fixed point of an element $(\neq 1)$ of $\langle x, P\rangle^{\Omega-I(P)}$ is contained in exactly one orbit of P. The number of P-orbits in $\Omega - I(P)$ is odd.*

Proof. Since the order of P is 16, by Lemma in [4] x commutes with four elements of P. Since P is semi-regular and x fixes four points of $\Omega - I(P)$, these points must be contained in the same P-orbit, say Δ. Put $Q=\langle x, P\rangle$. The order of Q is 32 and Q fixes Δ. For an element a of P if x commutes with a, then xa is of order 2, and if not, then xa is of order 4, since $(xax)a$ belongs to P and it is not the identity. Let xa be of order 4, then by 1) of i) $\alpha_1(xa)=2$ and xa has no fixed point on $\Omega-I(P)$. Therefore

$$\begin{aligned}\sum_{y\in Q}\alpha_1(y^\Delta) &= \alpha_1(1^\Delta)+\sum_{y'}{}'\,\alpha_1(y'^\Delta)\\ &= 16+\sum_{y'}{}'\,\alpha_1(y'^\Delta)\,,\end{aligned}$$

where y' ranges over all involutions of $Q-P$. On the other hand from Lemma 2

$$\sum_{y\in Q}\alpha_1(y^\Delta) = |Q^\Delta| = 32\,.$$

Hence $\sum_{y'}{}'\,\alpha_1(y'^\Delta)=16$. Since $Q-P$ has four involutions and these involutions have four fixed points in $\Omega-I(P)$ respectively, these 16 points are all contained in Δ. Hence Q is semi-regular on $\Omega-\{I(P)\cup\Delta\}$, in which any Q-orbit contains exactly two P-orbits. Thus the number of P-orbits in $\Omega-I(P)$ is odd.

iv) *G has an element of order* 8.

Proof. Let a be an element of order four and of the following form

$$a=(1\ 2\ 3\ 4)\cdots\,.$$

Then a normalizes a Sylow 2-subgroup of G_{1234}. We may assume that a normalizes P. From $N_G(P)^{I(P)}=A_6$, a must be of the following form

$$a=(1\ 2\ 3\ 4)\,(5\ 6)\cdots\,,$$

where $I(P)=\{1, 2, 3, 4, 5, 6\}$. By 1) of i) a fixes two points of $\Omega-I(P)$, and these points are contained in a P-orbit, say Δ. Put $Q=\langle P, a\rangle$. Then the order of Q is $4\cdot16$ and Δ is a Q-orbit. Suppose that Q has no element of order 8. From iii) any fixed point of an element $(\neq1)$ of $\langle P, a^2\rangle^{\Omega-I(P)}$ is contained in Δ. Let a' be any element of Pa or Pa^{-1}. Then a' is of the following form

$$(1\ 2\ 3\ 4)\,(5\ 6)\cdots \text{ or } (1\ 4\ 3\ 2)\,(5\ 6)\cdots\,.$$

We assumed that Q has no element of order 8. Hence a' is of order 4, and a' has exactly two fixed points. Since Q/P is a cyclic group of order 4, a'^2 belongs to $\langle P, a^2\rangle$ Therefore a' fixes two points of Δ.

From $Q=P+Pa+Pa^2+Pa^{-1}$

$$\begin{aligned}\sum_{x\in Q}\alpha_1(x^\Delta) &\geqq \sum_{x\in P}{}'\,\alpha_1(x^\Delta)+\sum_{x\in Pa}{}'\,\alpha_1(a^\Delta)+\sum_{x\in Pa^{-1}}{}'\,\alpha_1(x^\Delta)\\ &=16+2\cdot16+2\cdot16=5\cdot16.\end{aligned}$$

On the other hand by Lemma 2

$$\sum_{x\in Q}\alpha_1(x^{\Delta})=|Q|=4\cdot 16\,,$$

which is a contradiction. Thus Q has an element of order 8.

Since G is 4–fold transitive, by Lemma 2

$$\sum_{x\in G}\alpha_4(x)=\frac{1}{4}g\,,$$

and

$$\sum_{x\in G}\alpha_2(x)\cdot\alpha_4(x)=\frac{m\cdot g}{2\cdot 4}\,,$$

where $g=|G|$. From i) if $\alpha_4(x)\neq 0$, then $\alpha_4(x)\cdot\alpha_2(x)=2\cdot\alpha_4(x)$ or $\alpha_4(x)$. Since there is an element of order 8, from i) we have

$$\sum_{x\in G}\alpha_4(x)<\sum_{x\in G}\alpha_2(x)\cdot\alpha_4(x)<2\cdot\sum_{x\in G}\alpha_4(x)$$

and hence $1<\dfrac{m}{2}<2$. Thus $m=3$, and

$$\sum_{x\in G}\alpha_2(x)\cdot\alpha_4(x)=\frac{3}{8}g\,.$$

From two equations above, we obtain

$$\sum_{y}{}'\,\alpha_4(y)+\sum_{y'}{}'\,\alpha_4(y')=\frac{1}{4}g\,,$$

$$\sum_{y}{}'\,2\cdot\alpha_4(y)+\sum_{y'}{}'\,\alpha_4(y')=\frac{3}{8}g\,,$$

where y and y' range over all elements of order $4t$ and $8t$ (t: odd) respectively. Hence

$$\sum_{y}{}'\,\alpha_4(y)=\frac{1}{8}g\,.$$

On the other hand

$$\sum_{x\in G}\binom{\alpha_2(x)}{2}\cdot\alpha_4(x)=\frac{m'\cdot g}{2^2\cdot 2\cdot 4}\,.$$

Since an element of order $8t$ with t odd has only one 2–cycle, and an element of order $4t$ with t odd has two 2–cycles,

$$\sum_{x\in G}\binom{\alpha_2(x)}{2}\cdot\alpha_4(x)=\sum_{y}{}'\,\alpha_4(y)=\frac{1}{8}g\,,$$

Therefore

$$\frac{m'\cdot g}{2^2\cdot 2\cdot 4}=\frac{1}{8}g\,,$$

hence

$$m'=4\,.$$

From the remark of Lemma 2, G has four orbits on $\Omega^{(8)}=\{(i_1, i_2, j_1, j_2, k_1, k_2, k_3, k_4)\mid x=(i_1\, i_2)\,(j_1 j_2)\,(k_1\, k_2\, k_3\, k_4)\cdots\in G$, x is of order $4\}$. Since G is 4–fold transitive on Ω, $G_{1234}=H$ has four orbits on $\Omega^{(4)}=\{(k_1, k_2, k_3, k_4)\mid a=(1\ 2)\,(3\ 4)\,(k_1\, k_2\, k_3\, k_4)\cdots\in G$, a is of order $4\}$.

When H is regarded as a permutation group on $\Omega^{(4)}$, we denote it by H^*. If $(k_1, k_2, k_3, k_4)\in\Omega^{(4)}$, then there is an element a of G of the form

$$a=(1\ 2)\,(3\ 4)\,(k_1\, k_2\, k_3\, k_4)\cdots\,.$$

Since $a^{-1}=(1\ 2)\,(3\ 4)\,(k_1\, k_4\, k_3\, k_2)\cdots\in G$, we have eight points (k_1, k_2, k_3, k_4), (k_2, k_3, k_4, k_1), (k_3, k_4, k_1, k_2), (k_4, k_1, k_2, k_3), (k_1, k_4, k_3, k_2), (k_4, k_3, k_2, k_1), (k_3, k_2, k_1, k_4) and (k_2, k_1, k_4, k_3) of $\Omega^{(4)}$.

v) *Let* $(k_1, k_2, k_3, k_4)\in\Omega^{(4)}$. *Then* $(k_1, k_2, k_3, k_4)^{H^*}$ *and* $(k_2, k_3, k_4, k_1)^{H^*}$ *are the different* H^**-orbits.*

Proof. Since any 2–element of H is of order 2, H has no element as follows:

$$\begin{pmatrix} k_1\, k_2\, k_3\, k_4\cdots \\ k_2\, k_3\, k_4\, k_1\cdots\end{pmatrix}=(k_1\, k_2\, k_3\, k_4)\cdots\,.$$

Therefore $(k_1, k_2, k_3, k_4)^{H^*}\neq(k_2, k_3, k_4, k_1)^{H^*}$.

From now on we treat two cases $N_G(P)^{I(P)}=A_6$ and A_7 separately. For the proofs in these cases the result that the number of H^*-orbits is four is important.

A) $N_G(P)^{I(P)}=A_6\,.$

Let $I(P)=\{1, 2, 3, 4, 5, 6\}$. and $G_{1234}=H$. Then the points 5 and 6 are contained in H-orbits of odd length. Put $5^H=\Delta_1$ and $6^H=\Delta_2$.

Suppose that $\Delta_1=\Delta_2$. Since P-orbits in Ω—$I(P)$ are of even length, the length of Δ_1 is even, which is a contradiction. Therefore $\Delta_1\neq\Delta_2$. Furthermore the other H-orbits in Ω—$I(H)$ are all of even lengths.

From $N_G(P)^{I(P)}=A_6$ there is an element x of the following form

$$x=(1\ 2)\,(3)\,(4)\,(5\ 6)\cdots\,.$$

Since $x\in N_G(H)$, $\Delta_1^x=\Delta_2$. Hence $|\Delta_1|=|\Delta_2|$. Suppose that H-orbits in Ω—$I(H)$ are Δ_1 and. Δ_2. From iii) the number of P-orbitis in Ω—$I(P)$ is odd, and all P-orbits in $\Omega-I(P)$ are of the same length. Hence $|\Delta_1| \neq |\Delta_2|$, which is a contradiction. Therefore H has at least three orbits in $\Omega-I(H)$.

If $|\Delta_1|=|\Delta_2|=1$, then $|I(H)|=6$, contradicting the assumption that $|I(H)|=4$. Therefore $|\Delta_1|=|\Delta_2|>1$.

Let (k_1, k_2, k_3, k_4) be a point of $\Omega^{(4)}$. Then there is an element

$$a = (1\ 2)\,(3\ 4)\,(k_1\, k_2\, k_3\, k_4)\cdots$$

in G. We may assume that $a \in N_G(P)$ and a is of the form

$$a = (1\ 2)\,(3\ 4)\,(5)\,(6)\,(k_1\, k_2\, k_3\, k_4)\cdots .$$

By the assumption a is of order four, and from 1) of i) any point in $\Omega-\{1, 2, \cdots, 6\}$ appears in some 4–cycle of a. Since $\Delta_1^a=\Delta_1$ and $\Delta_2^a=\Delta_2$, we may assume that $\{i_1, i_2, i_3, i_4\} \subset \Delta_1$ and $\{j_1, j_2, j_3, j_4\} \subset \Delta_2$, where $a=(i_1\, i_2\, i_3\, i_4)\,(j_1 j_2 j_3 j_4)\cdots$. Then from v) $(i_1, i_2, i_3, i_4)^{H^*}$, $(i_2, i_3, i_4, i_1)^{H^*}$, $(j_1, j_2, j_3, j_4)^{H^*}$ and $(j_2, j_3, j_4, j_1)^{H^*}$ are all different H^*-orbits. Thus we have four H^*-orbits. But H has at least three orbits in $\Omega-I(H)$. Hence there is a 4–cycle $(l_1\, l_2\, l_3\, l_4)$ of a such that $\{l_1, l_2, l_3, l_4\} \not\subset \Delta_1 \cup \Delta_2$. Therefore $(l_1, l_2, l_3, l_4)^{H^*}$ is the different H^*-orbit from these four H^*-orbits. Hence we have five H^*-orbits, which is a contradiction.

Thus we have no group in this case.

B) $N_G(P)^{I(P)} = A_7$.

Let $I(P)=\{1, 2, 3, 4, 5, 6, 7\}$ and $H=G_{1234}$. Then from $N_G(P)^{I(P)}=A_7$, there is an element

$$x = (1)\,(2)\,(3)\,(4)\,(5\ 6\ 7)\cdots .$$

Since $x \in H$, three points 5, 6 and 7 belong to the same H-orbit, say Δ_1. Then Δ_1 is the only H-orbit in $\Omega-I(H)$ of odd length. If H has only one orbit in $\Omega-I(H)$, namely, H is transitive on $\Omega-I(H)$, then a stabilizer of one point in G satisfies the assumption of Case II. A), which is a contradiction. Therefore H has at least two orbits, say Δ_1 and Δ_2, in $\Omega-I(H)$.

Suppose that $|\Delta_1|>3$. Let $(k_1, k_2, k_4, k_3) \in \Omega^{(4)}$. Then there is an element

$$a = (1\ 2)\,(3\ 4)\,(k_1\, k_2\, k_3\, k_4)\cdots$$

of G. By the assumption a is of order four, and from 1) of i) the cycles of a are all 4–cycles except two 2–cycles and three 1–cycles. Since $a \in N_G(H)$, Δ_2^a is an H-orbit. Assume that $\Delta_2^a \neq \Delta_2$. Since the length of Δ_2 is even, $\Delta_2^a \neq \Delta_1$. We may assume that $k_1 \in \Delta_2$, and hence $k_2 \notin \Delta_1 \cup \Delta_2$. Then we shall show that $(k_1, k_2, k_3, k_4)^{H^*}$, $(k_2, k_3, k_4, k_1)^{H^*}$, $(k_1, k_4, k_3, k_2)^{H^*}$ and $(k_4, k_3, k_2, k_1)^{H^*}$ are all different H^*-orbits. From v) $(k_1, k_2, k_3, k_4)^{H^*} \neq (k_2, k_3, k_4, k_1)^{H^*}$ and $(k_1, k_4, k_3, k_2)^{H^*} \neq (k_4, k_3, k_2, k_1)^{H^*}$.

If $(k_1, k_2, k_3, k_4)^{H^*}=(k_1, k_4, k_3, k_2)^{H^*}$, then H has a following element

$$x=\begin{pmatrix} k_1 & k_2 & k_3 & k_4 & \cdots \\ k_1 & k_4 & k_3 & k_2 & \cdots \end{pmatrix}=(k_1)(k_3)(k_2\, k_4)\cdots .$$

Since the order of x is even, there is a Sylow 2–subgroup of H fixing the point k_1. By the conjugacy of Sylow 2–subgroups of H, k_1 must be contained in Δ_1, which is a contradiction. Therefore $(k_1, k_2, k_3, k_4)^{H^*}\neq(k_1, k_4, k_3, k_2)^{H^*}$. In the same way $(k_2, k_3, k_4, k_1)^{H^*}\neq(k_4, k_3, k_2, k_1)^{H^*}$.

If $(k_2, k_3, k_4, k_1)^{H^*}=(k_1, k_4, k_3, k_2)^{H^*}$, then H has a following element

$$\begin{pmatrix} k_2 & k_3 & k_4 & k_1 & \cdots \\ k_1 & k_4 & k_3 & k_2 & \cdots \end{pmatrix}=(k_1\, k_2)(k_3\, k_4)\cdots$$

But this is impossible, for $k_1\in\Delta_2$ and $k_2\notin\Delta_2$. Therefore $(k_2, k_3, k_4, k_1)^{H^*}\neq(k_1, k_4, k_3, k_2)^{H^*}$.

Since $a^2=(1)(2)(3)(4)(k_1\, k_3)(k_2\, k_4)\cdots$ belongs to H and $(k_4, k_3, k_2, k_1)^{a^2}=(k_2, k_1, k_4, k_3)$, $(k_4, k_3, k_2, k_1)^{H^*}=(k_2, k_1, k_4, k_3)^{H^*}$. Therefore in the same way $(k_1, k_2, k_3, k_4)^{H^*}\neq(k_2, k_1, k_4, k_3)^{H^*}=(k_4, k_3, k_2, k_1)^{H^*}$.

Thus we have four H^*-orbits. On the other hand since $|\Delta_1|>3$ and $\Delta_1{}^a=\Delta_1$, there is a 4–cycle of a, say $(j_1 j_2 j_3 j_4)$, such that $\{j_1, j_2, j_3, j_4\}\subset\Delta_1$. Then $(j_1, j_2, j_3, j_4)^{H^*}$ is different from these four H^*-orbits. Thus we have five H^*-orbits which is a contradiction. Therefore $\Delta_2{}^a=\Delta_2$.

If H has an orbit different from Δ_1 and Δ_2 in $\Omega—I(H)$, then as proved above a fixes these three orbits respectively. By v) H^* has at least six orbits, which is a contradiction. Thefore H-orbits in $\Omega—I(H)$ are Δ_1 and Δ_2.

Now we shall show that $|\Delta_1|>3$ leads to a contradiction. Since $\Delta_1{}^a=\Delta_1$ and $\Delta_2{}^a=\Delta_2$, we may assume that $\{i_1, i_2, i_3, i_4\}\subset\Delta_1$ and $\{j_1, j_2, j_3, j_4\}\subset\Delta_2$, where $a=(1\ 2)(3\ 4)(i_1\, i_2\, i_3\, i_4)(j_1\, j_2\, j_3\, j_4)\cdots$. Since $\Delta_1\neq\Delta_2$, by v) $(i_1, i_2, i_3, i_4)^{H^*}$, $(i_2, i_3, i_4, i_1)^{H^*}$, $(j_1, j_2, j_3, j_4)^{H^*}$ and $(j_2, j_3, j_4, j_1)^{H^*}$ are all different. We shall show that $(i_1, i_4, i_3, i_2)^{H^*}$ is different from these four H^*-orbits. From$\Delta_1\neq\Delta_2$, $(i_1, i_4, i_3, i_2)^{H^*}\neq(j_1, j_2, j_3, j_4)^{H^*}$ and $(j_2, j_3, j_4, j_1)^{H^*}$. If $(i_1, i_2, i_3, i_4)^{H^*}=(i_1, i_4, i_3, i_2)^{H^*}$, then H has a following element

$$x=\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & \cdots \\ i_1 & i_4 & i_3 & i_2 & \cdots \end{pmatrix}=(i_1)(i_3)(i_2\, i_4)\cdots .$$

Since x is of order even, there is a Sylow 2–subgroup of H fixing 1, 2, 3, 4 i_1 and i_3. Thus any Sylow 2–subgroup of $H_{i_1 i_3}$ is a Sylow 2–subgroup of H. On the other hand $a^2=(1)(2)(3)(4)(i_1\, i_3)(i_2\, i_4)\cdots$ normalizes $H_{i_1 i_3}$. Hence a^2 normalizes a Sylow 2–subgroup P' of $H_{i_1 i_2}$, and $a^{I(P')}=(1)(2)(3)(4)(i_1\, i_3)\cdots$, contrary to $N_G(P')^{I(P')}=A_7$. Thus $(i_1, i_2, i_3, i_4)^{H^*}\neq(i_1, i_4, i_3, i_2)^{H^*}$. If $(i_2, i_3, i_4, i_1)^{H^*}=(i_1, i_4, i_3, i_2)^{H^*}$, then H has a following element

$$h=\begin{pmatrix} i_2\, i_3\, i_4\, i_1 \cdots \\ i_1\, i_4\, i_3\, i_2 \cdots \end{pmatrix}=(i_1\, i_2)\,(i_3\, i_4)\cdots .$$

Then

$$ah=(1\ 2)\,(3\ 4)\,(i_1)\,(i_3)\,(i_2\, i_4)\cdots .$$

Since $\alpha_2(ah)\geqq 3$, from 1) ah is of order $2t$ with t odd. Put $b=(ah)^t$. Then b normalizes a Sylow 2–subgroup P'' of H. Since Δ_2 contains an odd number of P''-orbits, from iii) the four points of $I(b)$ are contained in Δ_2 and the three points i_1, i_3 and some one of $I(b)$ are contained in Δ_1. Since $b\in N_G(P'')$, $I(P'')\supset\{1, 2, 3, 4, i_1, i_3\}$. Therefore $a^2=(1)\,(2)\,(3)\,(4)\,(i_1\, i_3)\cdots$normalizes a Sylow 2–subgroup of $H_{i_1 i_3}$, which is also a Sylow 2–subgroup P''' of H. Thus $(a^2)^{I(P''')}=(1)\,(2)\,(3)\,(4)\,(i_1\, i_3)\cdots$, which contradicts the assumption $N_G(P''')^{I(P''')}=A_7$. Therefore $(i_2, i_3, i_4, i_1)^{H^*}\neq(i_1, i_4, i_3, i_2)^{H^*}$.

Thus H^* has at least five orbits, which is a contradiction. Therefore $|\Delta_1|=3$.

In the proof of Case II of [5. Theorem 2] we needed only the following condition: The number of the fixed points of an involution is seven, and every Sylow 2–subgroup of H fixes the same points. Therefore in the same way we have that G is M_{23}.

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References

[1] M. Hall: The Theory of Groups, Macmillan, New York, 1959.

[2] H. Wielandt: Finite Permutation Groups, Academic Press, New York, 1964.

[3] H. Nagao: Multiply Transitive Groups, California Institute of Technology, California, 1967.

[4] H. Nagao: *On multiply transitive groups* I, Nagoya Math. J. **27** (1966), 15–19.

[5] H. Nagao and T. Oyama: *On multiply transitive groups* II, Osaka J. Math. **2** (1965), 129–136.

[6] H. Nagao: *On multiply transitive groups* IV, Osaka J. Math. **2** (1965), 327–341.

[7] H. Nagao: *On multiply transitive groups* V, J. Algebra. **9** (1968), 240–248.

[8] R. Noda and T. Oyama: *On multiply transitive groups* VI, to appear in J. Algebra.